Mathematica Aeterna, Vol. 2, 2012, no. 1, 31-42

$(L, \odot)$-fuzzy topologies induced by functions

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Abstract

In this paper, we investigate the properties of $(L, \odot)$-fuzzy topologies and $(L, \odot)$-filters induced by functions on strictly two-sided, commutative quantale lattices $(L, \odot)$ and $(L, *)$. Furthermore, we study their convergence and functorial relations.

Mathematics Subject Classification: 03E72, 54A40,54B10

Keywords: Quantale lattice, $(L, \odot)$-fuzzy topologies, $(L, \odot)$-filters, LF-continuous, $(\mathcal{F}^x, *)$-neighborhood filter.

1 Introduction

Höhle and Šostak [6] introduced the notion of $(L, \odot)$-fuzzy topological spaces on a complete quasi-monoidal lattice (or GL-monoid) instead of a completely distributive lattice or an unit interval. Höhle and Sostak [6] introduced the concept of $(L, \odot)$-filters for a complete quasi-monoidal lattice L.

In this paper, we investigate the products of $(L, \odot)$-fuzzy topologies and $(L, \odot)$-filters induced by functions on strictly two-sided, commutative quantale lattices $(L, \odot)$ and $(L, *)$. Furthermore, we study relations among LF-continuous maps, filter convergence, $(\mathcal{F}^x, *)$-neighborhood filters and L-filter maps.

2 Preliminaries

Definition 2.1 [8] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $L = (L, \leq, \top, \bot)$ is a complete lattice where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(L2) $(L, \odot)$ is a commutative semigroup;
(L3) $a = a \odot \top$, for each $a \in L$;
(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Example 2.2 [8] (1) Each frame is a stsc-quantale. In particular, the unit interval $([0,1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(2) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a stsc-quantale.

(3) Every GL-monoid is a stsc-quantale.

(4) Define a binary operation $\odot$ on $[0,1]$ by $x \odot y = \max\{0, x+y-1\}$. Then $([0,1], \leq, \odot)$ is a stsc-quantale.

Definition 2.3 [6,8] A mapping $\tau : L^X \to L$ is called an $(L, \odot)$-*fuzzy topology* on X if it satisfies the following conditions:
(T1) $\tau(1_\emptyset) = \top$ and $\tau(1_X) = \top$,
(T2) $\tau(f \odot g) \geq \tau(f) \odot \tau(g)$, for each $f, g \in L^X$,
(T3) $\tau(\bigvee_{i\in\Gamma} f_i) \geq \bigwedge_{i\in\Gamma} \tau(f_i)$.
An $(L, \odot)$-fuzzy topology is called *enriched* if
(S) $\tau(\alpha \odot f) \geq \tau(f)$ for each $f \in L^X$ and $\alpha \in L$.
The pair (X, τ) is called an (resp. enriched)$(L, \odot)$-*fuzzy topological space*. $T_\odot(X)$ is a family of $(L, \odot)$-fuzzy topologies on X.

Let (X, τ_1) and (Y, τ_2) be two $(L, \odot)$-fuzzy topological spaces and a map $\phi : X \to Y$ called LF-continuous if $\tau_2(g) \leq \tau_1(\phi^{\leftarrow}(g))$ for all $g \in L^Y$.

Definition 2.4 [6,8] A mapping $\mathcal{F} : L^X \to L$ is called an $(L, \odot)$-*filter* on X if it satisfies the following conditions:
(F1) $\mathcal{F}(1_\emptyset) = \bot$ and $\mathcal{F}(1_X) = \top$,
(F2) $\mathcal{F}(f \odot g) \geq \mathcal{F}(f) \odot \mathcal{F}(g)$, for each $f, g \in L^X$,
(F3) if $f \leq g$, $\mathcal{F}(f) \leq \mathcal{F}(g)$.
An $(L, \odot)$-filter is called *stratified* if
(S) $\mathcal{F}(\alpha \odot f) \geq \alpha \odot \mathcal{F}(f)$ for each $f \in L^X$ and $\alpha \in L$.
The pair $(X, \mathcal{F})$ is called an (resp. a stratified)$(L, \odot)$-*filter space*. $F_\odot(X)$ is a family of $(L, \odot)$-filters on X.

Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be two $(L, \odot)$-filter spaces and $\phi : X \to Y$ called an L-filter map if $\mathcal{F}_2(g) \leq \mathcal{F}_1(\phi^{\leftarrow}(g))$ for all $g \in L^Y$.

Example 2.5 (1) Define a map $[x] : L^X \to L$ as $[x](f) = f(x)$. Then $[x]$ is a stratified $(L, \odot)$-filter on X.

(2) Define a map inf : $L^X \to L$ as $\inf(f) = \bigwedge_{x\in X} f(x)$. Then inf is a stratified $(L, \odot)$-filter on X.

Definition 2.6 [6] Let $(L,*)$ and $(L,\odot)$ be stsc-quantales. An operation $\odot$ dominates $*$ if it satisfies:

$$(x_1 * y_1) \odot (x_2 * y_2) \geq (x_1 \odot x_2) * (y_1 \odot y_2).$$

Example 2.7 (1) For any left-continuous t-norm $*$, $\wedge$ dominates $*$ because

$$(x_1 * y_1) \wedge (x_2 * y_2) \geq (x_1 \wedge x_2) * (y_1 \wedge y_2).$$

(2) Define t-norms as $x \odot y = \frac{xy}{x+y-xy}$ and $x * y = xy$. Then $\odot$ dominates $*$.

Lemma 2.8 [9] *Let $(L,\odot)$ and $(L,*)$ be stsc-quantales which induce two implications $a \to b = \bigvee\{c \mid a \odot c \leq b\}$ and $a \Rightarrow b = \bigvee\{c \mid a * c \leq b\}$, respectively. Let $\odot$ dominates $*$. For each $a,b,c,a_i,b_i \in L$, we have the following properties.*

(1) *If $b \leq c$, then $a \odot b \leq a \odot c$ and $a * b \leq a * c$.*
(2) *$a \odot b \leq c$ iff $a \leq b \to c$. Moreover, $a * b \leq c$ iff $a \leq b \Rightarrow c$.*
(3) *If $b \leq c$, then $a \to b \leq a \to c$ and $c \to a \leq b \to a$ for $\to \in \{\to, \Rightarrow\}$.*
(4) *$a * b \leq a \odot b$, $a \to b \leq a \Rightarrow b$ and $a * (b \odot c) \leq (a * b) \odot c$.*
(5) *$(a \Rightarrow b) \odot (c \Rightarrow d) \leq (a \odot c) \Rightarrow (b \odot d)$.*
(6) *$(b \Rightarrow c) \leq (a \odot b) \Rightarrow (a \odot c)$.*
(7) *$(b \to c) \leq (a \Rightarrow b) \to (a \Rightarrow c)$ and $(b \Rightarrow a) \leq (a \to c) \to (b \Rightarrow c)$*
(8) *$a_i \to b_i \leq (\bigwedge_{i\in\Gamma} a_i) \to (\bigwedge_{i\in\Gamma} b_i)$.*
(9) *$a_i \to b_i \leq (\bigvee_{i\in\Gamma} a_i) \to (\bigvee_{i\in\Gamma} b_i)$.*
(10) *$(c \Rightarrow a) * (b \to d) \leq (a \to b) \to (c \Rightarrow d)$.*

Theorem 2.9 [10] *Let (X,τ) be an $(L,\odot)$-fuzzy topology and $\{\mathcal{F}^x \mid x \in X\}$ a family of $(L,\odot)$-filters. An operation $*$ dominates $\odot$. We define a map $\mathcal{N}^x_\tau : L^X \to L$ as follows:*

$$\mathcal{N}^x_\tau(f) = \bigvee_{g \leq f} (\mathcal{F}^x(g) * \tau(g))$$

Then

(1) *$\mathcal{N}^x_\tau$ is an $(L,\odot)$-filter.*
(2) *If $\mathcal{F}^x$ is a stratified $(L,\odot)$-filter and τ is an enriched $(L,\odot)$-fuzzy topology , then $\mathcal{N}^x_\tau$ is a stratified $(L,\odot)$-filter*
(3) *If $\mathcal{F}^x \geq H^x$, then $\tau_{\mathcal{N}^x_\tau} \geq \tau$.*
(4) *If $\mathcal{F}^x \leq H^x$, then $\mathcal{N}^x_{\tau_F} \leq \mathcal{F}^x$.*

Definition 2.10 [10] In above theorem, a map $\mathcal{N}^x_\tau : L^X \to L$ is called $(\mathcal{F}^x,*)$-neighborhood filter induced by $\mathcal{F}^x, \tau$ and operation $*$. A family $\{\mathcal{N}^x_\tau \mid x \in X\}$ is called $(\mathcal{F}^x,*)$-neighborhood system.

Theorem 2.11 [10] *An operation $*$ dominates $\odot$. Let (X,τ_X) and (Y,τ_Y) be $(L,\odot)$-fuzzy topological spaces,$\{\mathcal{F}^x \mid x \in X\}$ and $\{\mathcal{F}^y \mid y \in Y\}$ two families of $(L,\odot)$-filters and $\psi : X \to Y$ be a map. Then for $h \in L^Y$,*

$$(\tau_Y(h) \to \tau_X(\psi^{\leftarrow}(h))) * (\mathcal{F}^{\psi(x)}(h) \to \mathcal{F}^x(\psi^{\leftarrow}(h)) \leq \mathcal{N}^{\psi(x)}_{\tau_Y}(h) \to \mathcal{N}^x_{\tau_X}(\psi^{\leftarrow}(h))$$

In particular, if $\psi : (X,\tau_X) \to (Y,\tau_Y)$ is LF continuous and $\psi : (X,\mathcal{F}^x) \to (Y,\mathcal{F}^{\psi(x)})$ is an L-filter map, then $\psi : (X,\mathcal{N}^x_{\tau_X}) \to (Y,\mathcal{N}^{\psi(x)}_{\tau_Y})$ is an L-filter map.

Theorem 2.12 [10] *An operation $*$ dominates $\odot$. Let $F = \{\mathcal{F}^x \in L^{L^X} \mid x \in X\}$ and $G = \{\mathcal{G}^x \in L^{L^X} \mid x \in X\}$ be two families of $(L,\odot)$-filters satisfying the condition $\mathcal{F}^x(f) * \mathcal{G}^x(g) = \perp$ for each $f \odot g = \perp$. We define $\mathcal{F}^x * \mathcal{G}^x : L^X \to L$ as follows:*

$$\mathcal{F}^x * \mathcal{G}^x(h) = \bigvee\{\mathcal{F}^x(f) * \mathcal{G}^x(g) \mid f \odot g \leq h\}.$$

Let τ_1, τ_2 be an $(L,\odot)$-fuzzy topologies on X. We define $\tau_1 * \tau_2 : L^X \to L$ as follows:

$$(\tau_1 * \tau_2)(h) = \bigvee\{\tau_1(f) * \tau_2(g) \mid f \odot g = h\}.$$

(1) $\mathcal{F}^x * \mathcal{G}^x$ is an $(L,\odot)$-filter on X which is finer than $\mathcal{F}^x$ and $\mathcal{G}^x$. If $* = \odot$, then $\mathcal{F}^x \odot \mathcal{G}^x$ is the coarsest $(L,\odot)$-filter on X which is finer than $\mathcal{F}^x$ and $\mathcal{G}^x$. Moreover, if $* = \odot$ and $\mathcal{F}^x = \mathcal{G}^x$, then $\mathcal{F}^x \odot \mathcal{F}^x = \mathcal{F}^x$.

(2) If $\mathcal{F}^x$ or $\mathcal{G}^x$ is a stratified $(L,\odot)$-filter, then $\mathcal{F}^x * \mathcal{G}^x$ is a stratified $(L,\odot)$-filter on X.

(3) $\tau_1 * \tau_2$ is an $(L,\odot)$-fuzzy topology on X which is finer than τ_1 and τ_2. If $* = \odot$, then $\tau_1 \odot \tau_2$ is the coarsest $(L,\odot)$-fuzzy topology on X which is finer than τ_1 and τ_2.

(4) If τ_1 or τ_2 is an enriched $(L,\odot)$-fuzzy topology, then $\tau_1 * \tau_2$ is an enriched $(L,\odot)$-fuzzy topology on X.

(5) $\tau_{F*G} \geq \tau_F * \tau_G$ where $F * G = \{\mathcal{F}^x * \mathcal{G}^x \in L^{L^X} \mid x \in X\}$.

(6) $\mathcal{N}^x_{\tau_1 \odot \tau_2} \geq \mathcal{N}^x_{\tau_1} \odot \mathcal{N}^x_{\tau_2}$.

Definition 2.13 [10] Let (X,τ) be an $(L,\odot)$-fuzzy topological space, $\mathcal{N}^x_\tau$ $(\mathcal{F}^x,*)$-neighborhood filter, $\mathcal{G}$ an $(L,\odot)$-filter , $f, g \in L^X$ and $x \in X$.

(1) x is called $(\mathcal{F}^x,*)$-*cluster point* of $\mathcal{G}$, denoted by $\mathcal{G}\infty x(\mathcal{F}^x,*)$, if for every $\mathcal{N}^x_\tau(f) * \mathcal{G}(g) \neq \perp$, we have $f \odot g \neq 1_\emptyset$.

(2) x is called $(\mathcal{F}^x,*)$-*limit point* of $\mathcal{G}$, denoted by $\mathcal{G} \to x(\mathcal{F}^x,*)$, if $\mathcal{N}^x_\tau \leq \mathcal{G}$.

We denote

$$clu_\tau(\mathcal{G})(\mathcal{F}^x,*) = \bigcup\{x \in X \mid x \text{ is } (\mathcal{F}^x,*)\text{-cluster point of } \mathcal{G}\},$$

$$lim_\tau(\mathcal{G})(\mathcal{F}^x,*) = \bigcup\{x \in X \mid x \text{ is } (\mathcal{F}^x,*)\text{-limit point of } \mathcal{G}\}.$$

Theorem 2.14 [10] *Let (X,τ) be an $(L,\odot)$-fuzzy topological space and $\mathcal{N}_\tau^x$ be $(\mathcal{F}^x,*)$-neighborhood filter. Let $\mathcal{F}$ and $\mathcal{G}$ be $(L,\odot)$-filters on X which $\mathcal{F}$ is coarser than $\mathcal{G}$. For each $x\in X$, the following properties hold:*

(1) *$\mathcal{N}_\tau^x(f)\to\mathcal{F}(f)\le\mathcal{N}_\tau^x(f)\to\mathcal{G}(f)$, for all $f\in L^X$.*

(2) *If $\mathcal{F}\to x(\mathcal{F}^x,*)$, then $\mathcal{G}\to x(\mathcal{F}^x,*)$.*

(3) *$lim_\tau(\mathcal{F})(\mathcal{F}^x,*)\le lim_\tau(\mathcal{G})(\mathcal{F}^x,*)$.*

(4) *If $\mathcal{G}\infty x(\mathcal{F}^x,*)$, then $\mathcal{F}\infty x(\mathcal{F}^x,*)$.*

(5) *$clu_\tau(\mathcal{G})(\mathcal{F}^x,*)\le clu_\tau(\mathcal{F})(\mathcal{F}^x,*)$.*

(6) *If $\mathcal{F}\to x(\mathcal{F}^x,*)$ and $(L,\odot)$-filter $\mathcal{F}*\mathcal{F}$ exists, then $\mathcal{F}\infty x(\mathcal{F}^x,*)$. In particular, if $\odot=*$ and $\mathcal{F}\to x(\mathcal{F}^x,\odot)$, then $\mathcal{F}\infty x(\mathcal{F}^x,\odot)$ and $lim_\tau(\mathcal{F})(\mathcal{F}^x,\odot)\le clu_\tau(\mathcal{F})(\mathcal{F}^x,\odot)$.*

Theorem 2.15 [10] *Let (X,τ) be an $(L,\odot)$-fuzzy topological space, $\mathcal{N}_\tau^x$ be $(\mathcal{F}^x,*)$-neighborhood filter and $\mathcal{F}$ an $(L,\odot)$-filter.*

Then: (1) *If $\mathcal{F}\infty x(\mathcal{F}^x,*)$, then $\mathcal{F}$ has a finer $(L,\odot)$-filter $\mathcal{G}$ such that $\mathcal{G}\to x(\mathcal{F}^x,*)$.*

(2) *If $\mathcal{F}$ has a finer $(L,\odot)$-filter $\mathcal{G}$ such that $\mathcal{G}\to x(\mathcal{F}^x,*)$ which an $(L,\odot)$-filter $\mathcal{F}*\mathcal{F}$ exists, then $\mathcal{F}\infty x(\mathcal{F}^x,*)$.*

(3) *If $\odot=*$ and $\mathcal{F}$ has a finer $(L,\odot)$-filter $\mathcal{G}$ such that $\mathcal{G}\to x(\mathcal{F}^x,*)$, then $\mathcal{F}\infty x(\mathcal{F}^x,*)$.*

3 $(L,\odot)$-fuzzy topologies induced by functions

Theorem 3.1 *Let $\phi:X\to Y$ be a map. Let $\mathcal{F}$ and $\mathcal{G}$ be $(L,\odot)$-filters on X and Y, respectively. Let τ_X and τ_Y be $(L,\odot)$-fuzzy topologies on X and Y, respectively.*

(1) *Let $\mathcal{F}$ be an $(L,\odot)$-filter on X. We define a map $\phi^{\Rightarrow}(\mathcal{F}):L^Y\to L$ as $\phi^{\Rightarrow}(\mathcal{F})(g)=\mathcal{F}(\phi^{\leftarrow}(g))$. Then $\phi^{\Rightarrow}(\mathcal{F})$ is the finest $(L,\odot)$-filter on Y for which each $\phi:(X,\mathcal{F})\to(Y,\mathcal{G})$ is an L-filter map.*

(2) *Let τ_X be an $(L,\odot)$-fuzzy topology on X. We define a map $\phi^{\Rightarrow}(\tau_X):L^Y\to L$ as $\phi^{\Rightarrow}(\tau_X)(g)=\tau_X(\phi^{\leftarrow}(g))$. Then $\phi^{\Rightarrow}(\tau_X)$ is the finest $(L,\odot)$-fuzzy topology on Y for which each $\phi:(X,\tau_X)\to(Y,\tau_Y)$ is an LF-continuous map.*

(3) *Let $\mathcal{G}$ be an $(L,\odot)$-filter on Y with $\mathcal{G}(g)=\perp$ for $\phi^{\leftarrow}(g)=1_\emptyset$. We define a map $\phi^{\Leftarrow}(\mathcal{G}):L^X\to L$ as $\phi^{\Leftarrow}(\mathcal{G})(f)=\bigvee\{\mathcal{G}(g)\mid\phi^{\leftarrow}(g)\le f\}$. Then $\phi^{\Leftarrow}(\mathcal{G})$ is the coarsest $(L,\odot)$-filter on Y for which each $\phi:(X,\mathcal{F})\to(Y,\mathcal{G})$ is an L-filter map.*

(4) *Let τ_Y be an $(L,\odot)$-fuzzy topology on Y. We define a map $\phi^{\Leftarrow}(\tau_Y):L^X\to L$ as $\phi^{\Leftarrow}(\tau_Y)(f)=\bigvee\{\tau_Y(g)\mid\phi^{\leftarrow}(g)=f\}$. Then $\phi^{\Leftarrow}(\tau_Y)$ is the coarsest $(L,\odot)$-fuzzy topology on Y for which each $\phi:(X,\tau_X)\to(Y,\tau_Y)$ is an LF-continuous map.*

Proof. (2) (T1) $\phi^{\Rightarrow}(\tau_X)(1_\emptyset) = \tau_X((1_\emptyset) = \top$ and $\phi^{\Rightarrow}(\tau_X)(1_Y) = \tau_X((1_X) = \top$.

(T2) For $f, g \in L^Y$, we have

$$\begin{aligned}\phi^{\Rightarrow}(\tau_X)(f \odot g) &= \tau_X(\phi^{\leftarrow}(f \odot g)) \\ &= \tau_X(\phi^{\leftarrow}(f) \odot \phi^{\leftarrow}(g)) \\ &\geq \tau_X(\phi^{\leftarrow}(f)) \odot \tau_X(\phi^{\leftarrow}(g)) \\ &= \phi^{\Rightarrow}(\tau_X)(f) \odot \phi^{\Rightarrow}(\tau_X)(g).\end{aligned}$$

(T3) For a family $\{f_i \in L^X \mid i \in \Gamma\}$, we have

$$\begin{aligned}\phi^{\Rightarrow}(\tau_X)(\bigvee_{i\in\Gamma} f_i) &= \tau_X(\phi^{\leftarrow}(\bigvee_{i\in\Gamma} f_i)) \\ &\geq \bigwedge_{i\in\Gamma} \tau_X(\phi^{\leftarrow}(f_i)) = \bigwedge_{i\in\Gamma} \phi^{\Rightarrow}(\tau_X)(f_i)\end{aligned}$$

Hence $\phi^{\Rightarrow}(\tau_X)$ is an $(L, \odot)$-topology on Y. Also, $\phi : (X, \tau_X) \to (Y, \phi^{\Rightarrow}(\tau_X))$ is an LF-continuous map. Since $\tau_Y(g) \leq \tau_X(\phi^{\leftarrow}(g)) = \phi^{\Rightarrow}(\tau_X)(g)$, then $\phi^{\Rightarrow}(\tau_X)$ is the finest $(L, \odot)$-topology on Y for which each $\phi : (X, \tau_X) \to (Y, \tau_Y)$ is an LF-continuous map.

(4) (T1) is easy.

(T2) For $f_1, f_2 \in L^X$, we have

$$\begin{aligned}&\phi^{\Leftarrow}(\tau_Y)(f_1) \odot \phi^{\Leftarrow}(\tau_Y)(f_2) \\ &= \bigvee\{\tau_Y(g_1) \mid \phi^{\leftarrow}(g_1) = f_1\} \odot \bigvee\{\tau_Y(g_2) \mid \phi^{\leftarrow}(g_2) = f_2\} \\ &\leq \bigvee\{\tau_Y(g_1) \odot \tau_Y(g_2) \mid \phi^{\leftarrow}(g_1 \odot g_2) = f_1 \odot f_2\} \\ &\leq \bigvee\{\tau_Y(h) \mid \phi^{\leftarrow}(h) = f_1 \odot f_2\} \\ &= \phi^{\Rightarrow}(\tau_Y)(f_1 \odot f_2).\end{aligned}$$

(T3) For a family $\{f_i \in L^X \mid i \in \Gamma\}$, we have

$$\begin{aligned}\bigwedge_{i\in\Gamma} \phi^{\Leftarrow}(\tau_Y)(f_i) &= \bigwedge_{i\in\Gamma} \bigvee\{\tau_Y(g_i) \mid \phi^{\leftarrow}(g_i) = f_i\} \\ &= \bigvee \bigwedge_{i\in\Gamma}\{\tau_Y(g_i) \mid \phi^{\leftarrow}(g_i) = f_i\} \\ &\leq \bigvee\{\tau_Y(\bigvee_{i\in\Gamma} g_i) \mid \phi^{\leftarrow}(\bigvee_{i\in\Gamma} g_i) = \bigvee_{i\in\Gamma} f_i\} \\ &\leq \phi^{\Leftarrow}(\tau_Y)(\bigvee_{i\in\Gamma} f_i)\end{aligned}$$

Hence $\phi^{\Leftarrow}(\tau_Y)$ is an $(L, \odot)$-fuzzy topology on X. Since $\tau_Y(g) \leq \phi^{\Leftarrow}(\tau_Y)(\phi^{\leftarrow}(g))$ for $g \in L^Y$, $\phi : (X, \phi^{\Leftarrow}(\tau_Y)) \to (Y, \tau_Y)$ is an LF-continuous map. Let $\phi : (X, \tau_X) \to (Y, \tau_Y)$ be an LF-continuous map. Then $\tau_Y(g) \leq \tau_X(\phi^{\leftarrow}(g))$. Thus $\phi^{\Leftarrow}(\tau_Y)(f) \leq \tau_X(f)$.

(1) and (3) are similarly proved as in (2) and (4), respectively.

Theorem 3.2 *Let $\phi : X \to Y$ be a function, $\mathcal{F}_i \in F_{\odot}(X)$ and $\mathcal{G}_i \in F_{\odot}(Y)$ for $i = \{1, 2\}$. Then we have the following properties.*

(1) *If $\mathcal{F}_1 * \mathcal{F}_2 \in F_{\odot}(X)$, then $\phi^{\Rightarrow}(\mathcal{F}_1) * \phi^{\Rightarrow}(\mathcal{F}_2) \in F_{\odot}(Y)$.*

(2) *$\phi^{\Rightarrow}(\mathcal{F}_1) * \phi^{\Rightarrow}(\mathcal{F}_2) \leq \phi^{\Rightarrow}(\mathcal{F}_1 * \mathcal{F}_2)$ with equality if ϕ is injective.*

(3) *If $(\mathcal{G}_1 * \mathcal{G}_2) \in F_{\odot}(Y)$ with $g = \bot$ for $\phi^{\leftarrow}(g) = 1_\emptyset$, $\phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2) \in F_{\odot}(X)$.*

(4) *$\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2) = \phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2)$.*

Proof. (1) For $g_1 \odot g_2 = 1_\emptyset$, $\phi^{\leftarrow}(g_1) \odot \phi^{\leftarrow}(g_2) = \phi^{\leftarrow}(g_1 \odot g_2) = 1_\emptyset$. Since $\mathcal{F}_1 * \mathcal{F}_2 \in F_\odot(X)$, $\phi^{\Rightarrow}(\mathcal{F}_1)(g_1) * \phi^{\Rightarrow}(\mathcal{F}_2)(g_2) = \mathcal{F}_1(\phi^{\leftarrow}(g_1)) * \mathcal{F}_2(\phi^{\leftarrow}(g_2)) = \bot$.

(2)

$$
\begin{aligned}
&(\phi^{\Rightarrow}(\mathcal{F}_1) * \phi^{\Rightarrow}(\mathcal{F}_2)(h) \\
&= \bigvee\{\phi^{\Rightarrow}(\mathcal{F}_1)(f_1) * \phi^{\Rightarrow}(\mathcal{F}_2)(f_2) \mid f_1 \odot f_2 \leq h\} \\
&\leq \bigvee\{\mathcal{F}_1(\phi^{\leftarrow}(f_1)) * \mathcal{F}_2(\phi^{\leftarrow}(f_2)) \mid \phi^{\leftarrow}(f_1) \odot \phi^{\leftarrow}(f_2) \leq \phi^{\leftarrow}(h)\} \\
&\leq (\mathcal{F}_1 * \mathcal{F}_2)(\phi^{\leftarrow}(h)) = \phi^{\Rightarrow}(\mathcal{F}_1 * \mathcal{F}_2)(h).
\end{aligned}
$$

Let ϕ be an injective function. Suppose there exists $h \in L^Y$ such that

$$(\phi^{\Rightarrow}(\mathcal{F}_1) * \phi^{\Rightarrow}(\mathcal{F}_2))(h) \not\geq \phi^{\Rightarrow}(\mathcal{F}_1 * \mathcal{F}_2)(h).$$

By the definition of $\phi^{\Rightarrow}(\mathcal{F}_1 * \mathcal{F}_2)$, there exist $f_1, f_2 \in L^X$ with $f_1 \odot f_2 \leq \phi^{\leftarrow}(h)$ such that

$$\phi^{\Rightarrow}(\mathcal{F}_1) * \phi^{\Rightarrow}(\mathcal{F}_2)(h) \not\geq \mathcal{F}_1(f_1) * \mathcal{F}_2(f_2).$$

Since ϕ is an injective function, $\phi^{\rightarrow}(f_1) \odot \phi^{\rightarrow}(f_2) = \phi^{\rightarrow}(f_1 \odot f_2) \leq \phi^{\rightarrow}(\phi^{\leftarrow}(h)) \leq h$. Thus,

$$(\phi^{\Rightarrow}(\mathcal{F}_1) * \phi^{\Rightarrow}(\mathcal{F}_2))(h) \geq \phi^{\Rightarrow}(\mathcal{F}_1)(\phi^{\rightarrow}(f_1)) * \phi^{\Rightarrow}(\mathcal{F}_2)(\phi^{\rightarrow}(f_2)) \geq \mathcal{F}_1(f_1) * \mathcal{F}_2(f_2).$$

It is a contradiction. Hence $\phi^{\Rightarrow}(\mathcal{F}_1) * \phi^{\Rightarrow}(\mathcal{F}_2) \geq \phi^{\Rightarrow}(\mathcal{F}_1 * \mathcal{F}_2)$.

(3) For each $f_1 \odot f_2 = 1_\emptyset$, $\phi^{\leftarrow}(g_1 \odot g_2) = 1_\emptyset$ implies $g_1 \odot g_2 = 1_\emptyset$. Since $(\mathcal{G}_1 * \mathcal{G}_2) \in F_\odot(Y)$, $\mathcal{G}_1(g_1) * \mathcal{G}_2(g_2) = \bot$. Hence $\phi^{\Leftarrow}(\mathcal{G}_1)(f_1) * \phi^{\Leftarrow}(\mathcal{G}_2)(f_2) = \bot$. Hence $\phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2) \in F_\odot(X)$.

(4) Suppose there exists $h \in L^Y$ such that

$$\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2)(h) \not\geq (\phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2))(h).$$

By the definition of $\phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2)$, there exist h_1, h_2 with $h_1 \odot h_2 \leq h$ such that

$$\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2)(h) \not\geq \phi^{\Leftarrow}(\mathcal{G}_1)(h_1) * \phi^{\Leftarrow}(\mathcal{G}_2)(h_2).$$

By the definitions of $\phi^{\Leftarrow}(\mathcal{G}_i)$, for each $i \in \{1, 2\}$, there exists $g_i \in L^Y$ with $\phi^{\leftarrow}(g_i) \leq h_i$ such that

$$\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2)(h) \not\geq \mathcal{G}_1(g_1) * \mathcal{G}_2(g_2).$$

Since $\phi^{\leftarrow}(g_1) \odot \phi^{\leftarrow}(g_2) = \phi^{\leftarrow}(g_1 \odot g_2) \leq h$,

$$\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2)(h) \geq (\mathcal{G}_1 * \mathcal{G}_2)(g_1 \odot g_2) \geq \mathcal{G}_1(g_1) * \mathcal{G}_2(g_2).$$

It is a contradiction. Hence

$$\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2) \geq \phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2).$$

Suppose there exists $k \in L^X$ such that

$$\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2)(k) \not\leq (\phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2))(k).$$

By the definition of $\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2)$, there exists g with $\phi^{\leftarrow}(g) \leq k$ such that

$$(\mathcal{G}_1 * \mathcal{G}_2)(g) \not\leq (\phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2))(k).$$

By the definition of $\mathcal{G}_1 * \mathcal{G}_2$, there exist $g_i \in L^Y$ with $g_1 \odot g_2 \leq g$ such that

$$\mathcal{G}_1(g_1) * \mathcal{G}_2(g_2) \not\leq (\phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2))(k).$$

Since $\phi^{\leftarrow}(g_1) \odot \phi^{\leftarrow}(g_2) = \phi^{\leftarrow}(g_1 \odot g_2) \leq k$,

$$(\phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2))(k) \geq \phi^{\Leftarrow}(\mathcal{G}_1)(\phi^{\leftarrow}(g_1)) * \phi^{\Leftarrow}(\mathcal{G}_2)(\phi^{\leftarrow}(g_2)) \geq \mathcal{G}_1(g_1) * \mathcal{G}_2(g_2).$$

It is a contradiction. Hence

$$\phi^{\Leftarrow}(\mathcal{G}_1 * \mathcal{G}_2) \leq \phi^{\Leftarrow}(\mathcal{G}_1) * \phi^{\Leftarrow}(\mathcal{G}_2).$$

Example 3.3 Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be sets, $(L = [0,1], \odot)$ the stsc-quantale with $x \odot y = 0 \vee (x + y - 1)$ and let $g_1, g_2 \in [0,1]^Y$ defined as $g_1(y_1) = 0.6, g_2(y_2) = 0.5$ and $g_2(y_1) = 0.5, g_2(y_2) = 0.2$. We define $([0,1], \odot)$-filters $\mathcal{G}_i : [0,1]^Y \to [0,1]$ as follows:

$$\mathcal{G}_1(g) = \begin{cases} 1, & \text{if } g = 1_Y, \\ 0.6, & \text{if } g_1 \leq g \neq 1_Y, \\ 0.3, & \text{if } g_1 \odot g_1 \leq g \not\geq g_1, \\ 0, & \text{otherwise.} \end{cases} \quad \mathcal{G}_2(g) = \begin{cases} 1, & \text{if } g = 1_Y, \\ 0.5, & \text{if } g_2 \leq g \neq 1_Y, \\ 0, & \text{otherwise.} \end{cases}$$

(1) If $* = \odot$, then we obtain $\mathcal{G}_1 \odot \mathcal{G}_2$ as follows:

$$\mathcal{G}_1 \odot \mathcal{G}_2(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.6, & \text{if } g_1 \leq g \neq 1_X, \\ 0.5, & \text{if } g_2 \leq g \not\geq g_1, \\ 0.3, & \text{if } g_1 \odot g_1 \leq g \not\geq g_2, \\ 0.1, & \text{if } g_1 \odot g_2 \leq g \not\geq g_1 \odot g_1, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we have $\phi^{\Leftarrow}(\mathcal{G}_1 \odot \mathcal{G}_2) = \phi^{\Leftarrow}(\mathcal{G}_1) \odot \phi^{\Leftarrow}(\mathcal{G}_2)$ as follows:

$$\phi^{\Leftarrow}(\mathcal{G}_1 \odot \mathcal{G}_2)(f) = \begin{cases} 1, & \text{if } f = 1_X, \\ 0.6, & \text{if } \phi^{\leftarrow}(g_1) \leq f \neq 1_X, \\ 0.5, & \text{if } \phi^{\leftarrow}(g_2) \leq f \not\geq \phi^{\leftarrow}(g_1), \\ 0.3, & \text{if } \phi^{\leftarrow}(g_1) \odot \phi^{\leftarrow}(g_1) \leq f \not\geq \phi^{\leftarrow}(g_2), \\ 0.1, & \text{if } \phi^{\leftarrow}(g_2) \odot \phi^{\leftarrow}(g_1) \leq f \not\geq \phi^{\leftarrow}(g_1) \odot \phi^{\leftarrow}(g_1), \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $* = \wedge$, then we obtain $\mathcal{G}_1 \wedge \mathcal{G}_2$ as follows:

$$\mathcal{G}_1 \wedge \mathcal{G}_2(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.6, & \text{if } g_1 \leq g \neq 1_X, \\ 0.5, & \text{if } g_1 \odot g_2 \leq g \not\geq g_1, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we have $\phi^{\Leftarrow}(\mathcal{G}_1 \wedge \mathcal{G}_2) = \phi^{\Leftarrow}(\mathcal{G}_1) \wedge \phi^{\Leftarrow}(\mathcal{G}_2)$ as follows:

$$\phi^{\Leftarrow}(\mathcal{G}_1 \odot \mathcal{G}_2)(f) = \begin{cases} 1, & \text{if } f = 1_X, \\ 0.6, & \text{if } \phi^{\leftarrow}(g_1) \leq f \neq 1_X, \\ 0.5, & \text{if } \phi^{\leftarrow}(g_2) \odot \phi^{\leftarrow}(g_1) \leq f \not\geq \phi^{\leftarrow}(g_1), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.4 *Let $\phi : X \to Y$ be a function, $\tau_X^i \in T_\odot(X)$ and $\tau_Y^i \in T_\odot(Y)$ for $i = \{1, 2\}$. Then we have the following properties.*
(1) $\phi^{\Rightarrow}(\tau_X^1) * \phi^{\Rightarrow}(\tau_X^2) \leq \phi^{\Rightarrow}(\tau_X^1 * \tau_X^2)$ *with equality if ϕ is bijective.*
(2) $\phi^{\Leftarrow}(\tau_Y^1 * \tau_Y^2) = \phi^{\Leftarrow}(\tau_Y^1) * \phi^{\Leftarrow}(\tau_Y^2)$.

Proof. (1) Let ϕ be a bijective function. Suppose there exists $h \in L^Y$ such that

$$(\phi^{\Rightarrow}(\tau_X^1) * \phi^{\Rightarrow}(\tau_X^2))(h) \not\geq \phi^{\Rightarrow}(\tau_X^1 * \tau_X^2)(h).$$

By the definition of $\phi^{\Rightarrow}(\tau_X^1 * \tau_X^2)$, there exist $f_1, f_2 \in L^X$ with $f_1 \odot f_2 = \phi^{\leftarrow}(h)$ such that

$$\phi^{\Rightarrow}(\tau_X^1) * \phi^{\Rightarrow}(\tau_X^2)(h) \not\geq \tau_X^1(f_1) * \tau_X^2(f_2).$$

Since ϕ is a bijective function, $\phi^{\rightarrow}(f_1) \odot \phi^{\rightarrow}(f_2) = \phi^{\rightarrow}(f_1 \odot f_2) = \phi^{\rightarrow}(\phi^{\leftarrow}(h)) = h$. Thus,

$$(\phi^{\Rightarrow}(\tau_X^1) * \phi^{\Rightarrow}(\tau_X^2))(h) \geq \phi^{\Rightarrow}(\tau_X^1)(\phi^{\rightarrow}(f_1)) * \phi^{\Rightarrow}(\tau_X^2)(\phi^{\rightarrow}(f_2)) \geq \tau_X^1(f_1) * \tau_X^2(f_2).$$

It is a contradiction. Hence $\phi^{\Rightarrow}(\tau_X^1) * \phi^{\Rightarrow}(\tau_X^2) \geq \phi^{\Rightarrow}(\tau_X^1 * \tau_X^2)$.

Theorem 3.5 *Let (X, τ_1) and (Y, τ_2) be $(L, \odot)$-fuzzy topological spaces. Let $\phi : X \to Y$ be a map. Let $\mathcal{N}_{\tau_1}^x$ and $\mathcal{N}_{\tau_2}^{\phi(x)}$ be $(\mathcal{F}^x, *)$-and $(\mathcal{F}^{\phi(x)}, *)$-neighborhood filters, respectively. For each $(L, \odot)$-filter $\mathcal{F} \in L^{L^X}$, $x \in X$ and $h \in L^Y$, we have the following statements:*
(1)

$$\begin{aligned} &(\tau_2(h) \to \tau_1(\phi^{\leftarrow}(h))) * (\mathcal{F}^{\phi(x)}(h) \to \mathcal{F}^x(\phi^{\leftarrow}(h))) \\ &\leq \mathcal{N}_{\tau_2}^{\phi(x)}(h) \to \mathcal{N}_{\tau_1}^x(\phi^{\leftarrow}(h)) \\ &\leq \Big(\mathcal{N}_{\tau_1}^x(\phi^{\leftarrow}(h)) \Rightarrow \mathcal{F}(\phi^{\leftarrow}(h))\Big) \Rightarrow \Big(\mathcal{N}_{\tau_2}^{\phi(x)}(h) \to \phi^{\Rightarrow}(\mathcal{F})(h)\Big) \end{aligned}$$

(2)

$$\begin{aligned}&(\tau_2(h) \to \tau_1(\phi^{\leftarrow}(h))) * (\mathcal{F}^{\phi(x)}(h) \to \mathcal{F}^x(\phi^{\leftarrow}(h)))\\&\le \mathcal{N}^{\phi(x)}_{\tau_2}(h) \to \mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(h))\\&\le \Big(\mathcal{N}^{\phi(x)}_{\tau_2}(h) * \phi^{\Rightarrow}(\mathcal{F})(g)\Big) \to \Big(\mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(h)) * \mathcal{F}(\phi^{\leftarrow}(g))\Big)\end{aligned}$$

(3) *If* $\phi : (X, \tau_1) \to (Y, \tau_2)$ *is LF-continuous and* $\phi : (X, \mathcal{F}^x) \to (Y, \mathcal{F}^{\phi(x)})$ *is an L-filter map for each* $x \in X$*, then* $\phi : (X, \mathcal{N}^x_{\tau_1}) \to (Y, \mathcal{N}^{\phi(x)}_{\tau_2})$ *is an L-filter map for each* $x \in X$.

(4) *Let* $\phi : (X, \tau_1) \to (Y, \tau_2)$ *be LF-continuous and* $\phi : (X, \mathcal{F}^x) \to (Y, \mathcal{F}^{\phi(x)})$ *an L-filter map for each* $x \in X$*. If* $\mathcal{F} \to x(\mathcal{F}^x, *)$*, then* $\phi^{\Rightarrow}(\mathcal{F}) \to \phi(x)(\mathcal{F}^{\phi(x)}, *)$*. Furthermore, we have* $\phi(lim_{\tau_1}(\mathcal{F})(\mathcal{F}^x, *)) \le lim_{\tau_2}(\phi^{\Rightarrow}(\mathcal{F}))(\mathcal{F}^{\phi(x)}, *)$.

(5) *Let* $\phi : (X, \tau_1) \to (Y, \tau_2)$ *be LF-continuous and* $\phi : (X, \mathcal{F}^x) \to (Y, \mathcal{F}^{\phi(x)})$ *an L-filter map for each* $x \in X$*. If* $\mathcal{F} \propto x(\mathcal{F}^x, *)$*, then* $\phi^{\Rightarrow}(\mathcal{F}) \propto \phi(x)(\mathcal{F}^{\phi(x)}, *)$*. Furthermore, we have* $\phi(clu_{\tau_1}(\mathcal{F})(\mathcal{F}^x, *)) \le clu_{\tau_2}(\phi^{\Rightarrow}(\mathcal{F}))(\mathcal{F}^{\phi(x)}, *)$.

Proof. (1) By Theorem 2.11 and Lemma 2.8(7), we have:

$$\begin{aligned}&\mathcal{N}^{\phi(x)}_{\tau_2}(h) \to \mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(h))\\&\le \Big(\mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(h)) \Rightarrow \mathcal{F}(\phi^{\leftarrow}(h))\Big) \Rightarrow \Big(\mathcal{N}^{\phi(x)}_{\tau_2}(h) \to \phi^{\Rightarrow}(\mathcal{F})(h))\Big)\end{aligned}$$

(2) It follows from Lemma 2.8(6).

(3) Since $\mathcal{N}^{\phi(x)}_{\tau_2}(h) \to \mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(h)) = \top$, we have $\mathcal{N}^{\phi(x)}_{\tau_2}(h) \le \mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(h))$.

(4) Let $\mathcal{F}$ be an $(L, \odot)$-filter and $x \in X$ such that $\mathcal{F} \to x(\mathcal{F}^x, *)$. Since $\mathcal{F} \to x(\mathcal{F}^x, *)$, we have $\mathcal{N}^x_{\tau_1} \le \mathcal{F}$. Since $\mathcal{N}^{\phi(x)}_{\tau_2}(g) \le \mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(g))$ from (1), we have

$$\mathcal{N}^{\phi(x)}_{\tau_2}(g) \le \mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(g)) \le \mathcal{F}(\phi^{\leftarrow}(g)) = \phi^{\Rightarrow}(\mathcal{F})(g).$$

Thus, $\phi^{\Rightarrow}(\mathcal{F}) \to \phi(x)(\mathcal{F}^{\phi(x)}, *)$.

(5) Let $\mathcal{N}^{\phi(x)}_{\tau_2}(h) * \phi^{\Rightarrow}(\mathcal{F})(g) \ne \bot$. By (2), since

$$\top = \Big(\mathcal{N}^{\phi(x)}_{\tau_2}(h) * \phi^{\Rightarrow}(\mathcal{F})(g)\Big) \to \Big(\mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(h)) * \mathcal{F}(\phi^{\leftarrow}(g))\Big),$$

we have

$$\mathcal{N}^x_{\tau_1}(\phi^{\leftarrow}(h)) * \mathcal{F}(\phi^{\leftarrow}(g)) \ge \mathcal{N}^{\phi(x)}_{\tau_2}(h) * \phi^{\Rightarrow}(\mathcal{F})(g) \ne \bot.$$

Since $\mathcal{F} \propto x(\mathcal{F}^x, *)$, we have $\phi^{\leftarrow}(h) \odot \phi^{\leftarrow}(g) \ne \bot$. Thus $h \odot g \ne \bot$.

Example 3.6 Let $X = \{x, y\}$ be a set and $f_1(x) = 0.6, f_1(y) = 0.5$. Let $(L = [0, 1], \odot = *)$ be stsc-quantales where $a \odot b = a * b = (a + b - 1) \vee 0$. Then

$\odot$ dominates $\odot$. We define $([0,1],\odot)$-fuzzy topologies $\tau_1, \tau_2 : [0,1]^X \to [0,1]$ as follows:

$$\tau_1(f) = \begin{cases} 1, & \text{if } f = \overline{0} \text{ or } \overline{1}, \\ 0.6, & \text{if } f = f_1, \\ 0.3, & \text{if } f = f_1 \odot f_1, \\ 0, & \text{otherwise,} \end{cases} \quad \tau_2(f) = \begin{cases} 1, & \text{if } f = \overline{0} \text{ or } \overline{1}, \\ 0.6, & \text{if } f = f_1, \\ 0.5, & \text{if } f = f_1 \odot f_1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Since $\mathcal{N}^x_{\tau_i}(f) = \bigvee_{g \le f}([x](g) \odot \tau_i(g))$ for $i \in \{1,2\}$, we obtain $([x],\odot)$-neighborhood filters $\mathcal{N}^x_{\tau_1} = \mathcal{N}^x_{\tau_2}, \mathcal{N}^y_{\tau_1} = \mathcal{N}^y_{\tau_2} : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{N}^x_{\tau_1}(f) = \begin{cases} 1, & \text{if } f = \overline{1}, \\ 0.2, & \text{if } f_1 \le f \ne \overline{1}, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{N}^y_{\tau_1}(f) = \begin{cases} 1, & \text{if } f = \overline{1}, \\ 0.1, & \text{if } f_1 \le f \ne \overline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

An identity map $id_X : (X,\tau_1) \to (X,\tau_2)$ is not LF-continuous because $0.5 = \tau_2(f_1 \odot f_1) \not\le \tau_1(f_1 \odot f_1) = 0.3$. We have the following properties:

(A) Since $\mathcal{N}^x_{\tau_2} = \mathcal{N}^x_{\tau_1}$, $\mathcal{N}^y_{\tau_2} = \mathcal{N}^y_{\tau_1}$ for each $x, y \in X$, then $id_X : (X, \mathcal{N}^x_{\tau_1}) \to (X, \mathcal{N}^x_{\tau_2})$ is an L-filter map for each $x \in X$.

(B) $\mathcal{F} \infty x([x],\odot)$ iff $id_X(\mathcal{F}) = \mathcal{F} \infty x([x],\odot)$ for each $x \in X$.

(C) $\mathcal{F} \to x([x],\odot)$ iff $id_X(\mathcal{F}) = \mathcal{F} \to x([x],\odot)$ for each $x \in X$.

(D) $lim_{\tau_1}(\mathcal{F})([x],\odot) = lim_{\tau_2}(\mathcal{F})([x],\odot)$ and $clu_{\tau_1}(\mathcal{F})([x],\odot) = clu_{\tau_2}(\mathcal{F})([x],\odot)$.

(2) Since $\mathcal{N}^x_{\tau_i}(f) = \bigvee_{g \le f}(\inf(g) \odot \tau_i(g))$ for $i \in \{1,2\}$, we obtain $(\inf,\odot)$-neighborhood filters $\mathcal{N}^x_{\tau_1} = \mathcal{N}^x_{\tau_2} = \mathcal{N}^y_{\tau_1} = \mathcal{N}^y_{\tau_2} : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{N}^y_{\tau_1}(x) = \begin{cases} 1, & \text{if } f = \overline{1}, \\ 0.1, & \text{if } f_1 \le f \ne \overline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

We have the following properties:

(E) Since $\mathcal{N}^x_{\tau_2} = \mathcal{N}^x_{\tau_1} = \mathcal{N}^y_{\tau_2} = \mathcal{N}^y_{\tau_1}$ for each $x, y \in X$, then $id_X : (X, \mathcal{N}^x_{\tau_1}) \to (X, \mathcal{N}^x_{\tau_2})$ is an L-filter map for each $x \in X$.

(F) $\mathcal{F} \infty x(\inf,\odot)$ iff $id_X(\mathcal{F}) = \mathcal{F} \infty x(\inf,\odot)$ for each $x \in X$.

(G) $\mathcal{F} \to x(\inf,\odot)$ iff $id_X(\mathcal{F}) = \mathcal{F} \to x(\inf,\odot)$ for each $x \in X$.

(H) $lim_{\tau_1}(\mathcal{F})(\inf,\odot) = lim_{\tau_2}(\mathcal{F})(\inf,\odot)$ and $clu_{\tau_1}(\mathcal{F})(\inf,\odot) = clu_{\tau_2}(\mathcal{F})(\inf,\odot)$.

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Received: December, 2010