

HEMIRINGS CHARACTERIZED BY INTERVAL VALUED  
 $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -FUZZY  $k$ -IDEALS

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**Abstract**

We define interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -ideals, interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideals, interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -bi-ideals and characterize  $k$ -regular and  $k$ -intra regular hemirings by the properties of these ideals.

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## 1 Introduction

Although there are many structures generalizing associative rings and distributive lattices but most famous among them are semirings (or half-rings) and near rings. Semirings are proved to be very useful for solving many problems in different areas of mathematics and information sciences. Some of them are optimization theory, graph theory, theory of discrete event dynamical systems, generalized fuzzy computation, automata theory, formal language theory, coding theory, analysis of computer programs, and so on (see [1, 6, 7, 8, 16]). Semirings were introduced by H. S. Vandiver in 1934 [24]. Semirings with commutative addition and additive identity are called hemirings.

In semirings ideals play an important role and are useful for many purposes,

but they do not coincide with those of the ring ideals. Due to the reason many results of ring theory have no analogues in semirings using only ideals. To manage this problem, in [9], Henriksen defined a class of ideals in semirings, called  $k$ -ideals.  $k$ -ideals of semirings have the property that if the semiring  $R$  is a ring then a subset of  $R$  is a  $k$ -ideal if and only if it is a ring ideal. Iizuka [10] defined a more restricted class of ideals in hemirings, called  $h$ -ideals. La Torre [13] thoroughly studied  $h$ -ideals and  $k$ -ideals.

After the introduction of fuzzy sets by Zadeh [25], there had been made many efforts to fuzzify the algebraic structures. Rosenfeld [22] initiated and defined fuzzy groups. In [2] J. Ahsan initiated the study of fuzzy semirings. Fuzzy  $k$ -ideals in semirings are studied in [5, 11, 12, 17].

The concept of "belongingness ( $\in$ )" and of "quasicoincidence ( $q$ )" of a fuzzy point with a fuzzy set are given in [20, 21]. Further this concept was carried out in [3, 4, 15]. In 1975 the concept of interval valued fuzzy sets was introduced by Zadeh [26], as a generalization of the notion of fuzzy sets. In [14], Ma and Zhan introduced the concept of interval valued  $(\in, \in \vee q)$ -fuzzy  $h$ -ideals in hemirings and developed some basic results. In [23] Sun et al characterized  $h$ -hemiregular and  $h$ -intra-hemiregular hemirings by the properties of their interval valued fuzzy left and right  $h$ -ideals. In [5] Ghosh introduced the concept of  $k$ -regular semirings. M. Shabir and R. Anjum [18] introduced the concept of  $k$ -intra-regular semirings and characterize  $k$ -regular and  $k$ -intra-regular semirings by the properties their fuzzy  $k$ -ideals. Authors [19] introduced the concept of interval valued fuzzy  $k$ -ideals, interval valued fuzzy  $k$ -quasi-ideals, interval valued fuzzy  $k$ -bi-ideals and characterize  $k$ -regular and  $k$ -intra-regular semirings by the properties of these ideals. In this paper we define interval valued  $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy  $k$ -ideals, interval valued  $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy  $k$ -quasi-ideals, interval valued  $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy  $k$ -bi-ideals and characterize different classes of hemirings by the properties of these ideals.

## 2 Preliminaries

For basic definitions see [7]. A left (right) ideal  $A$  of a hemiring  $R$  is called a left (right)  $k$ -ideal if for all  $x \in R$  and for any  $a, b \in A$  from  $x + a = b$ , it follows  $x \in A$  (cf. [5]). A bi-ideal  $B$  of a hemiring  $R$  is called a  $k$ -bi-ideal of  $R$  if for all  $x \in R$  and  $a, b \in B$  from  $x + a = b$ , it follows  $x \in B$  (cf. [17]).

The  $k$ -closure of a non-empty subset  $A$  of a hemiring  $R$  is denoted and defined as

$$\overline{A} = \{x \in R \mid x + a = b \text{ for some } a, b \in A\}.$$

A quasi-ideal  $Q$  of a hemiring  $R$  is said to be  $k$ -quasi-ideal of  $R$  if  $\overline{RQ} \cap \overline{QR} \subseteq Q$  and  $x + a = b \Rightarrow x \in Q$ , for all  $x \in R$  and  $a, b \in Q$  (cf. [17]). Every one

sided  $k$ -ideal of  $R$  is a  $k$ -quasi-ideal of  $R$  and every  $k$ -quasi-ideal is a  $k$ -bi-ideal of  $R$ . However, the converse is not true in general (cf. [17]).

### 3 Interval Valued Fuzzy Sets

If  $X$  is a universe, then a function  $\lambda : X \rightarrow [0, 1]$  is said to be fuzzy subset of  $X$ . Now let  $\mathcal{L}[0, 1]$  denotes the family of all closed subintervals of  $[0, 1]$  with minimal element  $[0, 0]$  and maximal element  $[1, 1]$  according to the partial order  $[\alpha, \alpha'] \leq [\beta, \beta']$  if and only if  $\alpha \leq \beta, \alpha' \leq \beta'$  defined on  $\mathcal{L}[0, 1]$  for all  $[\alpha, \alpha'], [\beta, \beta'] \in \mathcal{L}[0, 1]$ .

By an interval number  $\hat{a}$  we mean an interval  $[a^-, a^+] \in \mathcal{L}[0, 1]$ , where  $0 \leq a^- \leq a^+ \leq 1$ . If  $a^- = a^+ = a$ , then the interval  $[a^-, a^+] = [a, a]$  will be simply identified by the number  $a$ .

An interval valued fuzzy subset  $\hat{\lambda}$  of a hemiring  $R$  is a function  $\hat{\lambda} : R \rightarrow \mathcal{L}[0, 1]$ . We write  $\hat{\lambda}(x) = [\lambda^-(x), \lambda^+(x)] \subseteq [0, 1]$ , for all  $x \in R$  and  $\lambda^-, \lambda^+ : R \rightarrow [0, 1]$  such that for each  $x \in R$ ,  $0 \leq \lambda^-(x) \leq \lambda^+(x) \leq 1$ . For simplicity we write  $\hat{\lambda} = [\lambda^-, \lambda^+]$ . We denote the set of all interval valued fuzzy subsets of  $R$  by  $\mathcal{L}(R)$ .

Let  $A \subseteq R$ . Then the interval valued characteristic function  $\hat{\chi}_A$  of  $A$  is a function  $\hat{\chi}_A : R \rightarrow \mathcal{L}[0, 1]$  such that for all  $x \in R$

$$\hat{\chi}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Clearly the interval valued characteristic function of any subset of  $R$  is also an interval valued fuzzy subset of  $R$ . Note that  $\hat{\chi}_R(x) = 1$  for all  $x \in R$ .

An interval valued fuzzy subset  $\hat{\lambda}$  of  $R$  of the form

$$\hat{\lambda}(z) = \begin{cases} \hat{t} (\neq 0) & \text{if } z = x \\ 0 & \text{if } z \neq x. \end{cases}$$

is said to be fuzzy interval valued point with support  $x$  and value  $\hat{t}$  and is denoted by  $x_{\hat{t}}$ . Further  $x_{\hat{t}} \in \hat{\lambda}$  (read it as  $x_{\hat{t}}$  belongs to  $\hat{\lambda}$ ) if  $\hat{\lambda}(x) \geq \hat{t}$ . Also  $x_{\hat{t}}q\hat{\lambda}$  (read it as  $x_{\hat{t}}$  is quasi-coincident with  $\hat{\lambda}$ ) if  $\hat{\lambda}(x) + \hat{t} > 1$ .  $x_{\hat{t}} \in \vee q\hat{\lambda}$  means  $x_{\hat{t}} \in \hat{\lambda}$  or  $x_{\hat{t}}q\hat{\lambda}$  and  $x_{\hat{t}} \in \wedge q\hat{\lambda}$  means  $x_{\hat{t}} \in \hat{\lambda}$  and  $x_{\hat{t}}q\hat{\lambda}$ .  $x_{\hat{t}}\bar{\alpha}\hat{\lambda}$  means that  $x_{\hat{t}}\alpha\hat{\lambda}$  does not hold for  $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$ .

If  $\{\hat{\lambda}_i : i \in I\} \subseteq \mathcal{L}(R)$ , then for all  $x \in R$ ,

$$(\vee_i \hat{\lambda}_i)(x) = [\vee_i \lambda_i^-(x), \vee_i \lambda_i^+(x)] \quad \text{and} \quad (\wedge_i \hat{\lambda}_i)(x) = [\wedge_i \lambda_i^-(x), \wedge_i \lambda_i^+(x)]$$

Where  $\vee$  denotes the supremum and  $\wedge$  denotes the infimum. For  $\hat{\lambda}, \hat{\mu} \in \mathcal{L}(R)$ , then  $\hat{\lambda} \geq \hat{\mu}$  if and only if  $\hat{\lambda}(x) \geq \hat{\mu}(x)$ , for all  $x \in R$ .

### 3.1 Definition [18]

Let  $\hat{\lambda}, \hat{\mu} \in \mathcal{L}(R)$ . Then the  $k$ -intrinsic product of  $\hat{\lambda}$  and  $\hat{\mu}$  is denoted and defined as

$$\left(\hat{\lambda} \odot \hat{\mu}\right)(x) = \sup \left[ \left\{ \left( \bigwedge_{i=1}^m \left( \hat{\lambda}(a_i) \wedge \hat{\mu}(b_i) \right) \right) \wedge \left( \bigwedge_{j=1}^n \left( \hat{\lambda}(a'_j) \wedge \hat{\mu}(b'_j) \right) \right) \right\} \right]$$

for all  $x \in R$ , if  $x$  can be expressed as  $x + \sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j$ , and 0 if  $x$  cannot be expressed as  $x + \sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j$ .

### 3.2 Definition

Let  $\hat{\lambda}, \hat{\mu} \in \mathcal{L}(R)$ . Then for all  $x \in R$ , the fuzzy subset  $\hat{\lambda} + \hat{\mu}$  of  $R$  is defined as

$$\left(\hat{\lambda} + \hat{\mu}\right)(x) = \sup \{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\mu}(b_1) \wedge \hat{\mu}(b_2) \}$$

for all expressions  $x + (a_1 + b_1) = (a_2 + b_2)$  of  $x$  in  $R$ .

### 3.3 Lemma

Let  $R$  be a hemiring and  $A, B \subseteq R$ . Then we have

- (i)  $A \subseteq B$  if and only if  $\hat{\chi}_A \leq \hat{\chi}_B$ .
- (ii)  $\hat{\chi}_A \wedge \hat{\chi}_B = \hat{\chi}_{A \cap B}$ .
- (iii)  $\hat{\chi}_A \odot \hat{\chi}_B = \hat{\chi}_{\overline{AB}}$ .

## 4 Interval valued $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy $k$ -ideals

### 4.1 Definition

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  is said to be an interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy left (resp. right)  $k$ -ideal of  $R$  if for all  $x, y, a, b \in R$  and  $t, r \in (0, 1]$ ,

- (1)  $(x + y)_{\min\{t, r\}} \overline{\epsilon} \hat{\lambda} \rightarrow x_t \overline{\epsilon} \vee \overline{q} \hat{\lambda}$  or  $y_r \overline{\epsilon} \vee \overline{q} \hat{\lambda}$ .
- (2)  $(xy)_t \overline{\epsilon} \hat{\lambda} \rightarrow y_t \overline{\epsilon} \vee \overline{q} \hat{\lambda}$  (resp. (3)  $(xy)_t \overline{\epsilon} \hat{\lambda} \rightarrow x_t \overline{\epsilon} \vee \overline{q} \hat{\lambda}$ )
- (4)  $(x)_{\min\{t, r\}} \overline{\epsilon} \hat{\lambda} \rightarrow a_t \overline{\epsilon} \vee \overline{q} \hat{\lambda}$  or  $b_r \overline{\epsilon} \vee \overline{q} \hat{\lambda}$ , for all  $a, b, x \in R$  with  $x + a = b$ .

$\hat{\lambda} \in \mathcal{L}(R)$  is said to be an interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy  $k$ -ideal of  $R$  if it is both interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy left and right  $k$ -ideal of  $R$ .

### 4.2 Definition

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  is said to be an interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy  $k$ -bi-ideal of  $R$  if it satisfies (1), (4) and for all  $x, y, z \in R$ ,  $t, r \in (0, 1]$

- (5)  $(xy)_{\min\{t, r\}} \overline{\epsilon} \hat{\lambda} \rightarrow x_t \overline{\epsilon} \vee \overline{q} \hat{\lambda}$  or  $y_r \overline{\epsilon} \vee \overline{q} \hat{\lambda}$ .
- (6)  $(xyz)_{\min\{t, r\}} \overline{\epsilon} \hat{\lambda} \rightarrow x_t \overline{\epsilon} \vee \overline{q} \hat{\lambda}$  or  $z_r \overline{\epsilon} \vee \overline{q} \hat{\lambda}$ .

### 4.3 Definition

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  is said to be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal of  $R$  if it satisfies (1), (4) and for all  $x \in R$ ,  $t \in (0, 1]$

$$(7) x_{\hat{t}} \bar{\epsilon} \hat{\lambda} \rightarrow x_{\hat{t}} \bar{\epsilon} \vee \bar{q}(\hat{\lambda} \odot \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} \odot \hat{\lambda})$$

### 4.4 Theorem

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then (1) – (7) are equivalent to (1') – (7'). Where

$$(1') \max\{\hat{\lambda}(x+y), 0.5\} \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$$

$$(2') \max\{\hat{\lambda}(xy), 0.5\} \geq \hat{\lambda}(y)$$

$$(3') \max\{\hat{\lambda}(xy), 0.5\} \geq \hat{\lambda}(x)$$

$$(4') x+a=b \Rightarrow \max\{\hat{\lambda}(x), 0.5\} \geq \min\{\hat{\lambda}(a), \hat{\lambda}(b)\}$$

$$(5') \max\{\hat{\lambda}(xy), 0.5\} \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$$

$$(6') \max\{\hat{\lambda}(xyz), 0.5\} \geq \min\{\hat{\lambda}(x), \hat{\lambda}(z)\}$$

$$(7') \max\{\hat{\lambda}(x), , 0.5\} \geq \min\{(\hat{\lambda} \odot \hat{\mathcal{R}})(x), (\hat{\mathcal{R}} \odot \hat{\lambda})(x)\}$$

for all  $x, y, z, a, b \in R$ .

**Proof.** We prove (1)  $\Leftrightarrow$  (1'), other follows in analogues way.

(1)  $\Rightarrow$  (1') Suppose (1') does not hold, then there exists  $x, y \in R$  such that  $\max\{\hat{\lambda}(x+y), 0.5\} < \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$ . Then we can choose  $\hat{t}$  such that,  $\max\{\hat{\lambda}(x+y), 0.5\} < \hat{t} < \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$ . Which implies  $(x+y)_{\hat{t}} \bar{\epsilon} \hat{\lambda}, 0.5 < \hat{t} \leq 1, \hat{\lambda}(x) > \hat{t}$  and  $\hat{\lambda}(y) > \hat{t}$ . Then  $x_{\hat{t}} \in \wedge q \hat{\lambda}$  and  $y_{\hat{t}} \in \wedge q \hat{\lambda}$ . Which contradicts (1).

$$(1') \Rightarrow (1) \text{ Let } (x+y)_{\min\{\hat{t}, \hat{r}\}} \bar{\epsilon} \hat{\lambda}, \text{ then } \hat{\lambda}(x+y) < \min\{\hat{t}, \hat{r}\}.$$

Case I: If  $\hat{\lambda}(x+y) \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$ , then  $\min\{\hat{\lambda}(x), \hat{\lambda}(y)\} < \min\{\hat{t}, \hat{r}\}$ , which implies  $\hat{\lambda}(x) < \hat{t}$  or  $\hat{\lambda}(y) < \hat{r}$ . Then  $x_{\hat{t}} \bar{\epsilon} \hat{\lambda}$  or  $y_{\hat{r}} \bar{\epsilon} \hat{\lambda} \Rightarrow x_{\hat{t}} \bar{\epsilon} \vee \bar{q} \hat{\lambda}$  or  $y_{\hat{r}} \bar{\epsilon} \vee \bar{q} \hat{\lambda}$ .

Case II: If  $\hat{\lambda}(x+y) < \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$ , then  $0.5 \geq \min\{\hat{\lambda}(x), \hat{\lambda}(y)\}$ . Let for some  $t, r \in (0, 1], x_{\hat{t}} \in \hat{\lambda}, y_{\hat{r}} \in \hat{\lambda}$ . Then  $\hat{t} \leq \hat{\lambda}(x) \leq 0.5$  or  $\hat{r} \leq \hat{\lambda}(y) \leq 0.5$ , consequently  $x_{\hat{t}} \bar{q} \hat{\lambda}$  or  $y_{\hat{r}} \bar{q} \hat{\lambda}$ . Hence  $x_{\hat{t}} \bar{\epsilon} \vee \bar{q} \hat{\lambda}$  or  $y_{\hat{r}} \bar{\epsilon} \vee \bar{q} \hat{\lambda}$ . ■

By using Theorem 4.4 we have:

### 4.5 Theorem

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  is said to be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -ideal of  $R$  if and only if it satisfies (1'), (2'), (3') and (4').

### 4.6 Theorem

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  is said to be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -bi-ideal of  $R$  if and only if it satisfies (1'), (4'), (5') and (6').

#### 4.7 Theorem

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  is said to be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal of  $R$  if and only if it satisfies (1'), (4') and (7').

#### 4.8 Lemma

A non-empty subset  $A$  of  $R$  is a left  $k$ -ideal (resp. right  $k$ -ideal,  $k$ -bi-ideal,  $k$ -quasi-ideal) of  $R$  if and only if  $\hat{\chi}_A$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal (resp. right  $k$ -ideal,  $k$ -bi-ideal,  $k$ -quasi-ideal) of  $R$ .

**Proof.** Proof is straightforward. ■

#### 4.9 Definition

Let  $\hat{\lambda}, \hat{\mu} \in \mathcal{L}(R)$ . Then the fuzzy subsets  $\hat{\lambda} \vee 0.5$ ,  $\hat{\lambda} \wedge^{0.5} \hat{\mu}$ ,  $\hat{\lambda} \odot^{0.5} \hat{\mu}$  and  $\hat{\lambda} +^{0.5} \hat{\mu}$  of  $R$  are defined as

$$\begin{aligned} (\hat{\lambda} \vee 0.5)(x) &= \hat{\lambda}(x) \vee 0.5 \\ (\hat{\lambda} \wedge^{0.5} \hat{\mu})(x) &= (\hat{\lambda} \wedge \hat{\mu})(x) \vee 0.5 \\ (\hat{\lambda} \odot^{0.5} \hat{\mu})(x) &= (\hat{\lambda} \odot \hat{\mu})(x) \vee 0.5 \\ (\hat{\lambda} +^{0.5} \hat{\mu})(x) &= (\hat{\lambda} + \hat{\mu})(x) \vee 0.5 \end{aligned}$$

for all  $x \in R$ .

#### 4.10 Theorem

Let  $\hat{\lambda}$  be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal of a hemiring  $R$ , then  $\hat{\lambda} \vee 0.5$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal of  $R$ .

**Proof.** Suppose  $\hat{\lambda}$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal of  $R$  and  $a, b, x, y \in R$ . Then

$$\begin{aligned} ((\hat{\lambda} \vee 0.5)(x + y)) \vee 0.5 &= (\hat{\lambda}(x + y) \vee 0.5) \vee 0.5 \geq (\min\{\hat{\lambda}(x), \hat{\lambda}(y)\}) \vee 0.5 \\ &= \min\{\hat{\lambda}(x) \vee 0.5, \hat{\lambda}(y) \vee 0.5\} = \min\{(\hat{\lambda} \vee 0.5)(x), (\hat{\lambda} \vee 0.5)(y)\}. \end{aligned}$$

This shows that  $\max\{(\hat{\lambda} \vee 0.5)(x + y), 0.5\} \geq \min\{(\hat{\lambda} \vee 0.5)(x), (\hat{\lambda} \vee 0.5)(y)\}$

Similarly we can show

$$\max\{(\hat{\lambda} \vee 0.5)(xy), 0.5\} \geq (\hat{\lambda} \vee 0.5)(y) \text{ and}$$

$$\max\{(\hat{\lambda} \vee 0.5)(x), 0.5\} \geq \min\{(\hat{\lambda} \vee 0.5)(a), (\hat{\lambda} \vee 0.5)(b)\} \text{ for all } x + a = b \text{ in}$$

$R$ .

This shows that  $(\hat{\lambda} \vee 0.5)$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal of  $R$ . ■

Similarly we can show:

### 4.11 Theorem

If  $\hat{\lambda}$  is an interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy  $k$ -bi-ideal of a hemiring  $R$ , then  $(\hat{\lambda} \vee 0.5)$  is an interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy  $k$ -bi-ideal of  $R$ .

### 4.12 Lemma

Let  $A, B \subseteq R$ . Then

$$\hat{\chi}_A +^{0.5} \hat{\chi}_B = \hat{\chi}_{\overline{A+B}} \vee 0.5.$$

**Proof.** Let  $A, B$  be subsets of a hemiring  $R$  and  $x \in R$ . If  $x \in \overline{A+B}$  then there exist  $a, a' \in A$  and  $b, b' \in B$  such that  $x + (a + b) = (a' + b')$ . Then for all such expressions

$$\begin{aligned} & (\hat{\chi}_A +^{0.5} \hat{\chi}_B)(x) \\ &= \sup \{ \hat{\chi}_A(a) \wedge \hat{\chi}_A(a') \wedge \hat{\chi}_B(b) \wedge \hat{\chi}_B(b') \} \vee 0.5 \\ &= 1 \vee 0.5 = \hat{\chi}_{\overline{A+B}}(x) \vee 0.5. \end{aligned}$$

If  $x \notin \overline{A+B}$  then there do not exist  $a, a' \in A$  and  $b, b' \in B$  such that  $x + (a + b) = (a' + b')$ . Thus

$$(\hat{\chi}_A +^{0.5} \hat{\chi}_B)(x) = 0 \vee 0.5 = \hat{\chi}_{\overline{A+B}}(x) \vee 0.5. \text{ Hence } \hat{\chi}_A +^{0.5} \hat{\chi}_B = \hat{\chi}_{\overline{A+B}} \vee 0.5.$$

■

### 4.13 Lemma

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  satisfies conditions (1') and (4') if and only if it satisfies condition

$$(8') \quad \hat{\lambda} \vee 0.5 \geq \hat{\lambda} +^{0.5} \hat{\lambda}.$$

**Proof.** Suppose  $\hat{\lambda}$  satisfies conditions (1') and (4'). Let  $x \in R$ , then for all expressions  $x + (a_1 + b_1) = (a_2 + b_2)$  in  $R$ ,

$$\begin{aligned} & (\hat{\lambda} +^{0.5} \hat{\lambda})(x) = \sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \vee 0.5 \\ &= \sup \left\{ (\hat{\lambda}(a_1) \wedge \hat{\lambda}(b_1)) \wedge (\hat{\lambda}(a_2) \wedge \hat{\lambda}(b_2)) \right\} \vee 0.5 \\ &\leq \sup \left\{ \max(\hat{\lambda}(a_1 + b_1), 0.5) \wedge \max(\hat{\lambda}(a_2 + b_2), 0.5) \right\} \vee 0.5 \quad (\text{by condition} \\ &(1')) \\ &\leq \sup \left\{ (\hat{\lambda}(a_1 + b_1) \wedge \hat{\lambda}(a_2 + b_2)) \vee 0.5 \right\} \vee 0.5 \\ &\leq \hat{\lambda}(x) \vee 0.5 \quad (\text{by condition} \\ &(4')). \end{aligned}$$

$$\text{Thus } \hat{\lambda} +^{0.5} \hat{\lambda} \leq \hat{\lambda} \vee 0.5.$$

Conversely assume for  $\hat{\lambda} \in \mathcal{L}(R)$ , (8') is satisfied and to prove that (1') and (4') hold. Let  $x, y \in R$ . Then for all expressions  $(x + y) + (a_1 + b_1) = (a_2 + b_2)$  in  $R$ ,

$$\hat{\lambda}(x + y) \vee 0.5 = (\hat{\lambda} +^{0.5} \hat{\lambda})(x + y)$$

$$\begin{aligned}
&= \sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \vee 0.5 \\
&\geq \left\{ \hat{\lambda}(0) \wedge \hat{\lambda}(x) \wedge \hat{\lambda}(0) \wedge \hat{\lambda}(y) \right\} \vee 0.5 \quad (\text{because } (x+y)+(0+0) = (x+y)). \\
&\geq \{(\hat{\lambda}(x) \wedge \hat{\lambda}(y))\}
\end{aligned}$$

Thus condition (1') is satisfied.

Let  $a, b, x \in R$  such that  $x + a = b$ . Then

$$\begin{aligned}
&\hat{\lambda}(x) \vee 0.5 \geq (\hat{\lambda} +^{0.5} \hat{\lambda})(x) \\
&= \sup \left\{ \hat{\lambda}(a_1) \wedge \hat{\lambda}(a_2) \wedge \hat{\lambda}(b_1) \wedge \hat{\lambda}(b_2) \right\} \vee 0.5 \\
&\geq \{ \hat{\lambda}(a) \wedge \hat{\lambda}(0) \wedge \hat{\lambda}(b) \} \vee 0.5 \quad (\text{because } x + (a + 0) = (b + 0)) \\
&\geq \{ \hat{\lambda}(a) \wedge \hat{\lambda}(b) \}
\end{aligned}$$

Hence condition (4') is satisfied. ■

#### 4.14 Theorem

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right (resp. left)  $k$ -ideal of  $R$  if and only if  $\hat{\lambda}$  satisfies conditions

$$\begin{aligned}
(8') \quad &\hat{\lambda} +^{0.5} \hat{\lambda} \leq \hat{\lambda} \vee 0.5 \\
(9') \quad &\hat{\mathcal{R}} \odot^{0.5} \hat{\lambda} \leq \hat{\lambda} \vee 0.5 \quad (\text{resp. } \hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \leq \hat{\lambda} \vee 0.5).
\end{aligned}$$

**Proof.** Proof is straightforward by using Lemma 4.13. ■

#### 4.15 Theorem

Let  $\hat{\lambda} \in \mathcal{L}(R)$ . Then  $\hat{\lambda}$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal of  $R$  if and only if  $\hat{\lambda}$  satisfies conditions (7') and (8').

**Proof.** Proof is straight forward because by Lemma 4.13, conditions (1') and (4') are equivalent to condition (8'). ■

#### 4.16 Lemma

Every interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy one sided  $k$ -ideal of a hemiring  $R$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal of  $R$ .

**Proof.** Proof is straight forward so omitted. ■

#### 4.17 Lemma

Every interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal of  $R$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -bi-ideal of  $R$ .

**Proof.** Proof is straight forward so omitted. ■

#### 4.18 Remark

Converses of the Lemma 4.16 and Lemma 4.17 are not true in general.



### 4.19 Example

Let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,

$$R = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} : p, q, r, s \in \mathbb{N}_0 \right\} \text{ and } A = \left\{ \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} : p \in \mathbb{N}_0 \right\}.$$

Then  $R$  is hemiring under the usual binary operations of addition and multiplication of matrices, and  $A$  is  $k$ -quasi-ideal of  $R$  but  $A$  is not left (right)  $k$ -ideal of  $R$ . Then by Lemma 4.8,  $\hat{\chi}_A$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal of  $R$  and  $\hat{\chi}_A$  is not an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left (right)  $k$ -ideal of  $R$ .

### 4.20 Example

Let

$$\begin{aligned} R &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} : p, q \in \mathbb{R}^+, r \in \mathbb{N} \right\} \\ A &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} : p, q \in \mathbb{R}^+, r \in \mathbb{N}, p < q \right\} \\ B &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} : p, q \in \mathbb{R}^+, r \in \mathbb{N}, q > 3 \right\} \end{aligned}$$

Then  $R$  is hemiring under the usual binary operations of addition and multiplication of matrices, and  $A$  is right  $k$ -ideal,  $B$  is left  $k$ -ideal,  $AB$  is  $k$ -bi-ideal of  $R$  and it is not a  $k$ -quasi-ideal of  $R$ . Then by Lemma 4.8,  $\hat{\chi}_{AB}$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -bi-ideal of  $R$  and it is not an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal of  $R$ .

### 4.21 Lemma

If  $\hat{\lambda}$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right  $k$ -ideal and  $\hat{\mu}$  is an  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal of  $R$ , resp., then  $\hat{\lambda} \odot^{0.5} \hat{\mu} \leq \hat{\lambda} \wedge^{0.5} \hat{\mu}$ .

**Proof.** For given  $\hat{\lambda}$  and  $\hat{\mu}$  and for all expressions  $x = \sum_{i=1}^m a_i b_i = \sum_{j=1}^n c_j d_j$  of  $x$  in  $R$

$$\begin{aligned} & \left( \hat{\lambda} \odot^{0.5} \hat{\mu} \right) (x) \\ &= \sup \left\{ \left( \bigwedge_{i=1}^m \left( \hat{\lambda}(a_i) \wedge \hat{\mu}(b_i) \right) \right) \wedge \left( \bigwedge_{j=1}^n \left( \hat{\lambda}(c_j) \wedge \hat{\mu}(d_j) \right) \right) \right\} \vee 0.5 \\ &\leq \sup \left\{ \left( \bigwedge_{i=1}^m \left( \left( \hat{\lambda}(a_i b_i) \vee 0.5 \right) \wedge \left( \hat{\mu}(a_i b_i) \vee 0.5 \right) \right) \right) \wedge \left( \bigwedge_{j=1}^n \left( \left( \hat{\lambda}(c_j d_j) \vee 0.5 \right) \wedge \left( \hat{\mu}(c_j d_j) \vee 0.5 \right) \right) \right) \right\} \vee 0.5 \\ &\leq \sup \left\{ \left( \left( \bigwedge_{i=1}^m \left( \hat{\lambda}(a_i b_i) \wedge \hat{\mu}(a_i b_i) \right) \right) \wedge \left( \bigwedge_{j=1}^n \left( \hat{\lambda}(c_j d_j) \wedge \hat{\mu}(c_j d_j) \right) \right) \right) \right\} \vee 0.5 \end{aligned}$$

$$\begin{aligned}
& 0.5 \\
& = \sup \left\{ \left( \left( \bigwedge_{i=1}^m (\hat{\lambda}(a_i b_i) \wedge \hat{\mu}(a_i b_i)) \right) \wedge \left( \bigwedge_{j=1}^n (\hat{\lambda}(c_j d_j) \wedge \hat{\mu}(c_j d_j)) \right) \right) \vee 0.5 \right\} \\
& \leq \sup \left\{ \left( \left( \sum_{i=1}^m \hat{\lambda}(a_i b_i) \vee 0.5 \right) \wedge \left( \sum_{i=1}^m \hat{\mu}(a_i b_i) \vee 0.5 \right) \wedge \left( \sum_{j=1}^n \hat{\lambda}(c_j d_j) \vee 0.5 \right) \wedge \left( \sum_{j=1}^n \hat{\mu}(c_j d_j) \vee 0.5 \right) \right) \vee 0.5 \right\} \\
& = \sup \left\{ \left( \left( \sum_{i=1}^m \hat{\lambda}(a_i b_i) \right) \wedge \left( \sum_{i=1}^m \hat{\mu}(a_i b_i) \right) \wedge \left( \sum_{j=1}^n \hat{\lambda}(c_j d_j) \right) \wedge \left( \sum_{j=1}^n \hat{\mu}(c_j d_j) \right) \right) \vee 0.5 \right\} \vee 0.5 \\
& = \sup \left\{ \left( \sum_{i=1}^m \hat{\lambda}(a_i b_i) \right) \wedge \left( \sum_{i=1}^m \hat{\mu}(a_i b_i) \right) \wedge \left( \sum_{j=1}^n \hat{\lambda}(c_j d_j) \right) \wedge \left( \sum_{j=1}^n \hat{\mu}(c_j d_j) \right) \right\} \vee 0.5 \\
& = \sup \left\{ \left( \sum_{i=1}^m \hat{\lambda}(a_i b_i) \right) \wedge \left( \sum_{j=1}^n \hat{\lambda}(c_j d_j) \right) \wedge \left( \sum_{i=1}^m \hat{\mu}(a_i b_i) \right) \leq \left( \sum_{j=1}^n \hat{\mu}(c_j d_j) \right) \right\} \vee 0.5 \\
& \leq \left[ \sup \left\{ \left( \sum_{i=1}^m \hat{\lambda}(a_i b_i) \right) \wedge \left( \sum_{j=1}^n \hat{\lambda}(c_j d_j) \right) \wedge \left( \sum_{i=1}^m \hat{\mu}(a_i b_i) \right) \right\} \wedge \sup \left\{ \left( \sum_{i=1}^m \hat{\mu}(a_i b_i) \right) \wedge \left( \sum_{j=1}^n \hat{\mu}(c_j d_j) \right) \right\} \right] \vee 0.5 \\
& \leq \left( \left( \hat{\lambda}(x) \vee 0.5 \right) \wedge \left( \hat{\mu}(x) \vee 0.5 \right) \right) \vee 0.5 \\
& = \left( \hat{\lambda}(x) \wedge \hat{\mu}(x) \right) \vee 0.5 \\
& = \left( \hat{\lambda} \wedge^{0.5} \hat{\mu} \right) (x). \text{ Thus } \hat{\lambda} \odot^{0.5} \hat{\mu} \leq \hat{\lambda} \wedge^{0.5} \hat{\mu}. \blacksquare
\end{aligned}$$

## 5 $k$ -Regular Hemirings

In this section we characterize  $k$ -regular hemirings by the properties of interval valued  $(\overline{\varepsilon}, \overline{\varepsilon} \vee \overline{q})$ -fuzzy  $k$ -ideals, interval valued  $(\overline{\varepsilon}, \overline{\varepsilon} \vee \overline{q})$ -fuzzy  $k$ -bi-ideals and interval valued  $(\overline{\varepsilon}, \overline{\varepsilon} \vee \overline{q})$ -fuzzy  $k$ -quasi-ideals.

### 5.1 Definition [5]

A hemiring  $R$  is said to be  $k$ -regular if for each  $x \in R$ , there exist  $a, a' \in R$  such that  $x + xax = xa'x$ .

### 5.2 Lemma[5]

A hemiring  $R$  is  $k$ -regular if and only if for any right  $k$ -ideal  $I$  and any left  $k$ -ideal  $L$  of  $R$  we have  $\overline{IL} = I \cap L$ .

### 5.3 Lemma[18]

Let  $R$  be a hemiring. Then the following conditions are equivalent.

- (i)  $R$  is  $k$ -regular.

- (ii)  $Q = \overline{QRQ}$  for every  $k$ -quasi-ideal  $Q$  of  $R$ .
- (iii)  $B = \overline{BRB}$  for every  $k$ -bi-ideal  $B$  of  $R$ .

## 5.4 Theorem

For a hemiring  $R$  the following conditions are equivalent:

- (i)  $R$  is  $k$ -regular.
- (ii)  $\hat{\lambda} \odot^{0.5} \hat{\mu} = \hat{\lambda} \wedge^{0.5} \hat{\mu}$ , for every interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy right  $k$ -ideal  $\hat{\lambda}$  of  $R$  and every interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy left  $k$ -ideal  $\hat{\mu}$  of  $R$ .

**Proof.** (i)  $\Rightarrow$  (ii): Let  $\hat{\lambda}$  and  $\hat{\mu}$  be interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy right and left  $k$ -ideals of  $R$  respectively, then by Lemma 4.21,  $\hat{\lambda} \odot^{0.5} \hat{\mu} \leq \hat{\lambda} \wedge^{0.5} \hat{\mu}$ . Let  $x \in R$ , then there exist  $a_1, a_2 \in R$  such that  $x + xa_1x = xa_2x$ . Then for all expressions  $x + \sum_{i=1}^m a_i b_i = \sum_{j=1}^n c_j d_j$  in  $R$  we have

$$\begin{aligned}
& \left( \hat{\lambda} \odot^{0.5} \hat{\mu} \right) (x) \\
&= \sup \left\{ \left( \bigwedge_{i=1}^m \left( \hat{\lambda}(a_i) \wedge \hat{\mu}(b_i) \right) \right) \wedge \left( \bigwedge_{j=1}^n \left( \hat{\lambda}(c_j) \wedge \hat{\mu}(d_j) \right) \right) \right\} \vee 0.5 \\
&\geq \{ (\hat{\lambda}(x) \wedge \hat{\mu}(a_1x) \wedge \hat{\lambda}(x) \wedge \hat{\mu}(a_2x)) \vee 0.5 \} \\
&\geq \{ (\hat{\lambda}(x) \wedge (\hat{\mu}(a_1x) \vee 0.5) \wedge (\hat{\mu}(a_2x) \vee 0.5)) \vee 0.5 \} \\
&\geq \{ \hat{\lambda}(x) \wedge \hat{\mu}(x) \vee 0.5 \} \\
&= \left( \hat{\lambda} \wedge^{0.5} \hat{\mu} \right) (x).
\end{aligned}$$

So  $\hat{\lambda} \odot^{0.5} \hat{\mu} \geq \hat{\lambda} \wedge^{0.5} \hat{\mu}$ . Hence  $\hat{\lambda} \odot^{0.5} \hat{\mu} = \hat{\lambda} \wedge^{0.5} \hat{\mu}$ .

(ii)  $\Rightarrow$  (i): Let  $I$  be right  $k$ -ideal and  $L$  be left  $k$ -ideal of  $R$ . Then by Lemma 4.8,  $\hat{\chi}_I$  and  $\hat{\chi}_L$  are interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy right  $k$ -ideal and interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy left  $k$ -ideal of  $R$ . Then by hypothesis  $\hat{\chi}_I \odot^{0.5} \hat{\chi}_L = \hat{\chi}_I \wedge^{0.5} \hat{\chi}_L$ . Which implies  $\hat{\chi}_{\overline{IL}} \vee 0.5 = \hat{\chi}_{I \cap L} \vee 0.5$ . Hence  $\overline{IL} = I \cap L$ , so by Lemma 5.2,  $R$  is  $k$ -regular. ■

## 5.5 Theorem

For a hemiring  $R$ , the following conditions are equivalent.

- (i)  $R$  is  $k$ -regular.
- (ii)  $\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\lambda} \geq \hat{\lambda} \vee 0.5$  for every interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy  $k$ -bi-ideal  $\hat{\lambda}$  of  $R$ .
- (iii)  $\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\lambda} \geq \hat{\lambda} \vee 0.5$  for every interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy  $k$ -quasi-ideal  $\hat{\lambda}$  of  $R$ .

**Proof.** (i)  $\Rightarrow$  (ii): Let  $\hat{\lambda}$  be an interval valued  $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy  $k$ -bi-ideal of  $R$  and  $a \in R$ . Then as  $R$  is  $k$ -regular so there exist  $x_1, x_2 \in R$  such that

$a + ax_1a = ax_2a$ . Then for all expressions  $a + \sum_{i=1}^m a_i b_i = \sum_{j=1}^n c_j d_j$  of  $a$  in  $R$ , we have

$$\begin{aligned}
& (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\lambda})(a) \\
&= \sup \left\{ \left( \bigwedge_{i=1}^m (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}})(a_i) \wedge \hat{\lambda}(b_i) \right) \wedge \left( \bigwedge_{j=1}^n (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}})(c_j) \wedge \hat{\lambda}(d_j) \right) \right\} \vee \\
0.5 & \geq \{ (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}})(ax_1) \wedge \hat{\lambda}(a) \wedge (\hat{\lambda} \odot^{0.5} \hat{\mathcal{R}})(ax_2) \vee 0.5 \} \\
&= \left[ \left\{ \sup \left( \left( \bigwedge_{i=1}^m \hat{\lambda}(a'_i) \right) \wedge \left( \bigwedge_{j=1}^n \hat{\lambda}(c'_j) \right) \right) \vee 0.5 \right\} \wedge \hat{\lambda}(a) \wedge \right. \\
& \quad \left. \left\{ \left( \sup \left( \left( \bigwedge_{i=1}^m \hat{\lambda}(a''_i) \right) \wedge \left( \bigwedge_{j=1}^n \hat{\lambda}(c''_j) \right) \right) \right) \vee 0.5 \right\} \right] \vee 0.5 \\
& \quad \text{(for all expressions } ax_1 + \sum_{i=1}^m a'_i b'_i = \sum_{j=1}^n c'_j d'_j \text{ and } ax_2 + \sum_{i=1}^m a''_i b''_i = \\
& \quad \sum_{j=1}^n c''_j d''_j \text{ in } R) \\
& \geq \{ (\hat{\lambda}(ax_1a) \wedge \hat{\lambda}(a) \wedge \hat{\lambda}(ax_2a) \vee 0.5 \} \\
& \quad \text{(because } ax_1 + ax_1ax_1 = ax_2ax_1 \text{ and } ax_2 + ax_1ax_2 = ax_2ax_2) \\
& \geq \hat{\lambda}(a) \vee 0.5.
\end{aligned}$$

(ii)  $\Rightarrow$  (iii): This is straightforward by using Lemma 4.17.

(iii)  $\Rightarrow$  (i): Let  $Q$  be a  $k$ -quasi-ideal of  $R$ . Then by Lemma 4.8,  $\hat{\chi}_Q$  is an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal of  $R$ . Now by using hypothesis  $\hat{\chi}_Q \vee 0.5 \leq \hat{\chi}_Q \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\chi}_Q = C_{\overline{QRQ}} \vee 0.5 \Rightarrow Q \subseteq \overline{QRQ}$ . But as  $\overline{QRQ} \subseteq Q$ , so  $Q = \overline{QRQ}$ . Hence by Lemma 5.3,  $R$  is  $k$ -regular. ■

## 5.6 Theorem

For a hemiring  $R$ , the following conditions are equivalent.

(i)  $R$  is  $k$ -regular.

(ii)  $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\lambda}$  for every interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -bi-ideal  $\hat{\lambda}$  and every interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -ideal  $\hat{\mu}$  of  $R$ .

(iii)  $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\lambda}$  for every interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideal and every interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -ideal  $\hat{\mu}$  of  $R$ .

**Proof.** (i)  $\Rightarrow$  (ii): Let  $a \in R$ . Then there exist  $x, x' \in R$  such that  $a + axa = ax'a$ . Now for all expressions  $a + \sum_{i=1}^m a_i b_i = \sum_{j=1}^n c_j d_j$  of  $a$  in  $R$ , we have

$$\begin{aligned}
& (\hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\lambda})(a) \\
&= \sup \left\{ \left( \bigwedge_{i=1}^m (\hat{\lambda} \odot^{0.5} \hat{\mu})(a_i) \wedge \hat{\lambda}(b_i) \right) \wedge \left( \bigwedge_{j=1}^n (\hat{\lambda} \odot^{0.5} \hat{\mu})(c_j) \wedge \hat{\lambda}(d_j) \right) \right\} \vee \\
0.5 & \geq \{ (\hat{\lambda} \odot^{0.5} \hat{\mu})(ax) \wedge \hat{\lambda}(a) \wedge (\hat{\lambda} \odot^{0.5} \hat{\mu})(ax') \vee 0.5 \}
\end{aligned}$$



for all expressions  $a + \sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j$  of  $a$  in  $R$ , we have

$$\begin{aligned}
& \left( \hat{\lambda} \odot^{0.5} \hat{\mu} \right) (a) \\
&= \sup \left\{ \left( \bigwedge_{i=1}^m \left( \hat{\lambda} (a_i) \wedge \hat{\mu} (b_i) \right) \right) \wedge \left( \bigwedge_{j=1}^n \left( \hat{\lambda} (a'_j) \wedge \hat{\mu} (b'_j) \right) \right) \right\} \vee 0.5 \\
&\geq \{ \hat{\lambda} (a) \wedge \hat{\mu} (x_1 a) \wedge \hat{\mu} (x_2 a) \} \vee 0.5 \quad \text{because } a + ax_1 a = ax_2 a \\
&= \{ \hat{\lambda} (a) \wedge \hat{\mu} (a) \vee 0.5 \} \\
&= \left( \hat{\lambda} \wedge^{0.5} \hat{\mu} \right) (a).
\end{aligned}$$

So  $\hat{\lambda} \odot^{0.5} \hat{\mu} \geq \hat{\lambda} \wedge^{0.5} \hat{\mu}$ .

(ii)  $\Rightarrow$  (iii) This is trivial by Lemma 4.17.

(iii)  $\Rightarrow$  (i) Let  $\hat{\lambda}$  be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right  $k$ -ideal and  $\hat{\mu}$  be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal of  $R$ . Then by Lemma 4.16 and by (iii) we have  $\hat{\lambda} \odot^{0.5} \hat{\mu} \geq \hat{\lambda} \wedge^{0.5} \hat{\mu}$ . But by Lemma 4.21,  $\hat{\lambda} \odot^{0.5} \hat{\mu} \leq \hat{\lambda} \wedge^{0.5} \hat{\mu}$ , so  $\hat{\lambda} \odot^{0.5} \hat{\mu} = \hat{\lambda} \wedge^{0.5} \hat{\mu}$ . Hence by Theorem 5.4,  $R$  is  $k$ -regular.

(i)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v) can be proved similarly.

(i)  $\Rightarrow$  (vi) Let  $\hat{\lambda}$  be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right  $k$ -ideal,  $\hat{\mu}$  be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -bi-ideal and  $\hat{\nu}$  be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal of  $R$  and  $a \in R$ . Then there exist  $x_1, x_2 \in R$  such that  $a + ax_1 a = ax_2 a$ . Now for all expressions  $a + \sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j$  in  $R$ , we have

$$\begin{aligned}
& \left( \hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\nu} \right) (a) \\
&= \sup \left\{ \left( \bigwedge_{i=1}^m \left( \left( \hat{\lambda} \odot^{0.5} \hat{\mu} \right) (a_i) \wedge \hat{\nu} (b_i) \right) \right) \wedge \left( \bigwedge_{j=1}^n \left( \left( \hat{\lambda} \odot^{0.5} \hat{\mu} \right) (a'_j) \wedge \hat{\nu} (b'_j) \right) \right) \right\} \wedge 0.5 \\
&\geq \{ \left( \hat{\lambda} \odot^{0.5} \hat{\mu} \right) (a) \wedge \hat{\nu} (x_1 a) \wedge \hat{\nu} (x_2 a) \vee 0.5 \} \\
&\geq \sup \left\{ \left( \bigwedge_{i=1}^m \left( \hat{\lambda} (a_i) \wedge \hat{\mu} (b_i) \right) \right) \wedge \left( \bigwedge_{j=1}^n \left( \hat{\lambda} (a'_j) \wedge \hat{\mu} (b'_j) \right) \right) \right\} \vee 0.5 \wedge \hat{\nu} (a) \\
&\geq \{ \hat{\lambda} (ax_1) \wedge \hat{\lambda} (ax_2) \wedge \hat{\mu} (a) \wedge \hat{\nu} (x_1 a) \wedge \hat{\nu} (x_2 a) \vee 0.5 \} \\
&\geq \{ \hat{\lambda} (a) \wedge \hat{\mu} (a) \wedge \hat{\nu} (a) \vee 0.5 \} \\
&= \left( \hat{\lambda} \wedge^{0.5} \hat{\mu} \wedge^{0.5} \hat{\nu} \right) (a). \\
&\Rightarrow \hat{\lambda} \wedge^{0.5} \hat{\mu} \wedge^{0.5} \hat{\nu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu} \odot^{0.5} \hat{\nu}.
\end{aligned}$$

(vi)  $\Rightarrow$  (vii) Straightforward by Lemma 4.17.

(vii)  $\Rightarrow$  (i) Let  $\hat{\lambda}$  be an interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right  $k$ -ideal, and  $\hat{\nu}$  be an interval valued  $(\epsilon, \epsilon \vee q)$ -fuzzy left  $k$ -ideal of  $R$ . Then  $\hat{\lambda} \wedge^{0.5} \hat{\nu} = \hat{\lambda} \wedge^{0.5} \hat{\mathcal{R}} \wedge^{0.5} \hat{\nu} \leq \hat{\lambda} \odot^{0.5} \hat{\mathcal{R}} \odot^{0.5} \hat{\nu} \leq \hat{\lambda} \odot^{0.5} \hat{\nu}$ .

But  $\hat{\lambda} \odot^{0.5} h \leq \hat{\lambda} \wedge^{0.5} \hat{\nu}$ . Thus  $\hat{\lambda} \odot^{0.5} \hat{\nu} = \hat{\lambda} \wedge^{0.5} \hat{\nu}$ . Hence by Theorem 5.4,  $R$  is  $k$ -regular. ■

## 6 $k$ -Intra-Regular Hemirings

In this section we characterize  $k$ -intra-regular hemirings by the properties of interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -ideals, interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -bi-ideals and interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy  $k$ -quasi-ideals.

### 6.1 Definition [18]

A hemiring  $R$  is said to be  $k$ -intra-regular if for each  $x \in R$ , there exist  $a_i, a'_i, b_j, b'_j \in R$  such that  $x + \sum_{i=1}^m a_i x^2 a'_i = \sum_{j=1}^n b_j x^2 b'_j$ .

### 6.2 Lemma[18]

A hemiring  $R$  is  $k$ -intra-regular if and only if for any right  $k$ -ideal  $I$  and any left  $k$ -ideal  $L$  of  $R$  we have  $\overline{LI} \supseteq I \cap L$ .

### 6.3 Lemma [18]

For a hemiring  $R$ , the following conditions are equivalent:

- (i)  $R$  is both  $k$ -regular and  $k$ -intra-regular.
- (ii)  $B = \overline{B^2}$  for every  $h$ -bi-ideal  $B$  of  $R$ .
- (iii)  $Q = \overline{Q^2}$  for every  $h$ -quasi-ideal  $Q$  of  $R$ .

### 6.4 Lemma

For a hemiring  $R$  the following conditions are equivalent:

- (i)  $R$  is  $k$ -intra-regular.
- (ii) For every interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal  $\hat{\lambda}$  and for every interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right  $k$ -ideal  $\hat{\mu}$  of  $R$ ,  $\hat{\lambda} \wedge^{0.5} \hat{\mu} \leq \hat{\lambda} \odot^{0.5} \hat{\mu}$

**Proof.** (i)  $\Rightarrow$  (ii) Let  $\hat{\lambda}, \hat{\mu}$  be interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy left  $k$ -ideal and interval valued  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy right  $k$ -ideal of  $R$ , respectively. Let  $a \in R$ , then there exist  $x_i, x'_i, y_j, y'_j, z \in R$  such that  $a + \sum_{i=1}^m x_i a^2 x'_i = \sum_{j=1}^n y_j a^2 y'_j$ .









