Mathematica Aeterna, Vol. 2, 2012, no. 10, 839 - 845

Cone Metric Spaces
And
Fixed Point Theorems for Pair of Contractive Maps

A.K.Dubey

Department of Mathematics, Bhilai Institute of Technology,

Bhilai House, Durg 491001, India

e-mail : anilkumardby@ rediffmail.com

A. Narayan

Department of Mathematics, Bhilai Institute of Technology,

Bhilai House, Durg 491001, India

ashu-1959@rediffmail.com

Abstract

We review, generalize and prove some fixed point theorems for contractive maps in cone metric spaces.

Mathematics Subject Classification:54H25, 47H10.
Key Words: Fixed point; Cone metric space; Contractive mapping; Ordered Banach space.

1. Introduction

Very recently, Huang and Zhang [3] introduce the notion of cone metric spaces. He replaced the real numbers by ordering Banach space. He also gave the condition in the setting of cone metric spaces. They prove some fixed point theorems for contractive mappings by using normality of the cone.

We recall the definition of cone metric spaces and some properties of theirs [3].

Definition 1.1 [3] Let E be a real Banach space and P a subset of E. P is called a cone if and only if

(i) P is closed, nonempty, and $P \neq \{0\}$,
(ii) $a, b \in R, a, b \geqslant 0, x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leqslant y$ but $x \neq y$, while $x \ll y$ if and only if $y - x \in Int\, P$, Int P denotes the interior of P.

The cone P is called normal if there is a numer $K > 0$ such that for all $x, y \in E, 0 \leqslant x \leqslant y$ implies $\| x \| \leqslant K \| y \|$. The least positive number satisfying above is called the normal constant of P.

In the following, we always suppose E is a Banach space, P is a cone in E with $Int\ P \neq \phi\ and\ \leqslant$ is partial ordering with respect to P.

Definition 1.2 [3] - Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leqslant d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called cone metric on X, and (X, d) is called a cone metric space.

It is obvious that cone metric spaces generalize metric spaces.

Example 1.3 - Let $E = R^2, P = \{(x, y) \in E : x, y \geqslant 0\} \subset R^2, X = R\, and\, d : X \times X \to E$ such that $d(x, y) = (| x - y |, \propto | x - y |)$, where $\propto \geqslant 0$ is a constant. Then (X, d) is cone metric space.

Definition 1.4 [3] - Let (X, d) be a cone metric space. Let $\{x_n\}$be a sequence in X and $x \in X$. If for every $c \in E$ with $o \ll c$, there is N such that for all $n > N, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n \to \infty} x_n = x\ or\ x_n \to x (n \to \infty)$.

Definition 1.5 [3] - Let (X, d) be a cone metric space, $\{x_n\}$be a sequence in X. If for any $c \in E$ with $o \ll c$, there is N such that for all $n, m > N, d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Lemma 1.6 ([3], Lemmas 1 and 4) - Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$be a sequence in X. Then

(i) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to o\,(n \to \infty)$.
(ii) $\{x_n\}$ is a Cauchy sequence if and only if $d\,(x_n, x_m) \to o\,(n, m \to \infty)$.

Definition 1.7 [3] - Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 1.8 ([3], Lemmas 2 and 5) - Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X;

(i) **If** $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$is unique, obviously limit of $\{y_n\}$ is also unique.

(ii) **If** $x_n \to x,\ y_n \to y\,(n \to \infty)$. Then $d\,(x_n, y_n) \to d(x, y)(n \to \infty)$.

2. Main Results

In this section we shall prove some fixed point theorems for pair of contractive maps by using normality of the cone.

Theorem 2.1 - Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mappings $T_1, T_2 : X \to X$ satisfy the contractive condition $d\,(T_1x, T_2y) \leqslant kd(x, y)$, for all $x, y \in X$,

where $k \in [0, 1)$ is a constant. Then $T_1$and T_2 have a unique common fixed in X. And for any $x \in X$, iterative sequences $\left\{T_1^{2n+1}x\right\}$ *and* $\left\{T_2^{2n+2}x\right\}$ converge to the common fixed point.

Proof: Choose $x_o \in X$. Set $x_1 = T_1x_o, x_3 = T_1x_2 = T_1^3x_o, - - - - - - - -$ $x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o - - - - - - - - - -$

Similarly, we can have $x_2 = T_2x_1 = T_2^2x_o, x_4 = T_2x_3 = T_2^4x_o, - - - - - - - - -x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o - - - - -$

We have

$$d(x_{2n+1}, x_{2n}) = d(T_1x_{2n,}T_2x_{2n-1}) \leqslant kd(x_{2n}, x_{2n-1})$$
$$\leqslant k^2d(x_{2n-1}, x_{2n-2}) \leqslant ---- \leqslant k^{2n}d(x_1, x_0).$$

So far $n > m$

$$d(x_{2n,}x_{2m}) \leqslant d(x_{2n,}x_{2n-1}) + d(x_{2n-1,}x_{2n-2}) + -------- + d(x_{2m+1}, x_{2m})$$

$$\leqslant (k^{2n-1} + k^{2n-2} + ------ + k^{2m})d(x_{1,}x_0) \leqslant \frac{k^{2m}}{1-k}d(x_1, x_o)$$

We get $\| d(x_{2n}, x_{2m}) \| \leqslant \frac{k^{2m}}{1-k}K \| d(x_1, x_o) \|$.

This implies $d(x_{2n,}x_{2m}) \to 0(n, m \to \infty)$. Hence $\{x_{2n}\}$ is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_{2n} \to x^*(n \to \infty)$.

Since

$$d(T_1x^*, x^*) \leqslant d(T_1x_{2n}, T_1x^*) + d(T_1x_{2n}, x^*)$$

$$\leqslant kd(x_{2n}, x^*) + d(x_{2n+1}, x^*),$$

$\| d(T_1x^*, x^*) \| \leqslant K(k \| d(x_{2n}, x^*) \| + \| d(x_{2n+1}, x^*) \|) \to 0$. Hence $\| d(T_1x^*, x^*) \| = 0$. This implies $T_1x^* = x^*$. So $x^*$is a fixed point of T_1.

Now if $y^*$is another fixed fixed point of T_1, then

$$d(x^*, y^*) = d(T_1x^*, T_1y^*) \leqslant kd(x^*, y^*).$$

Hence $\| d(x^*, y^*) \| = 0$ and $x^* = y^*$. Therefore the fixed point of $T_1$is unique.

Similarly it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$. Thus $x^*$is the common fixed point of pair of maps T_1 and T_2. This completes the proof.

Corollary 2.2 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mappings $T_1, T_2 : X \to X$ satisfy for some positive integer n,

$d(T_1^{2n+1}x, T_2^{2n+2}y) \leqslant kd(x, y)$,for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then $T_1$and T_2 have a unique common fixed point in X.

Proof: From Theorem 2.1, T_1^{2n+1} has a unique fixed point x^*. But $T_1^{2n+1}(T_1x^*) = T_1(T_1^{2n+1}x^*) = T_1x^*$, so $T_1x^*$is also a fixed point of T_1^{2n+1}. Hence $T_1x^* = x^*, x^*$is a fixed point of T_1. Since the fixed point of T_1 is also fixed point of T_1^{2n+1}, the fixed point of T_1 is unique.

Similarly it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x*$. Thus x^* is common fixed point of T_1 and T_2.

Theorem 2.3 - Let (X, d) be a complete cone metric space, P a normal cone with normal constant K.

Suppose the mappings $T_1, T_2 : X \to X$ satisfy the contractive condition

$d(T_1x, T_2y) \leqslant k(d(T_1x, x) + d(T_2y, y))$, for all $x, y \in X, where\, k \in [0, \frac{1}{2})$ is a constant. Then $T_1$and T_2 have a unique common fixed point in X. And for any $x \in X$, iterative sequences $\{T_1^{2n+1}x\}$ *and* $\{T_2^{2n+2}x\}$ converge to the common fixed point.

Proof: Choose $x_o \in X$. *Set* $x_1 = T_1x_o, x_3 = T_1x_2 = T_1^3x_o, - - - - - -$ $x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o, - - - - -$

Similarly, we can have $x_2 = T_2x_1 = T_2^2x_o, x_4 = T_2x_3 = T_2^4x_o, - - - -$ $x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o, - - - - -$.

We have

$$d(x_{2n+1}, x_{2n}) = d(T_1x_{2n}, T_2x_{2n-1}) \leqslant k(d(T_1x_{2n}, x_{2n}) + d(T_2x_{2n-1}, x_{2n-1})).$$
$$= k(d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})).$$

So

$$d(x_{2n+1}, x_{2n}) \leqslant \frac{k}{1-k} d(x_{2n,}x_{2n-1}) = hd(x_{2n}, x_{2n-1}),$$

Where $h = \frac{k}{1-k}$. For $n > m$,

$$d(x_{2n,}x_{2m}) \leqslant d(x_{2n,}x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + - - - - - - - + d(x_{2m+1,}x_{2m})$$

$$\leqslant (h^{2n-1} + h^{2n-2} + - - - - - - + h^{2m})d(x_{1,}x_o)$$

$$\leqslant \frac{h^{2m}}{1-h} d(x_1, x_o)$$

We get $\parallel d(x_{2n,}x_{2m}) \parallel \leqslant \frac{h^{2m}}{1-h} K \parallel d(x_{1,}x_o) \parallel$.

This implies $d(x_{2n,}x_{2m}) \to o\,(n, m \to \infty)$. Hence $\{x_{2n}\}$ is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_{2n} \to x^*\,(n \to \infty)$. Since $d\,(T_1x^*, x^*) \leqslant d\,(T_1x_{2n}, T_1x^*) + d\,(T_1x_{2n}, x^*)$

$$\leqslant k\,(d\,(T_1x_{2n}, x_{2n}) + d\,(T_1x^*x^*)) + d(x_{2n+1}, x^*),$$

$$d\,(T_1x^*, x^*) \leqslant \frac{1}{1-k}\,(kd\,(T_1x_{2n}, x_{2n}) + d\,(x_{2n+1}, x^*)).$$

$\| d(T_1x^*, x^*) \| \leqslant K\frac{1}{1-k}\left(k \| d(x_{2n+1}, x_{2n}) \| + \| d(x_{2n+1}, x^*) \|\right) \to o.$

Hence $\| d(T_1x^*, x^*) \| = o$. This implies $T_1x^* = x^*$. So x^* is a fixed point of T_1.

Now if $y^*$is another fixed point of T_1, *then* $d(x^*, y^*) = d(T_1x^*, T_1y^*) \leqslant k(d(T_1x^*, x^*) + d(T_1y^*, y^*)) = o$.

Hence $x^* = y^*$. Therefore the fixed point of T_1 is unique.

Similarly, it can be established that $T_2x^* = x^*$. *Hence* $T_1x^* = x^* = T_2x^*$. *Thus* x^* is the common fixed point of pair of maps T_1 *and* T_2.

Theorem 2.4 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mappings $T_1, T_2 : X \to X$satisfy the contractive condition

$d(T_1x, T_2y) \leqslant k(d(T_1x, y) + d(T_2y, x))$, for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. Then T_1 *and* T_2 have a unique common fixed point in X. And for any $x \in X$, iterative sequences $\{T_1^{2n+1}x\}$ *and* $\{T_2^{2n+2}x\}$ converge to the common fixed point.

Proof: Choose $x_o \in X$. *Set* $x_1 = T_1x_o, x_3 = T_1x_2 = T_1^3x_o, ------ x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o, -----$

Similarly, we can have $x_2 = T_2x_1 = T_2^2x_o, x_4 = T_2x_3 = T_2^4x_o, ---- x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o, -----$.

We have

$$d(x_{2n+1}, x_{2n}) = d(T_1x_{2n}, T_2x_{2n-1}) \leqslant k(d(T_1x_{2n}, x_{2n-1}) + d(T_2x_{2n-1}, x_{2n}))$$

$$\leqslant k(d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})).$$

So

$$d(x_{2n+1}, x_{2n}) \leqslant \frac{k}{1-k}d(x_{2n}, x_{2n-1}) = hd(x_{2n}, x_{2n-1}),$$

Where $h = \frac{k}{1-k}$. *For* $n > m$,

$$d(x_{2n}, x_{2m}) \leqslant d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + ------ + d(x_{2m+1}, x_{2m})$$

$$\leqslant (h^{2n-1} + h^{2n-2} + ------- + h^{2m})d(x_1, x_o)$$

$$\leqslant \frac{h^{2m}}{1-h}d(x_1, x_o).$$

We get $\| d(x_{2n}, x_{2m}) \| \leqslant \frac{h^{2m}}{1-h}K \| d(x_1, x_o) \|$.

This implies $d(x_{2n}, x_{2m}) \to 0(n, m \to \infty)$. Hence $\{x_{2n}\}$is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_{2n} \to x^*(n \to \infty)$. Since $d(T_1x^*, x^*) \leqslant d(T_1x_{2n}, T_1x^*) + d(T_1x_{2n}, x^*)$

$$\leqslant k(d(T_1x^*, x_{2n}) + d(T_1x_{2n}, x^*)) + d(x_{2n+1}, x^*)$$

$$\leqslant k(d(T_1x^*, x^*) + d(x_{2n}, x^*) + d(x_{2n+1}, x^*)) + d(x_{2n+1}, x^*),$$

$$d(T_1x^*, x^*) \leqslant \tfrac{1}{1-k}(k(d(x_{2n}, x^*) + d(x_{2n+1}, x^*)) + d(x_{2n+1}, x^*)),$$

$$\| d(T_1x^*, x^*) \| \leqslant K \tfrac{1}{1-k}(k(\| d(x_{2n}, x^*) \| + \| d(x_{2n+1}, x^*) \|)$$

$$+ \| d(x_{2n+1}, x^*) \|) \to 0.$$

Hence $\| d(T_1x^*, x^*) \| = 0$. This implies $T_1x^* = x^*$. So $x^*$is fixed point of T_1.

Now if y^* is another fixed point of T_1, then,

$$d(x^*, y^*) = d(T_1x^*, T_1y^*) \leqslant k(d(T_1x^*, y^*) + d(T_1y^*, x^*))$$

$$= 2kd(x^*, y^*).$$

Hence $d(x^*, y^*) = 0, x^* = y^*$. Therefore the fixed point of T_1 is unique. Similarly it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is the common fixed point of pair of maps T_1 and T_2.

References

[1] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag 1985.

[2] B.E. Rhoades, A comparison of various definition of contractive mappings, Trans. Amer. Math. Soc. 266 (1977) 257-290.

[3] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 332(2007),1468-1476.

Received: November, 2012