Mathematica Aeterna, Vol. 2, 2012, no. 2, 89 - 100

Common Fixed Point Theorems for two mappings in D^*-Metric Spaces

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Abstract

In this paper, we present some new definitions of D^*-metric spaces and prove a common fixed point theorem for two mappings under the condition of weakly compatible mappings in complete D^*-metric spaces. Also we improved some fixed point theorems in complete D^*-metric spaces.

Mathematics Subject Classification: 54E40; 54E35; 54H25

Keywords: D^*-metric contractive mapping; Complete D^*-metric space; Common fixed point theorem.

1 Introduction

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [20], Jungck introduced more generalized commuting mappings, called *compatible* mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems (see, e.g.,([2, 3, 4, 5, 10, 12, 13, 21, 24, 25, 28]).

Dhage [6] introduced the notion of generalized metric or D-metric spaces and claimed that D-metrics defines a Hausdorff topology and that D-metric is sequentially continuous in all the three variables. Many authors used these claims for proving some fixed point theorems in D-metric spaces. Rhoades [20] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma [27] introduced the concept of D-compatibility of maps in D-metric space and proved some fixed point theorems using a contractive condition. Unfortunately, almost all theorems in D-metric spaces are not valid (see [17, 18, 19]). In this paper, we introduce D^*-metric which is a modification of the definition of D-metric introduced by Dhage [6] and prove some basic properties in D^*-metric spaces.

In this paper, (X, D^*) will denote a D^*-metric space, N the set of all natural numbers, and R^+ the set of all positive real numbers.

Definition 1.1 *Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function: $D^* : X^3 \longrightarrow R^+$ that satisfies the following conditions for each $x, y, z, a \in X$.*

(1) $D^*(x, y, z) \geq 0$,

(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,

(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$,(symmetry) where p is a permutation function,

(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a generalized metric (or D^*-metric) space.

Some examples of such a function are

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = R^n$ then we define

$$D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in R^+$.

(d) Let $X = R^+$. Define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise }, \end{cases}$$

(e) If $X = R$ then we define

$$D^*(x, y, z) = |x + y - 2z| + |y + z - 2x| + |z + x - 2y|$$

for every $x, y, z \in R$.

(f) If $X = R$ then we define

$$D^*(x, y, z) = |x + 2y - 3z| + |y + 2z - 3x| + |z + 2x - 3y|$$

for every $x, y, z \in R$.

Lemma 1.2 *Let (X, D^*) be a D^*-metric space. Then $D^*(x, x, y) = D^*(x, y, y)$.*

Proof. Form triangular inequality (4) we have that
(i) $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$
and similarly
(ii) $D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$.
(i),(ii) imply that $D^*(x, x, y) = D^*(x, y, y)$.

Let (X, D^*) be a D^*-metric space. We define the *open ball* $B_{D^*}(x, r)$ with center $x \in X$ and radius $r > 0$ as

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}.$$

Example 1.3 *Let $X = R$. Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in R$. Thus*

$$\begin{aligned} B_{D^*}(1, 2) &= \{y \in R : D^*(1, y, y) < 2\} \\ &= \{y \in R : |y - 1| + |y - 1| < 2\} \\ &= \{y \in R : |y - 1| < 1\} = (0, 2). \end{aligned}$$

Definition 1.4 *Let (X, D^*) be a D^*-metric space and $A \subset X$.*

(1) If for every $x \in A$ there exist $r > 0$ such that $B_{D^*}(x, r) \subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be D^*-bounded if there exists $r > 0$ such that $D^*(x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exist $n_0 \in N$ such that for each $n \geq n_0$ we have that

$$D^*(x, x_n, x_n) = D^*(x, x, x_n) < \epsilon. \quad (*)$$

This is equivalent with, for each $\epsilon > 0$ there exist $n_0 \in N$ such that for each $n, m \geq n_0$ we have that

$$D^*(x, x_n, x_m) < \epsilon. \quad (**)$$

Indeed, from $(*)$ we conclude that

$$D^*(x_n, x_m, x) = D^*(x_n, x, x_m) \leq D^*(x_n, x, x) + D^*(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Conversely, set $m = n$ in $(**)$ we have $D^*(x_n, x_n, x) < \epsilon$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exits $n_0 \in N$ such that $D^*(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The D^*-metric space (X, D^*) is said to be complete if every Cauchy sequence is convergent.

2 Preliminary Notes

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $r > 0$ such that $B_{D^*}(x, r) \subseteq A$. Let τ_{D^*} denote the family of all open subsets of X. Then τ_{D^*} is called the *topology induced* by the D^* metric.

Lemma 2.1 *Let (X, D^*) be a D^*-metric space. If $r > 0$, then ball $B_{D^*}(x, r)$ with center $x \in X$ and radius r is open set.*

Proof. Let $z \in B_{D^*}(x, r)$, hence $D^*(x, z, z) < r$. If set $D^*(x, z, z) = \delta$ and $r' = r - \delta$ then we prove that $B_{D^*}(z, r') \subseteq B_{D^*}(x, r)$. Let $y \in B_{D^*}(z, r')$, by triangular inequality we have $D^*(x, y, y) = D^*(y, y, x) \leq D^*(y, y, z) + D^*(z, x, x) < r' + \delta = r$. Hence $B_{D^*}(z, r') \subseteq B_{D^*}(x, r)$. That is ball $B_{D^*}(x, r)$ is open ball.

Definition 2.2 *Let (X, D^*) be a D^*- metric space. D^* is said to be continuous function on X^3 if*

$$\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z).$$

Whenever a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$ i.e.

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$$

Lemma 2.3 *Let (X, D^*) be a D^*- metric space. Then D^* is continuous function on X^3.*

Proof. Let $\{(x_n, y_n, z_n)\} \in X^3$ converges to a point $(x, y, z) \in X^3$ i.e.

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z.$$

Then for each $\epsilon > 0$ there exist $n_1, n_2, n_3 \in N$ such that for every $n \geq n_1$ we have $D^*(x, x, x_n) < \frac{\epsilon}{3}$, for every $n \geq n_2$ we have $D^*(y, y, y_n) < \frac{\epsilon}{3}$ and for every $n \geq n_3$ we have $D^*(z, z, z_n) < \frac{\epsilon}{3}$.
If set $n_0 = \max\{n_1, n_2, n_3\}$, then for every $n \geq n_0$ by triangular inequality we have

$$\begin{aligned} D^*(x_n, y_n, z_n) &\leq D^*(x_n, y_n, z) + D^*(z, z_n, z_n) \leq D^*(x_n, z, y) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\ &\leq D^*(z, y, x) + D^*(x, x_n, x_n) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\ &< D^*(x, y, z) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x, y, z) + \epsilon. \end{aligned}$$

Hence we have

$$D^*(x_n, y_n, z_n) - D^*(x, y, z) < \epsilon$$

and

$$\begin{aligned} D^*(x,y,z) &\le D^*(x,y,z_n)+D^*(z_n,z,z) \le D^*(x,z_n,y_n)+D^*(y_n,y,y)+D^*(z_n,z,z) \\ &\le D^*(z_n,y_n,x_n)+D^*(x_n,x,x)+D^*(y_n,y,y)+D^*(z_n,z,z) \\ &< D^*(x_n,y_n,z_n)+\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = D^*(x_n,y_n,z_n)+\epsilon. \end{aligned}$$

Thus,

$$D^*(x,y,z) - D^*(x_n,y_n,z_n) < \epsilon.$$

Therefore we have $|D^*(x_n,y_n,z_n) - D^*(x,y,z)| < \epsilon$, that is

$$\lim_{n\to\infty} D^*(x_n,y_n,z_n) = D^*(x,y,z)$$

Lemma 2.4 *Let (X,D^*) be a D^*-metric space. If the sequence $\{x_n\}$ in X converges to x,then x is unique.*

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist $n_1, n_2 \in N$ such that for every $n \ge n_1$ we have $D^*(x,x,x_n) < \frac{\epsilon}{2}$ and for every $n \ge n_2$ we have $D^*(y,y,x_n) < \frac{\epsilon}{2}$. If $n_0 = \max\{n_1,n_2\}$, then for every $n \ge n_0$ we have

$$D^*(x,x,y) \le D^*(x,x,x_n) + D^*(x_n,y,y) < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \varepsilon.$$

Hence $D^*(x,x,y) = 0$ is a contradiction. Thus, $x = y$.

Lemma 2.5 *Let (X,D^*) be a D^*-metric space. If the sequence $\{x_n\}$ in X is converges to x, then the sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $x_n \longrightarrow x$, for each $\epsilon > 0$ there exist $n_1, n_2 \in N$ such that for every $n \ge n_1$ we have $D^*(x_n,x_n,x) < \frac{\epsilon}{2}$ and for every $m \ge n_2$ we have $D^*(x,x_m,x_m) < \frac{\epsilon}{2}$. If $n_0 = \max\{n_1,n_2\}$, then for every $n,m \ge n_0$ we have
$D^*(x_n,x_n,x_m) \le D^*(x_n,x_n,x)+D^*(x,x_m,x_m) < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$.Hence sequence $\{x_n\}$ is a Cauchy sequence.

In 1998, Jungck and Rhoades [12] introduced the following concept of weak compatibility.

Definition 2.6 *Let A and S be mappings from a D^*-metric space (X,D^*) into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.*

Let (X,D^*) be a D^*-metric space, for $A,B,C \subseteq X$, define

$$\delta_{D^*}(A,B,C) = \sup\{D^*(a,b,c); a \in A, b \in B, c \in C\}.$$

If A consists of a single point a, we write $\delta_{D^*}(A,B,C) = \delta_{D^*}(a,B,C)$. If B and C also consists of a single point b and c respectively, we write $\delta_{D^*}(A,B,C) = D^*(a,b,c)$.

It follows immediately from the definition that

$$\begin{aligned} \delta_{D^*}(A,B,C) &= 0 \Longleftrightarrow A = B = C = \{a\}, \\ \delta_{D^*}(A,B,C) &= \delta_{D^*}(p\{A,B,C\}) \geq 0, \end{aligned}$$

(symmetry) where p is a permutation function, for all $A,B,C \subseteq X$. In particular for $\emptyset \neq A = B = C \subset X$,

$$\delta_{D^*}(A) = \sup\{D^*(a,b,c); a,b,c \in A\}.$$

It follows immediately from the definition that:

If $A \subseteq B$, then $\delta_{D^*}(A) \leq \delta_{D^*}(B)$.

Let $a_n = \delta_{D^*}(A_n)$ for $n \in N$ in which $A_n = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$ in D^*-metric space (X, D^*). Then

(1) since $A_n \supseteq A_{n+1}$, $a_n \leq a_{n+1}$,

(2) $D^*(x_l, x_m, x_k) \leq \delta_{D^*}(A_n) = a_n$ for every $l,m,k \geq n$,

(3) $0 \leq \delta_{D^*}(A_n) = a_n$ and $a_{n+1} \leq a_n$ for every $n \geq 1$.

Therefore, $\{a_n\}$ is decreasing and bounded for all $n \in N$, and so there exists an $0 \leq a$ such that $\lim_{n\to\infty} a_n = a$.

Lemma 2.7 *By above conditions let (X, D^*) be a D^*-metric space. If $\lim_{n\to\infty} a_n = 0$, then the sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $\lim_{n\to\infty} a_n = 0$. Thus for every $\epsilon > 0$, there exists a $n_0 \in N$ such that for every $n > n_0$, we have $|a_n - 0| < \epsilon$. That is $a_n = \delta_{D^*}(A_n) < \epsilon$. Then for $l,m,k \geq n > n_0$ we have

$$D^*(x_l, x_m, x_k) \leq \sup\{D^*(x_i, x_j, x_p) \mid x_i, x_j, x_p \in A_n\} = a_n < \epsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X.

3 Main Results

Theorem 3.1 *Let f and g be self-mappings of a complete D^*-metric space (X, D^*) satisfying the following conditions:*

(i) *$g(X) \subseteq f(X)$, and $f(X)$ is closed subset of X,*

(ii) *the pair (f,g) is weakly compatible,*

(iii) *$D^*(gx, gy, gz) \leq \phi(D^*(fx, fy, fz))$, for every $x,y,z \in X$,*

where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a nondecreasing continuous function with $\phi(t) < t$ for every $t > 0$.

Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (i), we can choose a point x_1 in X such that $y_0 = gx_0 = fx_1$ and $y_1 = gx_1 = fx_2$. There exists a sequence $\{y_n\}$ such that, $y_n = gx_n = fx_{n+1}$, for $n = 0, 1, 2, \cdots$. We prove that the sequence $\{y_n\}$ is a Cauchy sequence. Let $A_n = \{y_n, y_{n+1}, y_{n+2}, \cdots\}$ and $a_n = \delta_{D^*}(A_n)$, $n \in N$, then $\lim_{n\to\infty} a_n = a$ for some $a \geq 0$.

Put $x = x_{n+k}, y = x_{m+k}, z = x_{l+k}$ in (iii) for $k \geq 1$ and $m, n, l \geq 0$, we have

$$\begin{aligned} D^*(y_{n+k}, y_{m+k}, y_{l+k}) &= D^*(gx_{n+k}, gx_{m+k}, gx_{l+k}) \\ &\leq \phi(D^*(fx_{n+k}, fx_{m+k}, fx_{l+k})) \\ &= \phi(D^*(y_{n+k-1}, y_{m+k-1}, y_{l+k-1})). \end{aligned}$$

Since $D^*(y_{n+k-1}, y_{m+k-1}, y_{l+k-1}) \leq a_{k-1}$, for every $n, m, l \geq 0$ and ϕ is increasing in t, we get

$$D^*(y_{n+k}, y_{m+k}, y_{m+k}) \leq \phi(D^*(y_{n+k-1}, y_{m+k-1}, y_{l+k-1})).$$

Hence

$$\sup_{m,n,l\geq 0} \{D^*(y_{n+k}, y_{m+k}, y_{l+k}) \leq \phi(a_{k-1}).$$

Therefore, we have $a_k \leq \phi(a_{k-1})$. Letting $k \to \infty$, we get $a \leq \phi(a)$. If $a \neq 0$, then $a \leq \phi(a) < a$, which is a contradiction. Thus $a = 0$ and hence $\lim_{n\to\infty} a_n = 0$. Thus by Lemma 2.7 $\{y_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists a $u \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n+1} = u.$$

Let $f(X)$ is closed, there exist $v \in X$ such that $fv = u$. Now we show that $gv = u$. From inequality (iii) we have that

$$D^*(gx_n, gx_n, gv) \leq \phi(D^*(fx_n, fx_n, fv)).$$

Taking $n \longrightarrow \infty$, we get

$$D^*(u, u, gv) \leq \phi(D^*(0) = 0,$$

it implies $gv = u$.

Since the pair (f, g) are weakly compatible, hence we get, $gfv = fgv$. Thus $fu = gu$. exists Now we prove that $gu = u$. If set x_n, x_n, u replacing x, y, z respectively , in inequality (iii) we get

$$D^*(gx_n, gx_n, gu) \leq \phi(D^*(fx_n, fx_n, fu))$$

Taking $n \longrightarrow \infty$, we get

$$D^*(u,u,gu) \leq \phi(D^*(u,u,gu))$$

If $gu \neq u$, then $D^*((u,u,gu) < D^*(u,u,gu)$, is contradiction. Therefore,

$$fu = gu = u.$$

For the uniqueness, let u and u' be fixed points of f, g. Taking $x = y = u$ and $z = u'$ in (iii), we have

$$\begin{aligned} D^*(u,u,u') &= D^*(gu,gu,gu') \\ &\leq \phi(D^*(fu,fu,fu')) \\ &= \phi(D^*(u,u,u')) < D^*(u,u,u'), \end{aligned}$$

which is a contradiction. Thus we have $u = u'$.

Corollary 3.2 *Let f, g and h be self-mappings of a complete D^*-metric space (X, D^*) satisfying the following conditions:*
(i) *$g(X) \subseteq fh(X)$, and $fh(X)$ is closed subset of X,*
(ii) *the pair (fh, g) is weakly compatible and $fh = hf$, $gh = hg$*
(iii) *$D^*(gx,gy,gz) \leq \phi(D^*(fhx,fhy,fhz))$,*
for every $x, y, z \in X$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a nondecreasing continuous function with $\phi(t) < t$ for every $t > 0$.
Then f, g and h have a unique common fixed point in X.

Proof. By Theorem 3.1 there exist a fixed point $u \in X$ such that $fhu = gu = u$. Now, we prove that $hu = u$. If $hu \neq u$, then in (iii), we have

$$\begin{aligned} D^*(hu,u,u) &= D^*(hgu,gu,gu) \\ &= D^*(ghu,gu,gu) \\ &\leq \phi(D^*(fhhu,fhu,fhu)) = \phi(D^*(hu,u,u)) \\ &< D^*(hu,u,u), \end{aligned}$$

which is a contradiction. Thus we have $hu = u$. Therefore,

$$fu = fhu = u = hu = gu.$$

Corollary 3.3 *Let g be self-mapping of a complete D^*-metric space (X, D^*) satisfying the following condition:*

$$D^*(g^n x, g^n y, g^n z) \leq \phi(D^*(x,y,z)),$$

for every $x, y, z \in X$ and $n \in N$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a nondecreasing continuous function with $\phi(t) < t$ for every $t > 0$.
Then g have a unique common fixed point in X.

Proof. Replace f with I, the identity map, in Theorem 3.1. Hence the all conditions of Theorem 3.1 are hold and therefore there exists a unique $u \in X$ such that $g^n u = u$. Thus

$$g^n(gu) = g(g^n u) = gu.$$

Since u is unique, we have $gu = u$.

Corollary 3.4 *Let f and g be self-mappings of a complete D^*-metric space (X, D^*) satisfying the following condition:*
(i) *$g^n(X) \subseteq f^m(X)$, and $f^m(X)$ is closed subset of X,*
(ii) *the pair (f^m, g^n) is weakly compatible and $f^m g = g f^m$, $g^n f = f g^n$*
(iii) *$D^*(g^n x, g^n y, g^n z) \le \phi(D^*(f^m x, f^m y, f^m z))$,*
for every $x, y, z \in X$ and $n, m \in N$, where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a nondecreasing continuous function with $\phi(t) < t$ for every $t > 0$.
Then f and g have a unique common fixed point in X.

Proof. By Theorem 3.1 there exist a fixed point $u \in X$ such that $f^m u = g^n u = u$. On the other hand, we have

$$gu = g(g^n u) = g^n(gu) \ \ and \ \ gu = g(f^m u) = f^m(gu).$$

Since u is unique, we have $gu = u$. Similarly, we have $fu = u$.

Corollary 3.5 *Let (X, D^*) be a complete D^*-metric space and let $f_1, f_2, \cdots, f_n, g : X \longrightarrow X$ be maps that satisfy the following conditions:*

(a) $g(X) \subseteq f_1 f_2 \cdots f_n(X)$;

(b) the pair $(f_1 f_2 \cdots f_n, g)$ is weak compatible, $f_1 f_2 \cdots f_n(X)$ is closed subset of X;

(c) $D^*(gx, gy, gz) \le \phi(D^*(f_1 f_2 \cdots f_n(x), f_1 f_2 \cdots f_n(y), f_1 f_2 \cdots f_n(z)))$, *for all $x, y, z \in X$ and $n \in N$, where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a nondecreasing continuous function with $\phi(t) < t$ for every $t > 0$;*

(d) $g(f_2 \cdots f_n) = (f_2 \cdots f_n)g$,
$g(f_3 \cdots f_n) = (f_3 \cdots f_n)g$,
$\vdots$
$g f_n = f_n g$,
$f_1(f_2 \cdots f_n) = (f_2 \cdots f_n)f_1$,
$f_1 f_2(f_3 \cdots f_n) = (f_3 \cdots f_n)f_1 f_2$,
$\vdots$
$f_1 \cdots f_{n-1}(f_n) = (f_n)f_1 \cdots f_{n-1}$.

Then $f_1, f_2, \cdots, f_n, g$ have a unique common fixed point.

Proof. By Corollarly3.2, if set $f_1 f_2 \cdots f_n = f$ then f, g have a unique common fixed point in X. That is, there exists $x \in X$, such that $f_1 f_2 \cdots f_n(x) = g(x) = x$. We prove that $f_i(x) = x$, for $i = 1, 2, \cdots$. From (c), we have

$$D^*(g(f_2 \cdots f_n x), g(x), g(x)) \leq \phi(D^*(f_1 f_2 \cdots f_n(f_2 \cdots f_n x), f_1 f_2 \cdots f_n(x), f_1 f_2 \cdots f_n(x))).$$

By (d), we get

$$\begin{aligned} D^*(f_2 \cdots f_n x, x, x) &\leq \phi(D^*(f_2 \cdots f_n x, x, x)) \\ &< D^*(f_2 \cdots f_n x, x, x). \end{aligned}$$

Hence, $f_2 \cdots f_n(x) = x$. Thus , $f_1(x) = f_1 f_2 \cdots f_n(x) = x$.

Similarly, we have $f_2(x) = \cdots f_n(x) = x$.

Now, we give one example to validate Theorem 2.1.

Example 3.6 *Let (X, D^*) be a complete D^*-metric space, where $X = [0, 2]$ and*

$$D^*(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Define self-maps f and g on X as follows: $fx = \frac{x+1}{2}$ and $gx = \frac{x+5}{6}$, for all $x \in X$.

Let $\phi(t) = \frac{1}{2}t$. Then , we have

$$\begin{aligned} D^*(gx, gy, gz) &= \frac{1}{6}(|x - y| + |y - z| + |z - x|) \\ &\leq \frac{1}{4}(|x - y| + |y - z| + |x - z|) = \phi(D^*(fx, fy, fz). \end{aligned}$$

That is all conditions of Theorem 3.1 are holds and 1 is the unique common fixed point of f and g.

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Received: October, 2011