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III Laplace-Beltrami and (S,L)-type Rotational Surface

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Abstract

We study on the third Laplace-Beltrami operator of timelike rotational surface of (S,L)-type in three dimensional Minkowski space.

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1 Introduction

Rotational surfaces in Euclidean 3-space $\mathbb{E}^3$ have been studied since long time, and nice examples of such surfaces have been obtained [3]. On the other side, Minkowski 3-space $\mathbb{E}_1^3$ has more complicated geometric structures compared to $\mathbb{E}^3$. In particular, $\mathbb{E}_1^3$ has distinguished axes of rotation, namely, *spacelike*, *timelike* and *lightlike axes*. About the semi-Riemannian geometry, many nice books have been done such as [4].

A helicoidal surface and a rotational surface with lightlike profile curve have an isometric relation by Bour's theorem is shown by Güler [1] in $\mathbb{E}_1^3$. He also classified the spacelike (and timelike) helicoidal (and rotational) surfaces with lightlike profile curve of spacelike, timelike and lightlike axes.

In this paper, we give an study on the timelike rotational surface with lightlike profile curve of $(S,L)-type$ in Minkowski 3-space. In section 2, we recall some basic notions of the Lorentzian geometry. We give the definition of the timelike rotational surface with lightlike profile in section 3. In section 4, we study on the third Laplace-Beltrami operator of timelike rotational surface of $(S,L)-type$ in three dimensional Minkowski space.

2 Preliminaries

In this section, we will obtain a lightlike profile curve in Minkowski 3-space. In the rest of this paper we shall identify a vector (a,b,c) with its transpose $(a,b,c)^t$.

The Minkowski 3-space $\mathbb{E}_1^3$ is the Euclidean space $\mathbb{E}^3$ provided with the inner product

$$\langle \overrightarrow{p}, \overrightarrow{q} \rangle = p_1q_1 + p_2q_2 - p_3q_3,$$

where $\overrightarrow{p}, \overrightarrow{q} \in \mathbb{E}^3$. We say that a Lorentzian vector $\overrightarrow{p}$ is spacelike (resp. lightlike and timelike) if $\overrightarrow{p} = 0$ or $\langle \overrightarrow{p}, \overrightarrow{p} \rangle > 0$ (resp. $\overrightarrow{p} \neq 0$; $\langle \overrightarrow{p}, \overrightarrow{p} \rangle = 0$ and $\langle \overrightarrow{p}, \overrightarrow{p} \rangle < 0$). The norm of the vector is defined by $\|\overrightarrow{p}\| = \sqrt{|\langle \overrightarrow{p}, \overrightarrow{p} \rangle|}$. Lorentzian vector product $\overrightarrow{p} \times \overrightarrow{q}$ of $\overrightarrow{p}$ and $\overrightarrow{q}$ is defined as follows

$$\overrightarrow{p} \times \overrightarrow{q} = (p_2q_3 - q_2p_3)\, e_1 + (q_1p_3 - p_1q_3)\, e_2 + (q_1p_2 - p_1q_2)\, e_3.$$

Now we define a non-degenerate rotational surface in $\mathbb{E}_1^3$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}_1^3$, and let ℓ be a straight line in Π which does not intersect the curve γ. A *rotational surface* in $\mathbb{E}_1^3$ is defined as a non degenerate surface rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). If the axis l is spacelike in $\mathbb{E}_1^3$, then we may suppose that l is the line spanned by the vector $(1,0,0)$. The semi$-$orthogonal matrix given as follow is the subgroup of the Lorentzian group that fixes the above vectors as invariant

$$S(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(v) & \sinh(v) \\ 0 & \sinh(v) & \cosh(v) \end{pmatrix},\ v \in \mathbb{R}.$$

The matrix S can be found by solving the following equations simultaneously $S\ell = \ell$, $S^t\varepsilon S = \varepsilon$, where $\epsilon = diag(1,1,-1)$, $\det S = 1$.

A rotational surface in $\mathbb{E}_1^3$ with the spacelike axis which is spanned by the vector $(1,0,0)$ is $R(u,v) = S(v)\gamma(u)$. A surface in $\mathbb{E}_1^3$ is $timelike\ surface$ if the $\det \mathbf{I} = EG - F^2 < 0$, where E, F, G are the coefficients of the first fundamental form. Parametrization of the profile curve γ is given by

$$\gamma(u) = (\zeta(u), \mu(u), \eta(u)),$$

where $\zeta(u), \mu(u)$ and $\eta(u)$ are differentiable functions for all $u \in \mathbb{R} \setminus \{0\}$. If $\gamma(u)$ lightlike curve, $<\gamma', \gamma'>= 0$, and $\eta = \int \sqrt{\zeta'^2 + \mu'^2} du$.

An example of a *lightlike profile curve* in $\mathbb{E}^3_1$ is given by

$$\gamma(u) = (u^2, u, \int \sqrt{4u^2+1} du), \tag{1}$$

where $\int \sqrt{4u^2+1} du = 2^{-1} u \sqrt{4u^2+1} + 2^{-2} \sinh^{-1}(2u) + c$ ($c = const.$). We will use the lightlike profile curve γ in (1) in the next sections.

3 Timelike Rotational Surface of (S,L)-type

We give a rotational surface by types of axis and profile curve, and we write it as (axis's type, profile curve's type)−type; for example, $(S, L)-type$ mean that the surface has a spacelike axis and a lightlike profile curve. We give rotational surfaces with lightlike profile curve that are used to obtain the main theorem in this paper. If the profile curve γ is a *lightlike* curve, then the rotational surface is a *timelike* surface with *spacelike* axis and it has $(S, L)-type.$

When the axis ℓ is spacelike, there is a Lorentz transformation by which the axis ℓ is transformed to the $(1, 0, 0)$ axis of $\mathbb{E}^3_1$. If the profile curve is $\gamma(u) = (u^2, u, \eta(u))$, then timelike rotational surface can be written as

$$R(u, v) = \begin{pmatrix} u^2 \\ u \cosh(v) + \eta(u) \sinh(v) \\ u \sinh(v) + \eta(u) \cosh(v) \end{pmatrix}. \tag{2}$$

Proposition 1 *A timelike rotational surface with lightlike profile curve of (S,L)−type (see Fig. 1) is as follows*

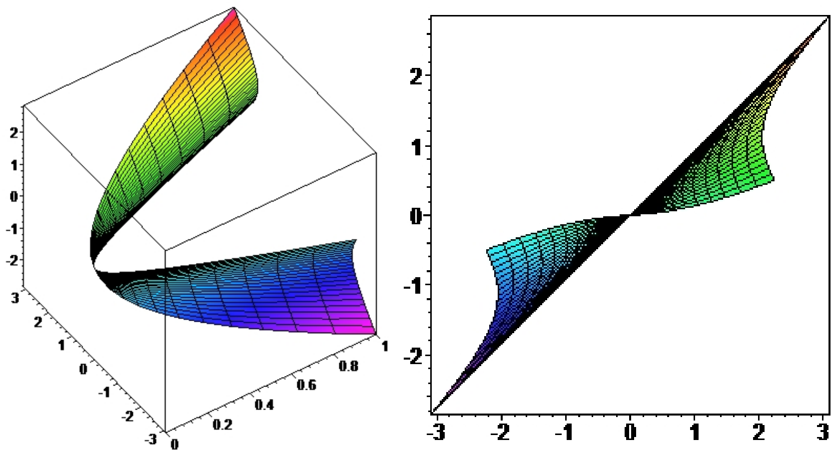
$$R(u, v) = \begin{pmatrix} u^2 \\ u \cosh(v) + (\frac{1}{2} u \sqrt{4u^2+1} + \frac{1}{4} \sinh^{-1}(2u)) \sinh(v) \\ u \sinh(v) + (\frac{1}{2} u \sqrt{4u^2+1} + \frac{1}{4} \sinh^{-1}(2u)) \cosh(v) \end{pmatrix}, \tag{3}$$

where $u, v \in \mathbb{R} \setminus \{0\}$.

Proposition 2 *If an (S,L)−type a timelike rotational surface with lightlike profile curve is as above, then its Gauss map is*

$$\mathbf{e} = \frac{1}{\sqrt{|\det I|}} \begin{pmatrix} u - \eta \eta' \\ -2u(u \cosh(v) + \eta \sinh(v)) \\ -2u(u \sinh(v) + \eta \cosh(v)) \end{pmatrix}, \tag{4}$$

where $\det I = -(\eta - u\eta')^2$, $\eta = \eta(u) = \frac{u\sqrt{4u^2+1}}{2} + \frac{\sinh^{-1}(2u)}{4}$, $u, v \in \mathbb{R} \setminus \{0\}$.



(a) (b)

Figure 1 Timelike rotational surface with lightlike profile curve of (S,L)$-$type

4 The Third Laplace-Beltrami

In this section, we study on the III Laplace-Beltrami operator of the timelike rotational surface with lightlike profile curve of *(S,L)*$-$type. We assume that the axis $\ell = (1, 0, 0)$ is a spacelike vector, profile curve $\gamma(u) = (u^2, u, \eta(u))$ is a lightlike curve, and $\eta(u) = 2^{-1}u\sqrt{4u^2+1} + 2^{-2}\sinh^{-1}(2u)$, $u \in \mathbb{R} \setminus \{0\}$.

Now, let $x = x(u^1, u^2)$ be a surface of 3$-$dimensional Minkowski space defined in domain D. The same for the functions ϕ, ψ. Let $n = n(u^1, u^2)$ be the normal vector of the surface. We write

$$g_{ij} = \langle x_i, x_j \rangle, \ b_{ij} = \langle x_{ij}, n \rangle, \ e_{ij} = \langle n_i, n_j \rangle. \tag{5}$$

The equations of Weingarten are

$$\begin{aligned} x_i &= b_{ij} e^{jr} n_r \\ &= -g_{ij} b^{jr} n_r, \end{aligned}$$

$$\begin{aligned} n_i &= -e_{ij} b^{jr} x_r \\ &= -b_{ij} g^{jr} x_r, \end{aligned}$$

where $x_i = \frac{\partial x}{\partial u^i}$. Then the first parameter Beltrami is defined

$$grad^{\mathbf{III}}(\phi, \psi) = e_i^{ik} \phi \psi_k.$$

Using following expressions

$$\begin{aligned} grad^{\mathbf{III}}(\phi) &= grad^{\mathbf{III}}(\phi,\phi) \\ &= e^{ik}\phi_i\psi_k, \end{aligned}$$

$$\begin{aligned} grad^{\mathbf{III}}\phi &= grad^{\mathbf{III}}(\phi,n) \\ &= e^{ik}\phi_i n_k, \end{aligned}$$

the second parameter Beltrami is defined

$$\Delta^{\mathbf{III}}\phi = -e^{ik} grad_k^{\mathbf{III}}\phi_i.$$

Using the last relation we get the expression the third Laplace-Beltrami operator of the function ϕ. So, we have the third fundamental form (see [2] for details) as follow

$$\Delta^{III}\phi = -\frac{\sqrt{|\det \mathbf{I}|}}{\det \mathbf{II}}\left[\frac{\partial}{\partial u}\left(\frac{Z\phi_u - Y\phi_v}{\sqrt{|\det \mathbf{I}|}\det \mathbf{II}}\right) - \frac{\partial}{\partial v}\left(\frac{Y\phi_u - X\phi_v}{\sqrt{|\det \mathbf{I}|}\det \mathbf{II}}\right)\right], \quad (6)$$

where the coefficients of the first (resp., second, and third) fundamental form of the function ϕ is E, F, G (resp., L, M, N, and X, Y, Z), $\det \mathbf{I} = EG - F^2$, $\det \mathbf{II} = LN - M^2$,

$$\begin{aligned} X &= EM^2 - 2FLM + GL^2, \\ Y &= EMN - FLN + GLM - FM^2, \\ Z &= GM^2 - 2FNM + EN^2. \end{aligned}$$

Theorem 3 *The III Laplace-Beltrami operator of the timelike rotational surface with lightlike profile curve of (S,L)−type (in (2)) is*

$$\Delta^{\mathbf{III}}R = \left(\Delta^{\mathbf{III}}R_1, \Delta^{\mathbf{III}}R_2, \Delta^{\mathbf{III}}R_3\right),$$

where

$$\begin{aligned} \Delta^{\mathbf{III}}R_1 &= \frac{1}{2u^2(\eta - u\eta')}(2u^2\eta\eta'^2 - 3u^3\eta' - u\eta^2\eta' + u^4\eta'' + 3u^2\eta \\ &\quad -u^2\eta^2\eta'' - \eta'^3), \end{aligned}$$

$$
\begin{aligned}
\Delta^{\mathbf{III}} R_2 \;=\; & \frac{1}{4u^4\left(\eta-u\eta'\right)^3}((u\eta'-\eta)((u^2\eta\eta'(u^2-\eta^2)(u\eta'-\eta) \\
& .(u\sinh(v)+\eta\cosh(v))\eta''' - u^2\eta(u^2-\eta^2)(u\eta'+\eta) \\
& .(u\sinh(v)+\eta\cosh(v))\eta''^2 - (-u^2(-3u\eta^2\sinh(v) \\
& -4\eta^3\cosh(v)+u^3\sinh(v)+2u^2\eta\cosh(v))\eta'^3 \\
& +(-2u^4\eta\sinh(v)-2u^2\eta^3\sinh(v)-4u\eta^4\cosh(v))\eta'^2 \\
& +((\cosh(v)-\sinh(v))\eta^4+2u\eta^3\cosh(v)-2(\cosh(v) \\
& -3.2^{-1}\sinh(v))u^2\eta^2+u^4\cosh(v))u\eta' \\
& +2(u-\eta)^2(u\sinh(v)+2^{-1}\eta\cosh(v))(u+\eta)^2)u\eta'' \\
& +(u\eta'-\eta)((u^4\cosh(v)-2u^3\eta\sinh(v))\eta'^3 \\
& +(2u^4\sinh(v)-6u^3\eta\cosh(v)+5u^2\eta^2\sinh(v) \\
& +4u\eta^3\cosh(v))\eta'^2+(-u^2\eta^2\cosh(v)+3u^4\cosh(v) \\
& -8u^3\eta\sinh(v)-\eta^4\cosh(v))\eta'+3u^4\sinh(v))) \\
& .(u\eta'-\eta)+(u\eta'-\eta)(u^2\eta^2\eta'^2(u^2-\eta^2)(u\cosh(v) \\
& +\eta\sinh(v))\eta''^2+3\eta\eta'((2.3^{-1}u\eta\sinh(v) \\
& -3^{-1}\eta^2\cosh(v)+u^2\cosh(v))u^2\eta'^2+(3^{-1}u^4\sinh(v) \\
& +2.3^{-1}\eta^4\sinh(v)-7.3^{-1}u^2\eta^2\sinh(v) \\
& -7.3^{-1}u^3\eta\cosh(v)+u\eta^3\cosh(v))\eta'+3^{-1}u\eta^3\sinh(v) \\
& +3^{-1}u^3\eta\sinh(v)+2.3^{-1}u^4\cosh(v))u\eta'' \\
& +u^5\eta'^4\cosh(v)+u^2(-2u\eta^2\sinh(v)-6u^2\eta\cosh(v) \\
& +u^3\sinh(v)+\eta^3\cosh(v))\eta'^3+(-2u^4\sinh(v)\eta \\
& +6u^2\eta^3\sinh(v)+8u^3\eta^2\cosh(v)+3u^5\cosh(v) \\
& -2u\eta^4\cosh(v)-\eta^5\sinh(v))\eta'^2+u(-3u^2\eta^2\sinh(v) \\
& -\eta^4\sinh(v)-7u^3\eta\cosh(v)+u^4\sinh(v))\eta' \\
& +u^2(\eta^3\sinh(v)+u\eta^2\cosh(v)+u^3\cosh(v)))u\eta'-\eta)),
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{III} R_3 \;\; = \;\; & \frac{1}{4u^4\left(\eta - u\eta'\right)^3}((u\eta' - \eta)^2((u^2\eta\eta'(u^2 - \eta^2)(u\eta' - \eta) \\
& (u\cosh(v) + \eta\sinh(v))\eta''' - u^2\eta(u^2 - \eta^2)(u\eta' + \eta) \\
& .(u\cosh(v) + \eta\sinh(v))\eta''^2 + u(u^2(-4\eta^3\sinh(v) \\
& + u^3\cosh(v) + 2u^2\eta\sinh(v) - 3u\eta^2\cosh(v))\eta'^3 \\
& + (4u\eta'^4\sinh(v) + 2u^4\eta\cosh(v) + 2u^2\eta^3\cosh(v))\eta'^2 \\
& - ((\sinh(v) - \cosh(v))\eta^4 + 2u\eta^3\sinh(v) \\
& + (-2\sinh(v) + \cosh(v))u^2\eta^2 + u^4\sinh(v))u\eta' \\
& - 2(u - \eta)^2(u\cosh(v) + 2^{-1}\eta\sinh(v))(u + \eta)^2)\eta'' \\
& + 2(u\eta' - \eta)((2^{-1}u^4\sinh(v) - u^3\eta\cosh(v))\eta'^3 \\
& + (u^3\cosh(v) - 3u^2\eta\sinh(v) + 5.2^{-1}u\eta^2\cosh(v) \\
& + 2\eta^3\sinh(v))u\eta'^2 + (-2^{-1}u^2\eta^2\sinh(v) + 3.2^{-1}u^4\sinh(v) \\
& - 4u^3\eta\cosh(v) - 2^{-1}\eta^4\sinh(v))\eta' + 3.2^{-1}u^4\cosh(v))) \\
& .((u\eta' - \eta) + (u^2\eta^2\eta'^2(u^2 - \eta^2)(u\sinh(v) \\
& + \eta\cosh(v))\eta''^2 + \eta\eta'((-u^2\eta^2\sinh(v) + 2u^3\eta\cosh(v) \\
& + 3u^4\sinh(v))\eta'^2 + (3u\eta^3\sinh(v) - 7u^3\eta\sinh(v) \\
& - 7u^2\eta^2\cosh(v) + u^4\cosh(v) + 2\eta^4\cosh(v))\eta' \\
& + 2u^4\sinh(v) + u\eta^3\cosh(v) + u^3\eta\cosh(v))u\eta'' \\
& + u^5\sinh(v)\eta'^4 + u^2(\eta^3\sinh(v) - 2u\eta^2\cosh(v) \\
& - 6u^2\eta\sinh(v) + u^3\cosh(v))\eta'^3 + (6u^2\eta^3\cosh(v) \\
& - \eta^5\cosh(v) - 2u\eta^4\sinh(v) + 8u^3\eta^2\sinh(v) \\
& - 2u^4\eta\cosh(v) + 3u^5\sinh(v))\eta'^2 + u(-3u^2\eta^2\cosh(v) \\
& - \eta^4\cosh(v) + u^4\cosh(v) - 7u^3\eta\sinh(v))\eta' \\
& + u^2(u\eta^2\sinh(v) + u^3\sinh(v) + \eta^3\cosh(v)))
\end{aligned}
$$

in Minkowski 3-space, $\eta(u) = \frac{u\sqrt{4u^2+1}}{2} + \frac{\sinh^{-1}(2u)}{4}$ *and* $u, v \in \mathbb{R} \setminus \{0\}$.

Proof. We consider the rotational surface (2). Components of the first fundamental form are

$$E = 0, \quad F = \eta - u\eta', \quad G = -u^2 + \eta^2,$$

and the second are

$$L = \frac{2(u - \eta\eta' + u\eta'\eta'')}{(\eta - u\eta')}, \quad M = -2u, \quad N = 0,$$

$\eta(u) = \frac{u\sqrt{4u^2+1}}{2} + \frac{\sinh^{-1}(2u)}{4}$ and $u, v \in R \setminus \{0\}$. Since $\det \mathbf{I} = -\left(\eta - u\eta'\right)^2 < 0$, $R(u,v)$ is a timelike surface. Therefore, we have

$$\Delta^{\mathbf{III}} R = -\left|\det \mathbf{I}\right|^{1/2} \left(\det \mathbf{II}\right)^{-1} \left(\frac{\partial A}{\partial u} - \frac{\partial B}{\partial v}\right),$$

where the matrices

$$A := \frac{GM^2 R_u - (GLM - FM^2)\, R_v}{-4u^2 F},$$

and

$$B := \frac{(GLM - FM^2)\, R_u - (-2FLM + GL^2)\, R_v}{-4u^2 F}.$$

After some computations we get $\Delta^{\mathbf{III}} R_1, \Delta^{\mathbf{III}} R_2$, and $\Delta^{\mathbf{III}} R_3$.

Corollary 4 *The mean curvature and the Gaussian curvature of the timelike rotational surface with lightlike profile curve of (S,L)−type in* (2) *is, respectively, as follow*

$$H = \frac{u^3 - 5u^2\eta\eta' + u^3\eta\eta'\eta'' + u\eta^2 + \eta^3\eta' - u\eta^3\eta'\eta'' + 2u^3\eta'^2}{(u\eta' - \eta)^3}, \tag{7}$$

and

$$K = -\frac{4u^2}{(u\eta' - \eta)^2}. \tag{8}$$

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