

# On the sequence related to Lucas numbers and its properties

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## Abstract

The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. In this article, we study a new generalization  $\{L_{k,n}\}$ , with initial conditions  $L_{k,0} = 2$  and  $L_{k,1} = 1$ , which is generated by the recurrence relation  $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$  for  $n \geq 2$ , where  $k$  is integer number. Some well-known sequence are special case of this generalization. The Lucas sequence is a special case of  $\{L_{k,n}\}$  with  $k = 1$ . Modified Pell-Lucas sequence is  $\{L_{k,n}\}$  with  $k = 2$ . We produce an extended Binet's formula for  $\{L_{k,n}\}$  and, thereby, identities such as Cassini's, Catalan's, d'Ocagne's, etc. using matrix algebra. Moreover, we present sum formulas concerning this new generalization.

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## 1 Introduction

In recent years, many interesting properties of classic Fibonacci numbers, classic Lucas numbers and their generalizations have been shown by researchers and applied to almost every field of science and art. For the rich and related applications of these numbers, one can refer to the nature and different areas of the science [3-11]. The classic Fibonacci  $\{F_n\}_{n \in \mathbb{N}}$  and Lucas  $\{L_n\}_{n \in \mathbb{N}}$  sequences are defined as, respectively,

$$F_0 = 0, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

and

$$L_0 = 2, L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2$$

where  $F_n$  and  $L_n$ , respectively, denotes the  $n$ th classic Fibonacci and Lucas numbers. Besides of the usual Fibonacci and Lucas numbers, many kinds of generalizations of these numbers have been presented in the literature [3, 8, 9, 11]. In [3], the  $k$ -Fibonacci sequence, say  $\{F_{k,n}\}_{n \in \mathbb{N}}$ , has been found by studying the recursive applications of two geometrical transformations used in the well-known four-triangle longest-edge(4TLE) partition and is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad 1 \leq n, k \in \mathbb{Z}$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1.$$

In that paper, many properties of these numbers have been obtained directly from elementary matrix algebra. Many properties of these numbers have been deduced and related with the so-called Pascal 2-triangle [4]. Additionally the authors of [5] defined  $k$ -Fibonacci hyperbolic functions as similar to hyperbolic functions and Fibonacci hyperbolic functions. In [6], authors studied 3-dimensional  $k$ -Fibonacci spirals from a geometric point of view.  $m$ -extension of the Fibonacci and Lucas  $p$ - numbers are defined in [8]. Afterwards, the continuous functions for the  $m$ -extension of the Fibonacci and Lucas  $p$ -numbers using the generalized Binet formulas have been obtained in that paper. The generating matrix, the Binet like formulas, applications to the coding theory and the generalized Cassini formula, i.e., of the Fibonacci  $p$ -numbers are given by Stakhov [10]. Stakhov and Rozin [9] showed that the formulas are similar to the Binet formulas given for the classical Fibonacci numbers, also defined to be of generalized Fibonacci and Lucas numbers or Fibonacci and Lucas  $p$ -numbers. As a similar study, in [10], it has been introduced the new continuous functions for the Fibonacci and Lucas  $p$ -numbers using Binet formulas. In [11], Civciv and Türkmen defined a new matrix generalization of the Fibonacci and Lucas numbers using essentially a matrix approach.

In this work, we define a new generalization of the classic Lucas sequence and give identities and sum formulas concerning this new generalization.

## 2 Main Results

Now, a new generalization of the classical Lucas sequence that its recurrence formula is depended on one parameter is introduced and some particular cases of this sequence are given.

**Definition 1** For any integer number  $k \geq 1$ , the  $k$ th Lucas sequence, say  $\{L_{k,n}\}_{n \in \mathbb{N}}$ , is defined by

$$L_{k,0} = 2, L_{k,1} = 1 \text{ and } L_{k,n} = kL_{k,n-1} + L_{k,n-2} \text{ for } n \geq 2. \quad (1)$$

The following table summarizes some special cases of  $n$ th  $k$ -Lucas numbers  $L_{k,n}$  :

$k$	$L_{k,n}$
1	Lucas numbers
2	Modified Pell-Lucas numbers

In [ 3 ], many properties of  $k$ - Fibonacci numbers from  $M = \begin{bmatrix} k-1 & 1 \\ k & 1 \end{bmatrix}$  using matrix algebra are obtained. Matrix methods are very useful tools to solve many problems for stemming from number theory. Now we will obtain some algebraic properties of  $k$ -Lucas numbers via the matrix  $M$ .

The follow proposition gives that the elements of the first row of  $n$ th power of  $M$  are  $k$ -Fibonacci and Lucas numbers.

**Proposition 1** Let  $M = \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix}$ . For any integer  $n \geq 1$  holds:

$$M^n = \begin{pmatrix} L_{k,n+1} - 3F_{k,n} & L_{k,n} - 2F_{k,n-1} \\ * & * \end{pmatrix}.$$

**Proof.** By induction: for  $n = 1$ :

$$M = \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix} = \begin{pmatrix} L_{k,2} - 3F_{k,1} & L_{k,1} - 2F_{k,0} \\ * & * \end{pmatrix}$$

since  $F_{k,0} = 0, F_{k,1} = 1, L_{k,1} = 1$  and  $L_{k,2} = k + 2$ . Let us suppose that the formula is true for  $n - 1$  :

$$M^{n-1} = \begin{pmatrix} L_{k,n} - 3F_{k,n-1} & L_{k,n-1} - 2F_{k,n-2} \\ * & * \end{pmatrix}.$$





















