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Existence and uniqueness theorems for a class of equations of interaction type between a vibrating structure and a fluid

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Abstract

The problems of the interaction between a vibrating structure and a fluid have been studied by many authors, see for example the interesting article [**7**] and their references. The principal objective of this work is to investigate the solvability of some problems of interaction between structure and fluid by a mathematical method based upon the analytic semigroups and fractional powers of operators and which can be applied to wider range of physical situations. In this paper, we develop this method on a three-dimensional model of interaction between a vibrating structure and a light fluid occupying a bounded domain in $I\!R^3$. This model was introduced in J. Sound Vibration 177 (1994) [**3**] by Filippi-Lagarrigue-Mattei for an one-dimensional clamped thin plate, extended by an infinite perfectly rigid baffle. Intissar and Jeribi have shown in J. Math. Anal. Appl. (2004) [**4**] the existence of a Riesz basis of generalized eigenvectors of this one-dimensional model. A two-dimensional model of the vibration and the acoustic radiation of a baffled rectangular plate in contact with a dense fluid was considered by Mattei in J. Sound Vibration (1996) [**9**].

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1 Introduction to the equations of our model

We consider a light fluid occupying

$$\Omega = \{(x,y,z) \in I\!R^3; 0 \le x \le a, 0 \le y \le b, 0 \le z \le h\}$$

a bounded domain of $I\!R^3$ with boundary $\partial\Omega$ and a plate displacing on $\Gamma_1 = \{(x,y,z) \in R^3; z = 0\} \bigcap \Omega$. We define Γ_2 by $\partial\Omega = \Gamma_1 \cup \Gamma_2$.
Let $p(x,y,z)$ be the pressure of the fluid and $u(x,y)$ the displacement of the structure then we seek nontrivial solutions of the following coupling problem

$$(\mathbf{P_{fluid}}) \begin{cases} -\Delta_{xyz} p(x,y,z) = \omega^2 p(x,y,z) \\ p_{|\Gamma_2} = 0 \\ \frac{\partial}{\partial z} p_{|\Gamma_2}(x,y,0) = \rho\omega^2 u(x,y) \end{cases}$$

and

$$(\mathbf{D_{plate}}) \begin{cases} \Delta_{xy}^2 u(x,y) - m\omega^2 u(x,y) + p(x,y,0) = f(x,y), (x,y) \in \Gamma_1 \\ u_{|\partial\Gamma_1} = 0 \\ \Delta_{xy} u_{|\partial\Gamma_1} = 0 \end{cases}$$

where
$\Delta_{xyz} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\Delta_{xy}^2 = \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$
- $f(x,y)$ is an excitation source.
- The mechanical parameters of the plate are its surface density m and its thickness h.
- The fluid is characterized by its density ρ and its wave-number ω.
- For an interesting construction of the theory of plates see [**10**], § 27, p.182-206 and the references therein.

Now let $\tilde{\Omega} = [0,a] \times [0,b]$ with the boundary $\partial\tilde{\Omega}$.
Taking into account the equations of the fluid and the equation of the structure-fluid coupling, we obtain the following problem

$$(\mathbf{P}) \begin{cases} p'' - Ap = 0 \\ p'(0) = \rho\omega^2 u \\ p(h) = 0 \end{cases}$$

where $p' = \frac{\partial}{\partial z} p$, $p'' = \frac{\partial^2}{\partial z^2} p$ and $A = -(\Delta_{xy} + \omega^2 I)$ is an unbounded operator acting on the Hilbert space $L_2(\tilde{\Omega})$ with the domain
$D(A) = \{v \in L_2(\tilde{\Omega}); Av \in L_2(\tilde{\Omega})$ and $v_{|\partial\tilde{\Omega}} = 0\}$.

The operator $-\Delta_{xy}$ is positive definite on its domain which consists of functions vanishing at the boundary of the rectangle $0 \leq x \leq a, 0 \leq y \leq b$.

We give a mathematical study of the above problem (**P**) in the next section.

2 Existence and uniqueness theorems of (P)

By $L_2(0, h, L_2(\tilde{\Omega}))$, we denote the Hilbert space of strongly measurable and square summable functions $z \longrightarrow p(z)$ from $[0, h]$ into $L_2(\tilde{\Omega})$, with norm $||| \, p \, |||^2 = \int_0^h || \, p \, ||^2 \, dz < \infty$ where $|| \, . \, ||$ is the norm of $L_2(\tilde{\Omega})$.

We consider the boundary value problem (**P**) and look for a solution of this problem in the space
$W_2^2(0, h, D(A), L_2(\tilde{\Omega})) = \{p \in L_2(0, h, L_2(\tilde{\Omega})); Ap \in L_2(0, h, L_2(\tilde{\Omega}))$ and $p'' \in L_2(0, h, L_2(\tilde{\Omega}))\}$ where $D(A)$ is equipped with its norm.

Remark 2.1 *a) The operator A is positive in the sense of Krein* [**8**] *i.e. that A has the property*

$$\exists c > 0; || \, (A + \beta I)^{-1} \, || \quad \leq \frac{c}{1 + \beta} \quad \forall \beta > 0 \tag{1}$$

b) A generates the analytic semigroup $e^{-zA}, z > 0$.

From this remark, we can (see for example [**11**]) define an interpolation space between $D(A^m)$ and $L_2(\tilde{\Omega})(m \in I\!N$ and $\theta \in]0, 1[)$ by

$$[L_2(\tilde{\Omega}), D(A^m)]_\theta = \{p \in L_2(\tilde{\Omega}); \int_0^\infty z^{2m(1-\theta)-1} \, || \, A^m e^{-zA} p \, ||^2_{L_2(\tilde{\Omega})} \, dz < \infty\}$$

Now we begin by recalling the definition of the Fourier multiplier and a theorem of Mikhlin-Schwartz ([**2**], p.1181) which we shall use in the study of the problem **(P)**.

Definition 2.2 *Let H be a Hilbert space and $L(H)$ denote the space of bounded linear operators acting in H. Let $\phi \in L_2(I\!R, H)$ and $F(\phi)$ its Fourier transform*

$$F(\phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} \phi(x) dx.$$

Let $T : I\!R \longrightarrow L(H)$ such that $\xi \longrightarrow T(\xi)$, the application T is called a Fourier multiplier if there exists $M > 0$ such that for $\phi \in L_2(I\!R, H)$ we have

$$|| \, F^{-1} T F(\phi) \, ||_{L_2(I\!R, H)} \quad \leq M \, || \, \phi \, ||_{L_2(I\!R, H)}$$

Theorem 2.3 *(Mikhlin-Schwartz* **[2]***)*
If $T : \mathbb{R} \longrightarrow L(H)$ is continuously differentiable and the following inequalities hold:
there exists $M > 0$ such that $\| T(\xi) \| \leq M$ and $\| T'(\xi) \| \leq \frac{M}{|\xi|}$ for all $\xi \neq 0$,
then $T(\xi)$ is a Fourier multiplier.

By virtue of this theorem, we obtain the following lemma

Lemma 2.4 *Let $T(\xi) = \xi\sqrt{A}(\xi^2 I + A)^{-1}$ where A is the operator of problem* (**P**) *and $\| \sqrt{A} \|$ denotes the norm associated to the domain $D(\sqrt{A})$, then*
i) $T(\xi)$ is a Fourier multiplier.
ii) There exists $M > 0$ such that $\| p' \|_{L_2(\mathbb{R},D(\sqrt{A}))} \leq M \| p \|_{W_2^2(\mathbb{R},D(A),L_2(\tilde{\Omega}))}$ for all $p \in W_2^2(\mathbb{R}, D(A), L_2(\tilde{\Omega}))$.

Proof

i) As A is positive in the sense of Krein, we get from (1)
$\| (A + \xi^2 I)^{-1} \| \quad \leq \frac{c}{1+\xi^2}$ and $\| T(\xi) \| = \| \xi\sqrt{A}(\xi^2 I + A)^{-1} \| \leq \frac{c|\xi| \|\sqrt{A}\|}{1+\xi^2}$
$\leq c \| \sqrt{A} \|$.
Now, we have $T'(\xi) = \sqrt{A}(\xi^2 I + A)^{-1} - 2\xi\sqrt{A}(\xi^2 I + A)^{-2}$ and therefore
$\| T'(\xi) \| \leq \| \sqrt{A}(\xi^2 I + A)^{-1} \| (1 + 2 \| \xi(\xi^2 I + A)^{-1} \|)$
$\leq (1 + c\frac{2|\xi|}{1+\xi^2}) \| \sqrt{A}(\xi^2 I + A)^{-1} \| \leq (1 + c) \| \sqrt{A}(\xi^2 I + A)^{-1} \|$
$\leq \frac{M}{|\xi|}$ where $M = c(1 + c) \| \sqrt{A} \|$.
We can see that $T(\xi) = \xi\sqrt{A}(\xi^2 I + A)^{-1}$ verifies the assumptions of the Mikhlin-Schwartz theorem and consequently, it is a Fourier multiplier.

ii) Let $p(z) \in W_2^2(\mathbb{R}, D(A), L_2(\tilde{\Omega})))$, the function $q(z) = p''(z) - Ap(z)$ is in $L_2(\mathbb{R}, L_2(\tilde{\Omega}))$ and its Fourier transform can therefore be written as
$F(q) = -\xi^2 F(p) - AF(p)$.Then $F(p) = -(\xi^2 I + A)^{-1} F(q)$. But,
$\| p' \|_{L_2(\mathbb{R},D(\sqrt{A}))} = \| \sqrt{A}p' \|_{L_2(\mathbb{R},L_2(\tilde{\Omega}))} = \| F^{-1}\sqrt{A}F(p') \|_{L_2(\mathbb{R},L_2(\tilde{\Omega}))} =$
$\| F^{-1}\sqrt{A}\xi F(p) \|_{L_2(\mathbb{R},L_2(\tilde{\Omega}))} = \| F^{-1}\sqrt{A}\xi(\xi^2 I + A)^{-1} F(q) \|_{L_2(\mathbb{R},L_2(\tilde{\Omega}))}$
$= \| F^{-1} T(\xi) F(q) \|_{L_2(\mathbb{R},L_2(\tilde{\Omega}))}$.

By virtue of the Mikhlin-Schwartz theorem and from property i) of this lemma, we get $\| p' \|_{L_2(\mathbb{R},D(\sqrt{A}))} \quad \leq M \| q \|_{L_2(\mathbb{R},L_2(\tilde{\Omega}))} = M \| p'' - Ap \|_{L_2(\mathbb{R},L_2(\tilde{\Omega}))}$
$\leq M(\| p'' \| + \| Ap \|)_{L_2(\mathbb{R},L_2(\tilde{\Omega}))}$.
This leads to $\| p' \|_{L_2(\mathbb{R},D(\sqrt{A}))} \leq M \| p \|_{W_2^2(\mathbb{R},D(A),L_2(\tilde{\Omega}))}$ which achieves the proof of lemma 2.4.

Theorem 2.5 *The function $p(z)$, solution of the equation $p''(z) - Ap(z) = 0$ belongs to $W_2^2(0, h, D(A),\ L_2(\tilde{\Omega}))$ if and only if $p(z) = e^{-z\sqrt{A}} p_1 + e^{-(h-z)\sqrt{A}} p_2$ where $p_1, p_2 \in [L_2(\tilde{\Omega}), D(\sqrt{A})]_{\frac{3}{4}}$.*

Proof

i) Let $p(z) \in W_2^2(I\!R, D(A), L_2(\tilde{\Omega}))$ be a solution of the equation

$$p''(z) - Ap(z) = 0.$$

Let us put $v_1(z) = p'(z) - \sqrt{A}p(z)$ then it is clear that
$v_1'(z) = p''(z) - \sqrt{A}p'(z) = p''(z) - \sqrt{A}(v_1(z) + \sqrt{A}p(z)) = -\sqrt{A}v_1(z)$
i.e.

$$v_1'(z) = -\sqrt{A}v_1(z) \tag{2}$$

Similarly, if we put $v_2(z) = p'(z) + \sqrt{A}p(z)$, we obtain

$$v_2'(z) = \sqrt{A}v_2(z) \tag{3}$$

As A is positive in the sense of Krein, we use the Krein theorem [**8**] on fractional powers of operators and we deduce that - $\sqrt{A}$ is the generator of an analytic semigroup and any solution of the Cauchy problems (2) and (3) have the form

$$v_1(z) = e^{-z\sqrt{A}}v_1(0) \tag{4}$$

and

$$v_2(z) = e^{-(h-z)\sqrt{A}}v_2(h) \tag{5}$$

where $v_1(0), v_2(h) \in [D(\sqrt{A}, L_2(\tilde{\Omega})]_{\frac{1}{2}}$
As $2\sqrt{A}p(z) = v_2(z) - v_1(z)$, then we get

$$p(z) = -\frac{1}{2}A^{-\frac{1}{2}}(v_1(z) - v_2(z)) = -\frac{1}{2}A^{-\frac{1}{2}}[e^{-z\sqrt{A}}v_1(0) - e^{-(h-z)\sqrt{A}}v_2(h)] \tag{6}$$

Therefore, p_1 and p_2 are given by

$$p_1 = -\frac{1}{2}A^{-\frac{1}{2}}v_1(0) \tag{7}$$

and

$$p_2 = \frac{1}{2}A^{-\frac{1}{2}}v_2(h) \tag{8}$$

Consequently, the operator $A^{-\frac{1}{2}}$ acts in a bounded manner from $L_2(\tilde{\Omega})$ into $D(\sqrt{A})$ and from $D(\sqrt{A})$ into D(A) where these spaces are equipped with their norms.

Then by virtue of the interpolation theorem [**8**], $A^{-\frac{1}{2}}$ acts in a bounded manner from$[D(\sqrt{A}), L_2(\tilde{\Omega})]_{\frac{1}{2}}$ into $[L_2(\tilde{\Omega},)D(A)]_{\frac{3}{4}}$.

ii) The sufficiency of this theorem is trivial.

Theorem 2.6 *For $u \in [L_2(\tilde{\Omega}), D(A)]_{\frac{1}{4}}$, the problem*

$$(\mathbf{P}) \begin{cases} p'' - Ap = 0 \\ p'(0) = \rho\omega^2 u \\ p(h) = 0 \end{cases}$$

has an unique solution $p(z) \in W_2^2(0, h, D(A),\ L_2(\tilde{\Omega}))$.

Proof

By virtue of theorem (2.5), a solution of the equation $p'' - Ap = 0$ has the representation $p(z) = e^{-z\sqrt{A}}p_1 + e^{-(h-z)\sqrt{A}}p_2$ with $p_1, p_2 \in [L_2(\tilde{\Omega}), D(A)]_{\frac{3}{4}}$

The boundary conditions must verify the following system

$$(\mathbf{S}) \begin{cases} -\sqrt{A}p_1 + \sqrt{A}e^{-h\sqrt{A}}p_2 = \rho\omega^2 u \\ e^{-h\sqrt{A}}p_1 + p_2 = 0 \end{cases}$$

From the second equation of the system ($\mathbf{S}$), we obtain $p_2 = -e^{-h\sqrt{A}}p_1$. From the first equation of the same system, we deduce that
$-\sqrt{A}(I + e^{-2h\sqrt{A}})p_1 = \rho\omega^2 u$
and therefore
$p_1 = -\rho\omega^2 A^{\frac{-1}{2}}(I + e^{-2h\sqrt{A}})^{-1}u$
and
$p_2 = \rho\omega^2 A^{\frac{-1}{2}}e^{-h\sqrt{A}}(I + e^{-2h\sqrt{A}})^{-1}u.$

Now we show that $p_1, p_2 \in [L_2(\tilde{\Omega}), D(A)]_{\frac{3}{4}}$

As $u \in [L_2(\tilde{\Omega}), D(A)]_{\frac{1}{4}}$ then we have $A^{\frac{-1}{2}}u \in [L_2(\tilde{\Omega}), D(\sqrt{A})]_{\frac{3}{4}}$.

The operator $(I + e^{-2h\sqrt{A}})^{-1}$ is bounded from $L_2(\tilde{\Omega})$ into $L_2(\tilde{\Omega})$ and by virtue of the analyticity of semigroup, it acts in bounded manner from D(A) into D(A).

Then by virtue of the interpolation theorem [**11**], the operator $(I+e^{-2h\sqrt{A}})^{-1}$ acts in bounded manner from the space $[L_2(\tilde{\Omega}), D(A)]_{\frac{3}{4}}$ into the $[L_2(\tilde{\Omega}), D(A)]_{\frac{1}{4}}$.

Consequently, $p_1 \in [L_2(\tilde{\Omega}), D(A)]_{\frac{3}{4}}$

As $p_2 = -e^{-h\sqrt{A}}p_1$, then similarly we have $p_2 \in [L_2(\tilde{\Omega}), D(A)]_{\frac{3}{4}}$

Therefore the solution of problem ($\mathbf{P}$) is :

$$p(z) = \rho\omega^2 A^{\frac{-1}{2}}(I + e^{-2h\sqrt{A}})^{-1}[e^{-2h\sqrt{A}}e^{z\sqrt{A}} - e^{-z\sqrt{A}}]u. \quad (9)$$

with

$$p(0) = \rho\omega^2 A^{\frac{-1}{2}}(I + e^{-2h\sqrt{A}})^{-1}[e^{-2h\sqrt{A}} - I]u. \quad (10)$$

As $A = -(\Delta_{xy} + \omega^2 I)$, we put $K_\omega = A^{\frac{-1}{2}}(I + e^{-2h\sqrt{A}})^{-1}[e^{-2h\sqrt{A}} - I]$ then from the equation (2.10) we get

$$p(x, y, 0) = \omega^2 \rho K_\omega u(x, y). \quad (11)$$

Now the system ($\mathbf{D_{plate}}$) of the plate movement can be written in this form

$$(\mathbf{P}_\omega) \begin{cases} \Delta_{xy}^2 u(x,y) - \omega^2[mI - \rho K_\omega]u(x,y) = f(x,y) & \text{on } \Gamma_1 \\ u = \Delta_{x,y}u = 0 \quad \text{on } \partial\Gamma_1 \end{cases}$$

In the following, we study the spectral properties of the above system $(\mathbf{P}_\omega)$. We begin by considering the operator $T = \Delta_{xy}^2$ acting on $L_2(\Gamma_1)$ with minimal domain $D(T) = \{u \in C^4(\Gamma_1); u = \Delta_{x,y}u = 0 \text{ on } \partial\Gamma_1\}$.
Let H be the closure of $D(T)$ equipped with its norm, then we have the following classical spectral properties of T (see **[10]**,theorem 3, p. 327)
1) $D(T)$ is dense in $L_2(\Gamma_1)$.
2) There exists an operator G acting from $L_2(\Gamma_1)$ into H such that
α) G is symmetric and positive.
β) G^{-1} is an extension of T.
γ) G is compact.
3) The functions $u_{mn}(x,y) = sin(\frac{m\pi x}{a})sin(\frac{n\pi y}{b}) \in D(T)$ because they are in $C^4(\Gamma_1)$ and they satisfy the boundary conditions
$u_{mn}(0,y) = u_{mn}(a,y) = u_{mn}(x,0) = u_{mn}(x,b) = 0$
and
$\Delta_{xy}u_{mn}(0,y) = \Delta_{xy}u_{mn}(a,y) = \Delta_{xy}u_{mn}(x,0) = \Delta_{xy}u_{mn}(x,b) = 0.$
4) $Tu_{mn}(x,y) = \lambda_{mn}u_{mn}(x,y)$ where $\lambda_{mn} = \pi^4(\frac{m^2}{a^2} + \frac{n^2}{b^2})^2$.

As K_ω is bounded we deduce that GK_ω is compact and if 0 is not an eigenvalue of the operator $T - \omega^2[mI - \rho K_\omega]$ then, for all $f \in L_2(\Gamma_1)$, the problem $(\mathbf{P}_\omega)$ has an unique solution in H.

For one-dimensional model the operator K_ω is a Hankel operator and the investigation of the existence of a generalized eigenvectors Riesz basis of the operator $T - \omega^2(mI - \rho K_\omega)$ is given in **[4]**.
In this work we see that 0 is not an eigenvalue of the Dirichlet problem on the rectangle associated to $-\Delta_{xy}$.
In fact, by combining the Green's formula for the Laplacian operator with the Dirichlet's condition we deduce that

$< -\Delta_{xy}u, u > = -\int_0^a \int_0^b u(x,y)\Delta_{xy}u(x,y)dxdy$
$= \int_0^a \int_0^b (\mid \frac{\partial u}{\partial x} \mid^2 + \mid \frac{\partial u}{\partial y} \mid^2)dxdy \geq 0.$
If $< -\Delta_{xy}u, u > = 0$ then of necessarly $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ since the integrand in the above integral is non negative. But then u is constant and by virtue of the boundary condition $u = 0$ on $\partial\Gamma_1$, then $u = 0$
As T with the domain $D(T) = \{u \in C^4(\Gamma_1); u = \Delta_{x,y}u = 0 \text{ on } \partial\Gamma_1\}$ is Δ_{xy}^2 then T is self adjoint and invertible.
The regularity of the operator T with this domain $D(T)$, allows us to use the Keldysh's theorem ([**6**], see below) or the results of ([**2**], theorem 12, p. 1204) to show the completeness of the generalized eigenvectors of the following eigenvalue problem

$$[T - \omega^2(mI - \rho K_\omega)]u(x,y) = \lambda u(x,y); u \in D(T), \lambda \in \mathbb{C}. \tag{12}$$

Let H be a separable Hilbert space, $L(H)$ the set of all bounded linear operators in H, and C_∞ the set of all compact operators in $L(H)$.

A linear operator A acting in H is called an operator with a discrete spectrum if the whole of its spectrum $\sigma(A)$ consists of eigenvalues of finite multiplicity, with the only possible limit point at infinity. If A is such an operator, then $N(r, A)$ denotes the distribution function of its eigenvalues; that is, the number of eigenvalues (counting multiplicity) in the disk $\mid \lambda \mid \leq r$.

A sufficient condition for an operator A to have a discrete spectrum is that its resolvent $R_\lambda(A) = (A - \lambda I)^{-1} \in C_\infty$ for at least one regular value λ (then $R_\lambda(A) \in C_\infty$ for every regular λ).

If A is selfadjoint, then its spectrum is discrete if and only if $R_\lambda(A) \in C_\infty$.

The following definition was introduced by Keldysh [**6**].

Definition 2.7 *An operator A is called an operator of finite order if it belongs to C_p for some $p \in (0, \infty)$ where C_p denote the set of compact operators such that $\sum_{n=1}^{\infty} s_n^p(A) < \infty$, $\{s_n\}_1^\infty$ are the eigenvalues of the operator $\sqrt{A^*A}$ numbered in decreasing order and with multiplicity taken into account.*

Theorem 2.8 *(Keldysh's theorem* [**6**]*)*
Let A be a normal operator of finite order acting on Hilbert space E whose spectrum lies on a finite number of rays $arg\lambda = \alpha_k (k = 1, ..., N), \alpha_{k+1} > \alpha_k$. If S is compact and $I + S$ is invertible then the system of generalized eigenvectors of $(I + S)A$ is complete in E.

Remark 2.9 *$T^{-1} \in C_p$ for all $p > \frac{1}{4}$.*

Now for $\lambda = 0$ the problem (12) can be written as following

$$[T - \omega^2(mI - \rho K_\omega)]u(x,y) = 0; u \in D(T). \tag{13}$$

or

$$T^{-1}(I - \frac{\rho}{m}K_\omega)u(x,y) = \frac{1}{m\omega^2}u(x,y); u \in D(T). \tag{14}$$

If $\frac{\rho}{m}$ is not in $\sigma(K_\omega)$, we can use the Keldysh's theorem to deduce the completeness of the eigenfunctions of problem (13)

For $\lambda \neq 0$, by applying the operator G to the equation (12), we get

$$[I - \omega^2 G(mI - \rho K_\omega)]u(x,y) = \lambda Gu(x,y); u \in D(T). \tag{15}$$

If $\frac{1}{\omega^2}$ is not an eigenvalue of the operator $G(mI - K_\omega)$, then the equation (13) can be written as

$$[I - \omega^2 G(mI - K_\omega)]^{-1}Gu(x,y) = \frac{1}{\lambda}u(x,y); u \in D(T). \tag{16}$$

In this case the operator $[I - \omega^2 G(mI - K)]^{-1}G$ satisfies the completeness criteria of the above Keldysh theorem **[6]** or the completeness criteria of Aimar-Intissar-Paoli theorem given in **[1]** (see also the chapter IV of **[5]**) and consequently the closure of the subspace spanned by all the generalized eigenvectors of $[T - \omega^2(mI - K_\omega)]$ corresponding to the eigenvalues which are different from zero is dense in $L_2(\Gamma_1)$.

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