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Solutions to a partial differential equation involving the provision of the pool of retail loans

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Abstract

A partial differential equation, which represents the provision of the pool of retail loans, is investigated. Solutions for the equation are given explicitly under certain circumstances. The provisions covering expected losses of collateralized retail lending due to default are measured by using the option approach.

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1 Introduction

In recent years, a lot of world banks have faced difficulties for a multitude of reasons. The major cause of the serious banking problems is directly related to lax credit standards for borrowers, poor portfolio risk management, or a lack of attention to changes in economic or other circumstances that can yield a deterioration in the credit standing of a bank's counterparties. In order to overcome this situation, the Basel Committee on Banking Supervision [5] issued the sound practices that specifically addressed establishing an appropriate credit risk environment and maintaining an proper credit administration, measurement and monitoring process. These practices are also employed in conjunction with a system in place for determining the adequacy of provisions. The Basel Committee is responsible for proposing regulatory requirements, including provisioning requirements for internationally active banks (see Basel II in [6]). Many international bank supervisors require their banks' provisioning systems to be forward looking for dealing with future changes in economic conditions that could cause unfavourable effects on the banks' credit exposures. The level of provisions will be in some senses dependent of the banks' forecasts of collateral value and other macroeconomic assumptions, regardless of the current losses, defaults and restructuring in their loans. Some countries' banks are required to determine provisions with reference to the losses, defaults and restructuring that have already happened in their loans, according to the detailed regulations on loan classifications with minimum provisioning requirements. The rationale behind issuing detailed regulatory parameters could be to level the playing field or make bank regulations more easily enforceable. In fact, a central feature of provisioning systems is typically to refer to losses that have already been incurred or are anticipated with a high degree of confidence. Provisioning rules may be different for several reasons. One is whether provisioning requirements only rely on losses of visible and identifiable events, or a establishing provisions for expected losses. Another reason is how banks are expected to factor in the value of collateral. Specific provisioning requirements are usually designed for certain portfolio segments such as retail loans including residential mortgage loans and credit card lending. Therefore, investigating the provision problems is needed in many banks and other related areas.

Under the framework of Basel II [6], banks are allowed to use the standardized approach or the internal ratings-based(IRB) approach to calculate regulatory capital charges for their exposures, including those in their portfolios. Usually, the IRB calculation of risk-weighted assets for credit exposures depends on four basic risk components. (1) probability of default (PD), which measures the likelihood that the borrower will default over a given time horizon. (2) loss-given-default (LGD), which measures the portion of the exposures. (3) exposure-at-default (EAD), which measures the bank's exposure at the time of default. (4) effective maturity. The IRB risk-weight functions for different classes of retail exposures are classified as residential mortgages, revolving credit and other retail loans. The expected losses defined in Basel II are $PD \times LGD \times EAD$ [2], in which the PD and LGD are assumed to be independently and the time horizon of PD and LGD is defined to be one year. However, it is noted that defaults are likely to be clustered during the times of economic distress and LGD may be correlated with default rates.

The effects of the correlation between PD and LGD on credit risk measures have been investigated by many scholars. In Frye's paper [8], a structural model is derived by using a single index based on the state of the economy and an risky idiosyncratic risk factor. The correlations between PD and LGD resulting from joint dependence of borrowers' assets and of collateral value on the systematic risk factor are established. A strong positive correlation between default rates and LGD for corporate bonds is shown empirically in [9]. These results confirm that the economic cycle can produce a double misfortune involving greater-than-average default rate and poor-than-average recoveries. Considering the collateral value uncertainty to LGD, Jokivuolle and Peura [12] derive a model of risky debt in which collateral value is correlated with PD of a borrower. The numerical analysis in [12] demonstrates the importance of factors such as collateral value volatility and the correlation between collateral value and the borrowers's asset value for the estimation of credit risk quantity. Using the simulation analysis, Altman et al. [1] find strong evidence of a positive correlation between PD and LGD. They explore that letting the correlation between PD and LGD to zero(as is usual practice), rather than to its estimated value, leads a reduction in the value-at-risk of at least one quarter. In addition to the correlation between PD and LGD, the time horizon defined as one year in Basel II should be corrected to consider long-term lending such as residential mortgage loans.

There are many research works to investigate the models for measuring provisions of a pool of collateralised retail loans which have the same collateral type (e.g. residential properties) and broadly the same loan-to-value ratio. The models follow the contingent claim approach of pricing options developed by Black and Scholes [2]. Morton's paper [14] is also the pioneer work in the pricing of corporate bonds applying the contingent claim framework. He treats default risk equivalent to a European put option on a firm's asset value and the firm's liability is the option strike. To extend the Merton model, structural models with more complex and dynamic liability structures have been investigated by Black and Cox [3], Briys and de Varence [4], Collin and Goldstein [7], Hui, Lo and Tsang [10].

Recently, Hui et al.[11] derive a closed form (exact solution) formula from the model as a function of the collateral value and PD to measure the provision of a pool of collateralised retail loans. It is found that two stochastic variables are explicitly correlated in the provision model. In [11], the model parameters such as the volatility, correlation and drift of PD are time dependent in the derivation. It is pointed in [11] that the model fitting well with the data typically available for banks can be applied to measure provisions of retail lending secured by collateral.

The objective of our work is to study a partial differential equation model, which measures provisions of a pool of collateralised retail loans. Although the model we will investigate is the same with that discussed in [11], two closed form solutions different from those presented in [11] are found. We thus get a rather complete structures for the provision of the pool of retail loans.

2 Model for measuring provisions

In order to make our paper self-contained, we briefly state how the partial differential equation model which measures provisions of a pool of retail loans is established in [11]. It is assumed in [11] that the provision is equivalent to the option premium which is equal to the outstanding loan value minus collateral value multiplied by the PD of the borrowers in the pool. The collateral value is therefore one of the two variable. Another variable is PD. The pool is composed of loans with the same collateral type and the same loan to value ratio.

The PD's continuous stochastic movement, which is denoted by D, is driven by a mean-reversion lognormal diffusion process and satisfies the stochastic differential equation

$$\frac{dD}{D} = \kappa_D(t)[\ln \theta_D(t) - \ln D]dt + \sigma_D(t)dz_D. \tag{1}$$

The parameter $\theta_D(t)$ is the average of D. The parameter $\kappa_D(t)$ determines the speed of adjustment of PD toward its mean number $\theta_D(t)$. $\sigma_D(t)$ is the volatility of D and z_D is a standard Wiener process. The model parameters are time dependent.

Let V denote the collateral value securing the loans in the poor. V is assumed to follow a lognormal diffusion process

$$\frac{dV}{V} = \mu_V(t)dt + \sigma_V(t)dz_V, \tag{2}$$

where $\mu_V(t)$ is the rate of V, $\sigma_V(t)$ is the volatility. This process is considered to be valid for financial collateral such as equities and physical collateral such as real estate collateral. The Brownian motion increments dz_D and dz_V satisfy the following relation

$$dz_D dz_V = \rho(t)dt. \tag{3}$$

Applying the Ito's lemma to (1)-(3), we derive that $P(D, V, t)$ of the provision of the poor of retail loans satisfies the partial differential equation

$$\begin{aligned}\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_D^2(t)D^2\frac{\partial^2 P}{\partial D^2} + \frac{1}{2}\sigma V^2(t)V^2\frac{\partial^2 P}{\partial V^2} + \rho(t)\sigma D(t)\sigma V(t)DV\frac{\partial^2 P}{\partial D\partial V} \\ +[\kappa D(t)(\ln \theta_D(t) - \ln D)]D\frac{\partial P}{\partial D} + (r-q)V\frac{\partial P}{\partial V} - rP = 0,\end{aligned} \tag{4}$$

where r is the risk-free rate and q is the dividend rate of V. It is noted that physical collateral could be analogous to a stock providing a known dividend yield. The owner of the collateral may receive a yield equivalent to a "dividend yield". As the loans in the pool broadly possess the same loan-to-value ratio, the pool can be viewed as an aggregated loan. The final assumption imposed on the provision is thus specified as

$$P(D, V, T) = Dmax(L - V, 0), \tag{5}$$

where L is the outstanding amount of the loans in the poor and $\max(L - V, 0)$ is equivalent to the standard payoff of a put option. We transform Eq.(1)

to a forward-in-time equation. Setting $\tau = T - t$. The transformed partial differential equation for P is in the form

$$\frac{\partial P}{\partial \tau} = \frac{1}{2}\sigma D^2(\tau)D^2\frac{\partial^2 P}{\partial D^2} + \frac{1}{2}\sigma V^2(\tau)V^2\frac{\partial^2 P}{\partial V^2} + \rho(\tau)\sigma_D(\tau)\sigma_V(\tau)DV\frac{\partial^2 P}{\partial D\partial V}$$
$$+[\kappa_D(\tau)(\ln\theta_D(\tau) - \ln D)]D\frac{\partial P}{\partial D} + (r-q)V\frac{\partial P}{\partial V} - rP. \tag{6}$$

The transformed initial condition is thus specified as

$$P(D, V, \tau) = Dmax(L - V, 0). \tag{7}$$

By the theory of partial differential equations, we know that there may have many solutions of Eq.(6) subject to the initial condition (7). The main contributions we obtain in this paper are that we derive two special solutions which are different from the solution acquired in [11].

Theorem 2.1 *Partial differential equation (6) associated with assumption (7) has a solution in the form*

$$P(D, V, \tau) = D^c\exp[(\int_0^\tau \alpha(\tau')d\tau' - r\tau\Big)] \times [LN(-\frac{z}{\sqrt{2c}})$$
$$-V\exp[\Big(\int_0^\tau c\rho(\tau')\sigma_D(\tau')\sigma_V(\tau')d\tau' - (q-r)\tau)]N(-\frac{z+2c_1}{\sqrt{2c_1}})], \tag{8}$$

where

$$\alpha(\tau) = \frac{1}{2}c(c-1)\sigma_D^2(\tau), \tag{9}$$

$$c_1(\tau) = \int_0^\tau \frac{\sigma_V^2(\tau')}{2}d\tau', \tag{10}$$

$$z(V,\tau) = \ln(\frac{V}{L}) + (r-q)\tau - c_1(\tau) + \int_0^\tau \rho(\tau')\sigma_D(\tau')\sigma_V(\tau')cd\tau', \tag{11}$$

where c is an arbitrary constant and N is the cumulative normal distribution function.

Proof. The solution of $P(D, V, \tau)$ can be written in the form

$$P(D, V, \tau) = D^cF(V,\tau)\exp[\int_0^\tau \alpha(\tau')d\tau']. \tag{12}$$

By selecting the appropriate $\alpha(\tau')$, Eq(6) is converted into heat conduction equation. Since

$$P_\tau = \exp[\int_0^\tau \alpha(\tau')d\tau'][\alpha(\tau)D^cF + cD^{c-1}D_\tau F + D^cF_\tau], \tag{13}$$

$$P_D = \exp[\int_0^\tau \alpha(\tau')d\tau']cD^{c-1}F, \tag{14}$$

$$P_{DD} = \exp[\int_0^\tau \alpha(\tau')d\tau']c(c-1)D^{c-2}F, \tag{15}$$

$$P_V = \exp[\int_0^\tau \alpha(\tau')d\tau']D^c F_V, \tag{16}$$

$$P_{VV} = \exp[\int_0^\tau \alpha(\tau')d\tau']D^c F_{VV}, \tag{17}$$

$$P_{DV} = \exp[\int_0^\tau \alpha(\tau')d\tau']cD^{c-1}F_V, \tag{18}$$

taking Eq(13)-(18) into Eq.(6), we obtain

$$\begin{aligned} F_\tau = &\frac{1}{2}\sigma_V^2 V^2 F_{VV} + [r - q + \rho(\tau)\sigma_D(\tau)\sigma_V(\tau)]VF_V - rF \\ &-[\alpha(\tau) + \frac{c}{D}D_t - \frac{1}{2}c(c-1)\sigma_D^2(\tau) - c\kappa_D(\ln\theta_D - \ln D)]F. \end{aligned} \tag{19}$$

Setting $\alpha(\tau) + \frac{c}{D}D_t - \frac{1}{2}c(c-1)\sigma_D^2(\tau) - c\kappa_D(\ln\theta_D - \ln D) = 0$, we have $\alpha(\tau) = \frac{1}{2}c(c-1)\sigma_D^2(\tau)$. The corresponding initial condition is given by

$$F(V, \tau = 0) = Lmax[1 - (V/L), 0]. \tag{20}$$

Using the result in [13] derives that $F(V, \tau)$ is given by

$$F(V,\tau) = \int_{-\infty}^{+\infty} K(V,\tau; y, 0)F(y, 0)dy = L\int_{-\infty}^{0} K(V,\tau; y, 0)[1 - \exp(y)], \tag{21}$$

where

$$\begin{aligned} &y = \ln(\frac{V}{L}), \\ &K(V,\tau,y,0) = \frac{\exp(-r\tau)}{\sqrt{4\pi c_1(\tau)}}\exp[-\frac{[y - z(V,\tau)]^2}{4c_1(\tau)}], \end{aligned} \tag{22}$$

where $z(V,\tau)$ and $c_1(\tau)$ are defined in Eqs.(10) and (11). The integral in Eq.(21) can be evaluated to a closed-form solution of

$$F(V,\tau) = L\exp(-rt) \times [N(-\frac{z}{\sqrt{2c_1}}) - \exp(z + c_1)N(-\frac{z + 2c_1}{\sqrt{2c_1}})]. \tag{23}$$

After substituting Eq.(23) into Eq.(12), we obtain the solution of Eq.(6).

Theorem 2.2 *Partial differential equation (6) associated with assumption (7) has a solution in the form*

$$\begin{aligned} P(D,V,\tau) = &D^{\eta^2} exp[(\int_0^\tau \alpha(\tau')d\tau' - r\tau)] \times [LN(-\frac{z}{\sqrt{2c}}) \\ &-V\exp[(\int_0^\tau \rho(\tau')\sigma_D(\tau')\sigma_V(\tau')\eta^2(\tau)d\tau' - (q-r)\tau)]N(-\frac{z+2c_1}{\sqrt{2c_1}})], \end{aligned} \tag{24}$$

where

$$\alpha(\tau) = (2\kappa_D(\tau)\ln D - \frac{1}{2}\sigma_D^2(\tau'))\eta^2(\tau) + \frac{1}{2}\sigma_D^2(\tau')\eta^4(\tau), \tag{25}$$

$$\eta = \exp(-\int_0^\tau \kappa_D(\tau')d\tau'), \tag{26}$$

$$c_1(\tau) = \int_0^\tau \frac{\sigma_V^2(\tau')}{2}d\tau', \tag{27}$$

$$z(V,\tau) = \ln(\frac{V}{L}) + (r-q)\tau - c_1(\tau) + \int_0^\tau \rho(\tau')\sigma_D(\tau')\sigma_V(\tau')\eta^2(\tau')d\tau'. \tag{28}$$

Proof. The solution of $P(D,V,\tau)$ can be written in the form

$$P(D,V,\tau) = D^{\eta^2(\tau)}F(V,\tau)\exp[\int_0^\tau \alpha(\tau')d\tau']. \tag{29}$$

By selecting the appropriate $\alpha(\tau)$ and $\eta(\tau)$, Eq(6) is converted into heat conduction equation. Since

$$P_\tau = \exp[\int_0^\tau \alpha(\tau')d\tau' + \eta^2(\tau)\ln D][F_\tau + (2\eta(\tau)\eta(\tau)_\tau \ln D + \eta^2(\tau)D^{-1}D_\tau + \alpha(\tau))F], \tag{30}$$

$$P_D = \exp[\int_0^\tau \alpha(\tau')d\tau' + \eta^2(\tau)\ln D]\eta^2(\tau)D^{-1}F, \tag{31}$$

$$P_{DD} = \exp[\int_0^\tau \alpha(\tau')d\tau' + \eta^2(\tau)\ln D](\eta^4(\tau) - \eta^2(\tau))D^{-2}F, \tag{32}$$

$$P_V = \exp[\int_0^\tau \alpha(\tau')d\tau' + \eta^2(\tau)\ln D]F_V, \tag{33}$$

$$P_{VV} = \exp[\int_0^\tau \alpha(\tau')d\tau' + \eta^2(\tau)\ln D]F_{VV}, \tag{34}$$

$$P_{DV} = \exp[\int_0^\tau \alpha(\tau')d\tau' + \eta^2(\tau)\ln D]\eta^2(\tau)D^{-1}F_V, \tag{35}$$

taking (30)-(35) into Eq.(6), we have

$$F_\tau = \frac{1}{2}\sigma_V^2V^2F_{VV} + [r - q + \rho(\tau)\sigma_D(\tau)\sigma_V(\tau)]VF_V - rF - [\alpha(\tau) + 2\eta(\tau)\eta(\tau)_\tau \ln D - \frac{1}{2}\sigma_D^2(\tau)(\eta^4(\tau) - \eta^2(\tau))]F. \tag{36}$$

Letting $\alpha(\tau)+2\eta(\tau)\eta(\tau)_\tau \ln D - \frac{1}{2}\sigma_D^2(\tau)(\eta^4(\tau)-\eta^2(\tau)) = 0, \eta = \exp(-\int_0^\tau \kappa_D(\tau')d\tau')$ yields $\alpha(\tau) = (2\kappa_D(\tau)\ln D - \frac{1}{2}\sigma_D^2(\tau'))\eta^2(\tau) + \frac{1}{2}\sigma_D^2(\tau')\eta^4(\tau)$. The corresponding initial condition is given by

$$F(V,\tau=0) = Lmax[1 - (V/L), 0]. \tag{37}$$

Using the result in [13] derives that $F(V,\tau)$ is given by

$$F(V,\tau) = \int_{-\infty}^{+\infty} K(V,\tau;y,0)F(y,0)dy = L\int_{-\infty}^{0} K(V,\tau;y,0)[1-\exp(y)], \tag{38}$$

where

$$y = \ln(\frac{V}{L}),$$
$$K(V, \tau; y, 0) = \frac{\exp(-r\tau)}{\sqrt{4\pi c_1(\tau)}} \exp[-\frac{[y - z(V, \tau)]^2}{4c_1(\tau)}], \tag{39}$$

$z(V, \tau)$ and $c_1(\tau)$ are defined in Eqs.(10) and (11). The integral in Eq.(21) is evaluated to a closed-form solution of

$$F(V, \tau) = L \exp(-rt) \times [N(-\frac{z}{\sqrt{2c_1}}) - \exp(z + c_1)N(-\frac{z + 2c_1}{\sqrt{2c_1}})]. \tag{40}$$

After substituting Eq.(40) into Eq.(29), we obtain the solution (24).

The closed-form solution of Eq.(6) is composed of a put-option solution. The put-option part is a decreasing function of the collateral value V and influenced by loan to value ratio(L/V). Also, the value of D affects the provision of the pool as a multiplication factor.

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