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Numerical Solution Of Linear Integral Equation Using Flatlet Oblique Multiwavelets

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Abstract

This paper is concerned the using of biorthogonal Flatlet oblique multiwavelet system to solve linear Fredholm integral equations. The biorthogonality and high vanishing moments properties of this system result in efficient and accurate solutions. Finally, numerical results and the absolute errors for some test problems with known solutions are presented.

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1 Introduction

Wavelet theory is relatively new and is an emerging area in mathematical research. In recent years, wavelets have found their way into different fields of science and engineering. Wavelet analysis assumed significance due to the successful applications in signal and image processing during the eighties. The smooth orthonormal basis obtained by the translation and dilation of a single function in a hierarchical fashion proved very useful to develop compression algorithms for signals and images up to a chosen threshold of relevant amplitudes [9, 10].

The advantages of multiwavelets, as extensions form scalar wavelets and their promising features have resulted in an increasing trend to study them. Features such as orthogonality, compact support, symmetry, high order vanishing moments and simple structure make multiwavelets useful both in theory and in applications such as signal compression and denoising [11, 12, 14, 15]. The use of multiwavelet basis leads to sparse representation for a wide class of differential, integral and integro-differential equations due to moments of the simple functions involved. In some works such as [3], representations of operators are constructed by using multiwavelets with the goal of developing adaptive solvers for linear and nonlinear partial differential equations. The use of operator modeling converts differential equations to systems of algebraic equations. It is known that short support and high vanishing moments are the two most important features of a biorthogonal Multiwavelet System(BMS). The dual multiwavelets used in this paper provide a high order of polynomial reproduction and offer specified approximation order near the borders. Moreover, their important smoothness properties transform into polynomial reproductions in the discrete setting. Since the true solution of the examples are power series, a higher m might allow for a higher order Taylor approximation of the solution. We refer the readers to [13, 11, 7] for more information about constructions and samples of BMS and wavelets. Using multiwavelets basis provides a better approximation for problems having smooth solutions. It should be noted that despite the fact that smoothness is a suitable feature for problems such as signal processing and image decomposition, thus far, non-smooth wavelets have provided desirable results in certain numerical methods [3, 8].

In this paper, we use Flatlet multiwavelets system [5] with multiplicity m and derive an algorithm to solving linear Fredholm integral equation of the form

$$y(x) - \int_0^1 K(x,t)y(t)dt = f(x), \qquad 0 \leq x \leq 1 \tag{1}$$

where f and K, are known functions and y is the unknown function to be found.

This paper is organized as follows: In Section 2, we describe the formulation of the biorthogonal Flatlet Multiwavelet System on $[0,1]$. In Section 3, the proposed method is used to solve the Fredholm integral equations (FIE). In section 4, we report our computational results and demonstrate the accuracy of the proposed numerical scheme by presenting numerical examples. Section 5, ends this paper with a brief conclusion.

2 Flatlet Multiwavelet System

A flatlet multiwavelet system [5, 13] with multiplicity $m+1$ consist of $m+1$ scaling functions and $m+1$ wavelet defined on $[0,1]$. The simplest example ($m=0$) for the flatlet family is identical to the Haar wavelets. To construct higher order flatlet multiwavelet system, we can follow the same procedures as Haar wavelets. The scaling functions in this system are defined as a set of $(m+1)$ unit constant functions $\phi_0(x), ..., \phi_m(x)$ divided equally into $m+1$ intervals on $[0,1]$ by

$$\phi_i(x) = \begin{cases} 1 & \frac{i}{m+1} \leq x < \frac{i+1}{m+1}, \\ & \\ 0 & \text{otherwise} \end{cases}, i = 0, 1, ..., m. \tag{2}$$

Let $m+1$ functions $\psi_0(x), ..., \psi_m(x)$ be flatlet wavelets corresponding to flatlet scaling functions defined on $[0,1]$. We construct corresponding wavelet by using a two-scale relation which will be introduced next. First for simplicity, we put flatlet scaling functions and wavelets into two vector functions

$$\Phi(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_i(x) \\ \vdots \\ \phi_m(x) \end{bmatrix}, \Psi(x) = \begin{bmatrix} \psi_0(x) \\ \vdots \\ \psi_i(x) \\ \vdots \\ \psi_m(x) \end{bmatrix}. \tag{3}$$

Now the two-scale relations for the flatlet multiwavelet system may be expressed as

$$\Phi(x) = \mathbf{P} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \Psi(x) = \mathbf{Q} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}. \tag{4}$$

where P and Q are $(m+1) \times 2(m+1)$ matrices. Rewriting the two-scale relations (2.3) in the matrix form, yields

$$\begin{bmatrix} \Phi(x) \\ \Psi(x) \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \tag{5}$$

which is called the reconstruction relation. Also the coefficients matrix in (5) is called reconstruction matrix(RCM) which is invertible[5]. Because of the simplicity of the flatlet scaling functions, the matrix P in the two-scale relations (4) is obtained as

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & & & & 0 \\ & & 1 & 1 & & \\ & & & \ddots & & \\ 0 & & & & 1 & 1 \end{bmatrix}. \tag{6}$$

For computing the $2(m+1)^2$ entries of matrix Q we need $2(m+1)^2$ independent conditions. There are many possibilities in choosing the conditions to be used that different flatlet multiwavelet systems with different properties. In this sequel, we use the $\frac{(m+1)(m+2)}{2}$ orthonormality conditions

$$\int_0^1 \psi_i(x)\psi_j(x)dx = \delta_{i,j}, \qquad i,j = 0,1,...,m, \tag{7}$$

and also $\frac{(m+1)(3m+2)}{2}$ vanishing moment conditions

$$\int_{-\infty}^{\infty} \psi_i(x)x^j dx = 0, \quad i = 0,1,...,m, \quad j = 0,1,...,m+i. \tag{8}$$

By using (2) and (4), equation (8) can be written to the following system of linear equations

$$\sum_{k=0}^{2(m+1)} \{(k+1)^{j+1} - (k)^{j+1}\}q_{j,k} = 0, \qquad j = 0,...,m+i. \tag{9}$$

By solving (7) – (9), the unknown matrix Q and so $\Psi(x)$ are obtained. As an example, for the first order flatlet basis functions

$$\phi_0(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_1(x) = \begin{cases} 1, & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

The matrix Q is computed as

$$\mathbf{Q} = \pm \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix}. \tag{11}$$

Which implies that the associated multiwavelets, are not unique. A simple form of multiwavelets for the above example may be given as

$$\psi_0(x) = \sqrt{2} \begin{cases} \frac{1}{2} & 0 \le x < \frac{1}{4}, \\ -\frac{1}{2} & \frac{1}{4} \le x < \frac{3}{4}, \\ \frac{1}{2} & \frac{3}{4} \le x < 1, \\ 0 & \text{otherwise} \end{cases},$$

$$\psi_1(x) = \sqrt{10}\begin{cases} \frac{1}{10} & 0 \le x < \frac{1}{4}, \\ -\frac{3}{10} & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{3}{10} & \frac{1}{2} \le x < \frac{3}{4}, \\ -\frac{1}{10} & \frac{3}{4} \le x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Also the second order flatlet multiwavelets system are

$$\phi_i(x) = \begin{cases} 1 & \frac{i}{3} \le x < \frac{i+1}{3}, \\ 0 & \text{otherwise} \end{cases}, i = 0, 1, 2,$$

$$\psi_0(x) = \sqrt{10}\begin{cases} \frac{1}{6} & 0 \le x < \frac{1}{6}, \\ -\frac{7}{30} & \frac{1}{6} \le x < \frac{1}{3}, \\ -\frac{2}{15} & \frac{1}{3} \le x < \frac{1}{2}, \\ \frac{2}{15} & \frac{1}{2} \le x < \frac{2}{3}, \\ \frac{7}{30} & \frac{2}{3} \le x < \frac{5}{6}, \\ -\frac{1}{6} & \frac{5}{6} \le x < 1, \\ 0 & \text{otherwise} \end{cases},$$

$$\psi_1(x) = \sqrt{14}\begin{cases} \frac{1}{14} & 0 \le x < \frac{1}{6}, \\ -\frac{3}{14} & \frac{1}{6} \le x < \frac{1}{3}, \\ \frac{1}{7} & \frac{1}{3} \le x < \frac{2}{3}, \\ -\frac{3}{14} & \frac{2}{3} \le x < \frac{5}{6}, \\ \frac{1}{14} & \frac{5}{6} \le x < 1, \\ 0 & \text{otherwise} \end{cases},$$

$$\psi_0(x) = \sqrt{14}\begin{cases} -\frac{1}{42} & 0 \leq x < \frac{1}{6}, \\ \frac{5}{42} & \frac{1}{6} \leq x < \frac{1}{3}, \\ -\frac{5}{21} & \frac{1}{3} \leq x < \frac{1}{2}, \\ \frac{5}{21} & \frac{1}{2} \leq x < \frac{2}{3}, \\ -\frac{5}{42} & \frac{2}{3} \leq x < \frac{5}{6}, \\ \frac{1}{42} & \frac{5}{6} \leq x < 1, \\ 0 & \text{otherwise} \end{cases}. \tag{13}$$

3 Biorthogonal Flatlet Multiwavelet System

The $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$ be dual scaling and wavelet vector functions in biorthogonal flatlet multiwavelet system(BFMS), respectively as

$$\tilde{\Phi}(x) = \begin{bmatrix} \tilde{\phi}_0(x) \\ \vdots \\ \tilde{\phi}_i(x) \\ \vdots \\ \tilde{\phi_m}(x) \end{bmatrix}, \tilde{\Psi}(x) = \begin{bmatrix} \tilde{\psi}_0(x) \\ \vdots \\ \tilde{\psi}_i(x) \\ \vdots \\ \tilde{\psi_m}(x) \end{bmatrix}. \tag{14}$$

Note that according to the biorthogonality conditions in BFMS we must have

$$\langle \tilde{\phi}_i, \phi_j \rangle = \int_0^1 \tilde{\phi}_i(x)\phi_j(x)dx = \delta_{i,j}, \tag{15}$$

$$\langle \tilde{\psi}_i, \psi_j \rangle = \int_0^1 \tilde{\psi}_i(x)\psi_j(x)dx = \delta_{i,j}, \tag{16}$$

$$\langle \tilde{\psi}_i, \phi_j \rangle = \int_0^1 \tilde{\psi}_i(x)\phi_j(x)dx = 0, \tag{17}$$

$$i, j = 0, 1, ..., m.$$

Now we introduce $\tilde{\phi}_i(x)$ and $\tilde{\psi}_i(x)$ as polynomials and piecewise polynomials of degree m respectively, by

$$\tilde{\phi}_i(x) = \begin{cases} a_{i1} + a_{i2}x + ... + a_{i,m+1}x^m & 0 \leq x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

$$\tilde{\psi}_i(x) = \begin{cases} b_{i1}^1 + b_{i2}^1 x + ... + b_{i,m+1}^1 x^m & 0 \leq x < \frac{1}{2}, \\ b_{i1}^2 + b_{i2}^2 x + ... + b_{i,m+1}^2 x^m & \frac{1}{2} \leq x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Based on biorthogonal conditions $(15)-(17)$, we show that coefficients $a_{i,j}, b_{i,j}^1$ and $b_{i,j}^2$, $i = 0, ..., m$, and $j = 1, ..., m+1$, in (18) and (19) are uniquely determined.

Lemma 3.1. *(See [5]) Let $A = [a_{i,j}]_{n\times n}$ be a square matrix with $a_{i,j} = p_{i-1}(j)$, a polynomial of exact degree $i-1$, then A is invertible.*

Theorem 3.2. *(See[5]) For oblique flatlet multiwavelets, The dual functions defined in* (18) *and* (19) *are uniquely determined.*

For example, the dual multiwavelets corresponding to (10) and (12) are computed as

$$\begin{aligned}
\tilde{\phi}_0(x) &= \begin{cases} 3-4x, & 0 \leq x < 1, \\ 0, & \text{otherwise}, \end{cases} \\
\tilde{\phi}_1(x) &= \begin{cases} -1+4x, & 0 \leq x < 1, \\ 0, & \text{otherwise}, \end{cases} \\
\tilde{\psi}_0(x) &= \begin{cases} 2\sqrt{2}(1-4x), & 0 \leq x < \frac{1}{2}, \\ -2\sqrt{2}(3-4x), & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise}, \end{cases} \\
\tilde{\psi}_1(x) &= \begin{cases} \sqrt{10}(1-4x), & 0 \leq x < \frac{1}{2}, \\ \sqrt{10}(3-4x), & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned} \tag{20}$$

Also the dual multiwavelets corresponding to (13) are computed as

$$\tilde{\phi}_0(x) = \begin{cases} \frac{11}{2} - 18x + \frac{27}{2}x^2, & 0 \leq x < 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$\tilde{\phi}_1(x) = \begin{cases} \frac{-7}{2} - 27x + 27x^2, & 0 \leq x < 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$\tilde{\phi}_2(x) = \begin{cases} 1 - 9x + \frac{27}{2}x^2, & 0 \leq x < 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$\tilde{\psi}_0(x) = \begin{cases} \sqrt{10}(\frac{7}{4} - \frac{33}{2}x + 27x^2), & 0 \leq x < \frac{1}{2}, \\ -\sqrt{10}(\frac{49}{4} - \frac{75}{2}x + 27x^2), & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$\tilde{\psi}_1(x) = \begin{cases} \sqrt{14}(\frac{9}{4} - \frac{45}{2}x + \frac{81}{2}x^2), & 0 \leq x < \frac{1}{2}, \\ \sqrt{14}(\frac{81}{4} - \frac{117}{2}x + \frac{81}{2}x^2), & \frac{1}{2} \leq x < 1, \\ 0 & \text{otherwise} \end{cases}$$
$$\tilde{\psi}_3(x) = \begin{cases} -\sqrt{14}(1 - 12x + 27x^2), & 0 \leq x < \frac{1}{2}, \\ \sqrt{14}(16 - 42x + 27x^2), & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise,} \end{cases} \tag{21}$$

A function $f(x)$ defined in $[0, 1]$ may be approximated by the flatlet multiwavelets [11] as

$$f(x) \simeq \Theta^T.\mathbf{f}, \tag{22}$$

or

$$f(x) \simeq \tilde{\Theta}^T.\tilde{\mathbf{f}}, \tag{23}$$

where

$$\Theta(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_m(x) \\ \psi_0(x) \\ \vdots \\ \psi_i(2^l x - k) \\ \vdots \\ \psi_m(2^J x - 2^J + 1) \end{bmatrix}, \tilde{\Theta}(x) = \begin{bmatrix} \tilde{\phi}_0(x) \\ \vdots \\ \tilde{\phi}_m(x) \\ \tilde{\psi}_0(x) \\ \vdots \\ \tilde{\psi}_i(2^l x - k) \\ \vdots \\ \tilde{\psi}_m(2^J x - 2^J + 1) \end{bmatrix}, \tag{24}$$

and $\mathbf{f}$, $\tilde{\mathbf{f}}$ are N-vectors as

$$\begin{aligned} \mathbf{f} &= [c'_0, ..., c'_m, d_{0,0,0}, ..., d_{i,l,k}, ..., d_{m,J,2^J-1}]^T, \\ \tilde{\mathbf{f}} &= [\tilde{c}'_0, ..., \tilde{c}'_m, \tilde{d}_{0,0,0}, ..., \tilde{d}_{i,l,k}, ..., \tilde{d}_{m,J,2^J-1}]^T, \end{aligned}$$

in which $N = 2^J(m+1)$. Also, a two variable function $g(x, y)$ can be approximated by flatlet multiwavelets [11] as

$$g(x, y) \simeq \Theta^T(y).\mathbf{G}.\Theta(x), \tag{25}$$

where

$$[\mathbf{G}]_{i,j} = \int_0^1 \int_0^1 g(x, y)\tilde{\theta}_i(x)\tilde{\theta}_j(y)dxdy, i, j = 1, 2, ..., N.$$

Note that from (14), the vector $\tilde{\Theta}(x)$ in (24) can express as

$$\tilde{\Theta}(x) = \begin{bmatrix} \tilde{\Phi}(x) \\ \tilde{\Psi}(x) \\ \vdots \\ \tilde{\Psi}(2^i x) \\ \vdots \\ \tilde{\Psi}(2^i x - 2^i + 1) \\ \vdots \\ \tilde{\Psi}(2^{J+1}x - 2^{J+1} + 1) \end{bmatrix}, \tag{26}$$

and from (24), it can express as

$$\tilde{\Theta}(x) = \mathbf{Q}\Lambda(x), \tag{27}$$

where

$$\Lambda(x) = \begin{bmatrix} \tilde{\Phi}(x) \\ \tilde{\Phi}(2x) \\ \tilde{\Phi}(2x-1) \\ \vdots \\ \tilde{\Phi}(2^i x) \\ \tilde{\Phi}(2^i x - 1) \\ \vdots \\ \tilde{\Phi}(2^{i+1}x - 2^{i+1} + 2) \\ \tilde{\Phi}(2^{i+1}x - 2^{i+1} + 1) \\ \vdots \\ \tilde{\Phi}(2^{J+1}x - 2^{J+1} + 2) \\ \tilde{\Phi}(2^{J+1}x - 2^{J+1} + 1) \end{bmatrix},$$

and $\mathbf{Q}'$ is a block matrix of the form

$$\begin{bmatrix} \mathbf{I} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \tilde{\mathbf{Q}} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{\mathbf{Q}} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \tilde{\mathbf{Q}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & \tilde{\mathbf{Q}} \end{bmatrix}$$

where $\mathbf{I}$ is the identity matrix of rank $m+1$, the 0th in the below $\mathbf{I}$ are $(m+1)\times(m+1)$ zero matrices and the other 0s are $(m+1)\times 2(m+1)$ zero matrices. Note that this block matrix can be separated as $2^{J+1}-1$ block column and block rows. So using (27), we can rewrite (22) as

$$f(x) \simeq \Lambda^T.\mathbf{Q}^T.\mathbf{f}. \tag{28}$$

4 Solving The Fredholm Integral Equations(FIE)

In this section we solve the linear integral equation (1) by utilizing BFMS. For this purpose, we give two methods as following.

The First method
By using (22) and (25), we approximate the functions involved by

$$\begin{aligned} y(x) &\simeq \mathbf{y}^T.\tilde{\Theta}(x), \\ k(x,t) &\simeq \tilde{\Theta}^T(t).K.\tilde{\Theta}(x), \\ f(x) &\simeq \mathbf{f}^T.\tilde{\Theta}(x), \end{aligned} \tag{29}$$

where $\mathbf{y}$ is unknown vector and $\mathbf{f}$ is a known vector with 2^{j+1} entries and $\mathbf{K}$ is a known $2^{j+1}\times 2^{j+1}$ matrix as

$$\mathbf{f}_i = \int_0^1 f(x)\Theta(x)dx, \tag{30}$$

and

$$\mathbf{K}_{i,j} = \int_0^1 \left(\int_0^1 k(x,t)\Theta(t)dt\right)\Theta(x)dx. \tag{31}$$

By substituting (29) into (1),we get

$$y^T.\tilde{\Theta}(x) - \int_0^1 \tilde{\Theta}^T(t).\mathbf{K}.\tilde{\Theta}(x).\mathbf{y}^T.\tilde{\Theta}(t)dt = \mathbf{f}^T.\tilde{\Theta}(x), \tag{32}$$

or

$$\mathbf{y}^T\tilde{\Theta}(x) - \mathbf{y}^T\left[\int_0^1 \tilde{\Theta}(t)\tilde{\Theta}(t)dt\right]\mathbf{K}\tilde{\Theta}(x) - \mathbf{f}^T\tilde{\Theta}(x) = 0. \tag{33}$$

Let

$$\mathbf{P} = \int_0^1 \tilde{\Theta}(t)\tilde{\Theta}(t)dt, \tag{34}$$

so $\mathbf{P}$ is a $2^{j+1} \times 2^{j+1}$ matrix. By substituting (34) into (33), we get

$$\mathbf{y}^T - \mathbf{y}^T\mathbf{P}\mathbf{K} - \mathbf{f}^T = 0, \tag{35}$$

Eq. (35) gives a system of linear equations with 2^{j+1} equations and 2^{j+1} unknowns as

$$\mathbf{y}^T(\mathbf{I} - \mathbf{P}\mathbf{K}) = \mathbf{f}^T, \tag{36}$$

which can be solved to find the vector $\mathbf{y}$, so the unknown function $y(x)$ can be found using Eq. (29).

The Second method
In this method we calculate the coefficients using dual (FOM) and approximate $\mathbf{y}(\mathbf{x})$ by flatlet multiwavelets, as bellow.

$$\begin{aligned} y(x) &\simeq \mathbf{y}^T.\Theta(x), \\ k(x,t) &\simeq \Theta^T(t).K.\Theta(x), \\ f(x) &\simeq \mathbf{f}^T.\Theta(x), \end{aligned} \tag{37}$$

where $\mathbf{y}$ is unknown vector and $\mathbf{f}$ is a known vector with 2^{j+1} entries and $\mathbf{K}$ is a known $2^{j+1} \times 2^{j+1}$ matrix as

$$\mathbf{f}_i = \int_0^1 f(x).\tilde{\Theta}_i(x)dx, \tag{38}$$

and

$$\mathbf{K}_{i,j} = \int_0^1 \left(\int_0^1 k(x,t).\tilde{\Theta}_i(t)dt\right)\tilde{\Theta}_j(x)dx. \tag{39}$$

By substituting (37) in (1), we get

$$\mathbf{y}^T.\Theta(x) - \int_0^1 \Theta^T(t).\mathbf{K}.\Theta(x).\mathbf{y}^T.\Theta(t)dt = \mathbf{f}^T.\Theta(x). \tag{40}$$

or

$$\mathbf{y}^T\Theta(x) - \mathbf{y}^T \left[\int_0^1 \Theta(t)\Theta(t)dt\right] \mathbf{K}\Theta(x) - \mathbf{f}^T\Theta(x) = 0, \tag{41}$$

Let

$$\mathbf{P} = \int_0^1 \Theta(t)\Theta(t)dt, \tag{42}$$

so we have

$$\mathbf{y}^T - \mathbf{y}^T\mathbf{PK} - \mathbf{f}^T = 0, \tag{43}$$

Eq.(43) gives a system of linear equations with 2^{j+1} equations and 2^{j+1} unknowns as

$$\mathbf{y}^T(\mathbf{I} - \mathbf{PK}) = \mathbf{f}^T, \tag{44}$$

which can be solved to find the unknown function $y(x)$.

5 Computational results and test problems

In this section, some numerical examples are presented to illustrate the validity and the merits of **BFMS** technique. One main merit of this technique is the generation of a sparse matrix. This advantage is illustrated in Examples $1-4$. Suppose $T = (I - PK)^T$ and $U = \mathbf{y}^T$ and $F = \mathbf{f}^T$, then Eqs. (35) and (44) result the linear system $TU = F$. We approximate T with a matrix T_ε whose elements are difined by formula

$$T_{\varepsilon_{i,j}} = \begin{cases} T_{i,j}, & |T_{i,j}| \geq \varepsilon \\ 0 & \text{otherwise} \end{cases}, \tag{45}$$

where the threshold ε is chosen so that a desired precision τ is maintained [2, 17]

$$\|T_\varepsilon - T\| \leq \tau\|T\|. \tag{46}$$

Here the norm $\|.\|$ is the row-sum norm. The threshold ε is given by $\varepsilon = \tau\|T\|/n$. In this paper we use the threshold parameter $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ and see that by increasing the value of the threshold parameter, the sparsity and error values are increased regarding Eqs. (45) and (46).

The percent sparsity (S_ε) of matrix T_ε is then defined by [16, 17]

$$S_\varepsilon = \frac{N_0 - N_\varepsilon}{N_0} \times 100\%, \tag{47}$$

Table 1: Sparsity and MaxError for $m = 2$ (The first method)

	Thresholdparameter(ε)	Sparsity(S_ε)(%)	MaxError
	0	91.8	8.4×10^{-5}
$J = 4$	10^{-4}	91.8	8.4×10^{-5}
	10^{-3}	94.01	9.1×10^{-5}
	10^{-2}	97.0	1.1×10^{-3}
	0	95.8	1.1×10^{-5}
$J = 5$	10^{-4}	96.5	1.1×10^{-5}
	10^{-3}	97.9	1.3×10^{-5}
	10^{-2}	98.7	1.0×10^{-3}

where N_0 is the total number of elements T_ε and N_ε is the number of the nonzero elements T_ε. We define the maximum error as $MaxError = max|y_{exact} - y_{app}|$ to calculate the errors in Tables $1 - 6$. Now two presented methods are applied to some examples with known exact solutions. The computations were carried out for different values of m and J. Comparing the value of MaxErrors for tow presented methods we can observe that the first method have better results than the second method. Futhermore, the other advantages of the method are its simplicity and small computations costs which result from the sparsity of the associated matrices.

Example 5.1. *Consider the following integral equation,*

$$\begin{cases} y(x) - \int_0^1 (x^2 + t^2) y(t) dt = f(x), \\ y(0) = 1, \\ f(x) = x^5 + 5x^4 - \frac{7}{6}x^2 - 2x + \frac{55}{168}. \end{cases} \tag{48}$$

The exact solution of this equation is

$$y(x) = x^5 + 5x^4 - 2x + 1.$$

Tables 1 and 2 show the sparsity and MaxError for $m = 2$, $J = 4, 5$ for different values of thresholding parameter, for the first and the second presented methods respectively. Figure 1 shows the plots of errors for two presented methods for $m = 2$ and $J = 4$ and Figure 2 shows the plots of the matrix elements for $m = 2$ and $J = 4$ after thresholding for the first method.

Example 5.2. *Consider the following integral equation,*

$$\begin{cases} y(x) - \int_0^1 (x^2 + t^2) y(t) dt = f(x), \\ y(0) = 4, \end{cases} \tag{49}$$

Table 2: Sparsity and MaxError for $m = 2$ (The second method)

	Thresholdparameter(ε)	Sparsity(S_ε)(%)	MaxError
	0	0	9.3×10^{-2}
$J = 4$	10^{-4}	88.5	9.3×10^{-2}
	10^{-3}	94.01	9.5×10^{-2}
	10^{-2}	97.0	5.6×10^{-2}
	0	0	3.8×10^{-2}
$J = 5$	10^{-4}	95.5	3.7×10^{-2}
	10^{-3}	97.9	4.8×10^{-2}
	10^{-2}	98.7	8.2×10^{-2}

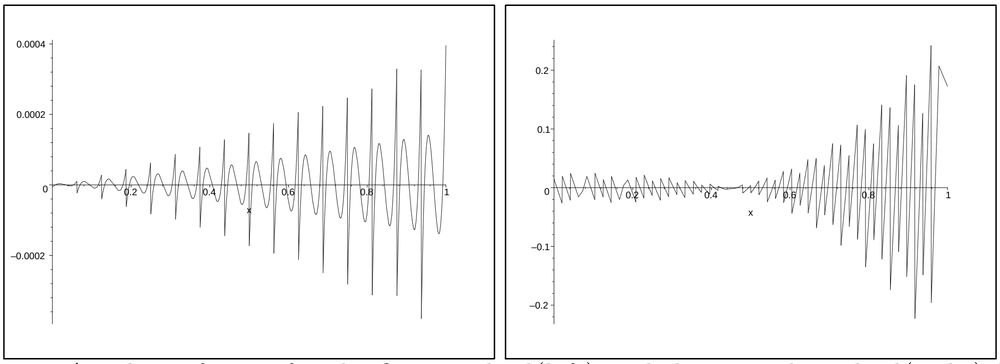

Figure 1: Plots of error for the first method(left) and the second method(right) for Example 1.

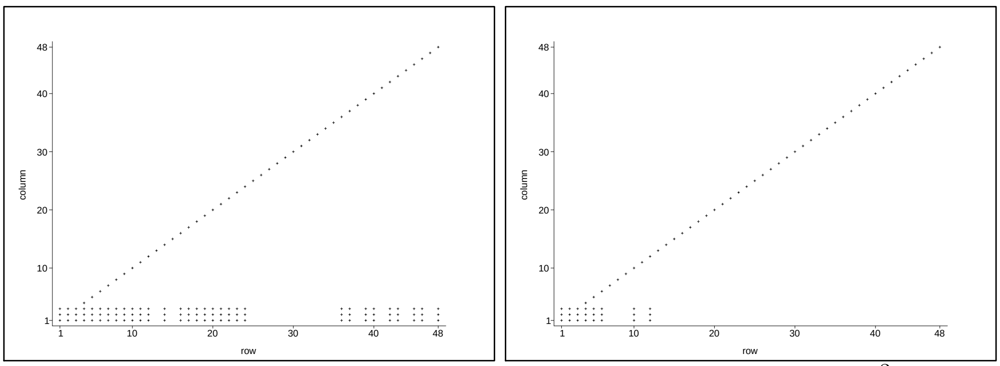

Figure 2: Plots of spars matrix after thresholding with $\varepsilon = 10^{-3}$(left) $\varepsilon = 10^{-2}$(right) for Example 1.

Table 3: Sparsity and MaxError for $m = 2$ for Example 2.

	Thresholdparameter(ε)	Sparsity(S_ε)(%)	MaxError
	0	95.8	5.7×10^{-6}
$J = 5$	10^{-4}	96.5	5.7×10^{-6}
	10^{-3}	97.9	3.3×10^{-5}
	10^{-2}	98.7	1.4×10^{-4}
	0	97.9	7.1×10^{-7}
$J = 6$	10^{-4}	98.5	7.9×10^{-7}
	10^{-3}	99.2	3.3×10^{-5}
	10^{-2}	98.7	1.4×10^{-5}

Here the forcing function f is selected such that

$$y(x) = \sqrt{x} + \frac{1}{1+x} + 2x\sqrt{x} + 3,$$

is the exact solution of problem (5.5). *Table 3 shows the sparsity and MaxError for $m = 2, J = 5, 6$ for different values of thresholding parameter, using the first method presented in the previous section. Figure 3 shows the plot of error for two presented methods for $m = 2$ and $J = 5$.*

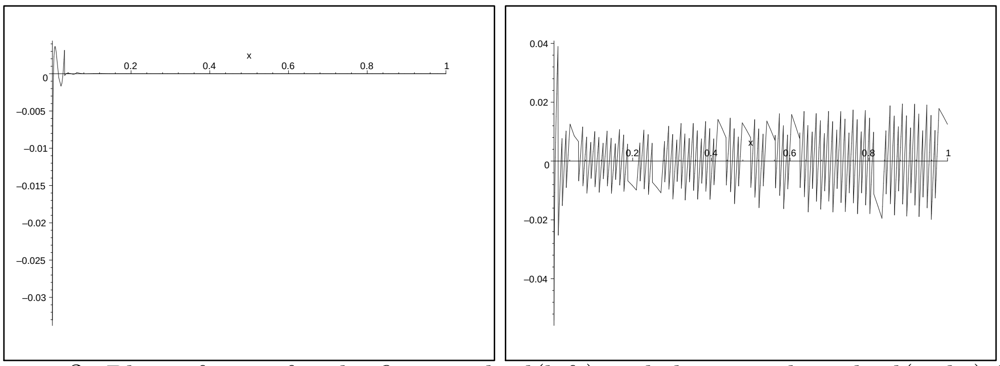

Figure 3: Plots of error for the first method(left) and the second method(right) for $m = 2$ and $J = 5$ for Example 2.

Example 5.3. *Consider the following integral equation,*

$$\begin{cases} y(x) - \int_0^1 (x+t)y(t)dt = f(x), \\ y(0) = e + 1, \end{cases} \tag{50}$$

Table 4: Sparsity and MaxError for $m = 2$ for Example 3.

	Thresholdparameter(ε)	Sparsity(S_ε)(%)	MaxError
	0	95.8	9.8×10^{-6}
$J = 5$	10^{-4}	96.8	9.8×10^{-6}
	10^{-3}	97.9	1.2×10^{-5}
	10^{-2}	98.9	1.0×10^{-3}
	0	97.9	1.1×10^{-6}
$J = 6$	10^{-4}	98.4	1.1×10^{-6}
	10^{-3}	99.2	3.7×10^{-6}
	10^{-2}	98.7	1.0×10^{-3}

Here the forcing function f is selected such that

$$y(x) = e^{2x+1} + \frac{1}{x+1},$$

is the exact solution of problem (5.6). *Table 4 shows the sparsity and MaxError for* $m = 2, J = 5, 6$ *for different values of thresholding parameter, using the first method presented in the previous section. Figure 4 shows the plot of error for two presented methods for* $m = 2$ *and* $J = 5$ *and Figure 5, and 6 show the plots of the matrix elements for* $m = 2$ *and* $J = 5$ *after thresholding.*

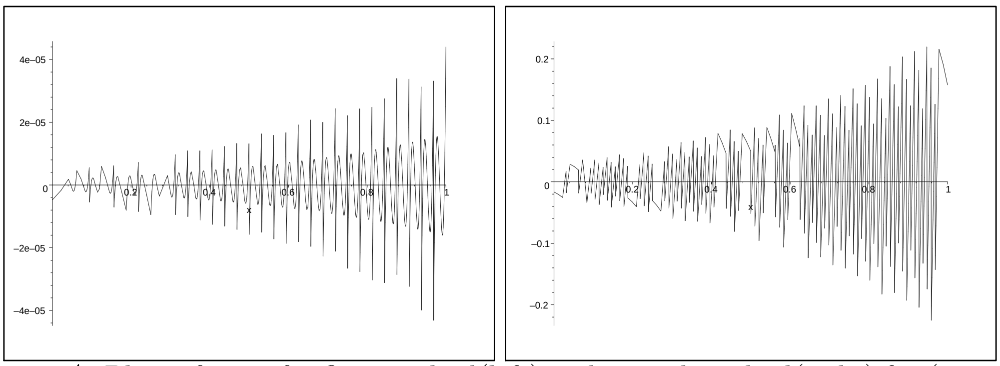

Figure 4: Plots of error for first method(left) and second method(right) for ($m = 2$) and ($J = 5$) for Example 3.

Example 5.4. *Consider the following integral equation,*

$$\begin{cases} y(x) & -\int_0^1 (t-x)^3 y(t)dt = f(x), \\ y(0) & = 1, \\ f(x) & = x^2(1-x^2) + \frac{2}{15}x^3 - \frac{1}{4}x^2 + \frac{6}{35}x - \frac{1}{24}. \end{cases} \tag{51}$$

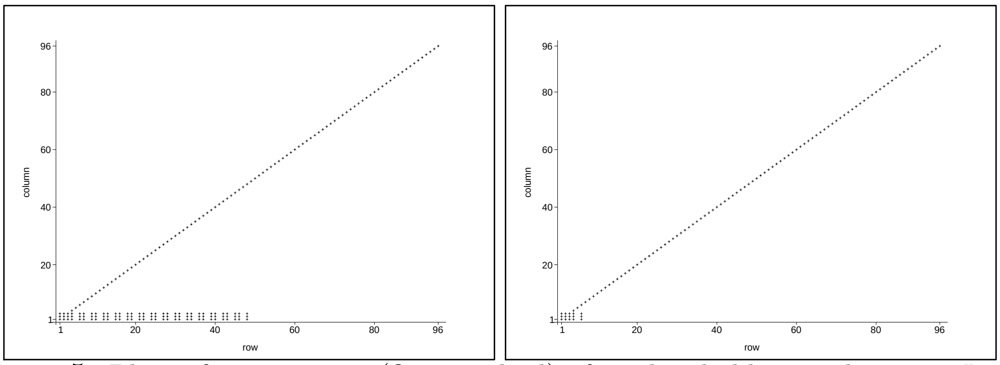

Figure 5: Plots of spars matrix(first method) after thresholding with$m = 2, J = 5$ and $\varepsilon = 10^{-3}$(left) $\varepsilon = 10^{-2}$(right) for Example 3.

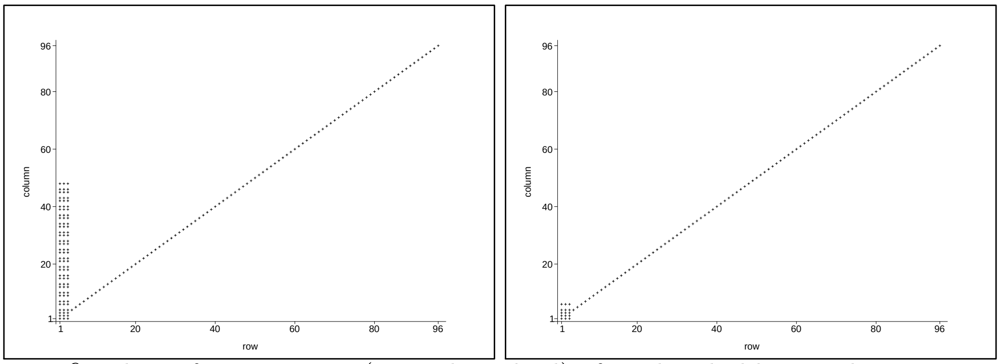

Figure 6: Plots of spars matrix(second method) after thresholding with$m = 2, J = 5$ and $\varepsilon = 10^{-3}$(left) $\varepsilon = 10^{-2}$(right) for Example 3.

The exact solution of this equation is

$$y(x) = x^2(1 - x^2).$$

Table 5,6 shows the sparsity and MaxError for $m = 2, J = 5, 6$ for different values of thresholding parameter, using tow method presented in the previous section. Figure 7 shows the plot of error in tow methods and Figure 8, and 9 show the plots of the matrix elements for $m = 2$ and $J = 5$ after thresholding.

6 Conclusion

In this paper the Flatlet oblique multiwavelets are employed to solve the Fredholm integral equations. Considering the properties of these wavelets and using dual functions, the solution of the integral equations is converted to the solution of a sparse linear system of algebraic equations. Using the spare structure of this linear system, the given problem is solved and memory times are

Table 5: Sparsity and MaxError for $m = 2$ for Example 4(the first method).

	Thresholdparameter(ε)	Sparsity(S_ε)(%)	MaxError
	0	63.9	1.5×10^{-6}
$J = 5$	10^{-4}	97.0	2.8×10^{-5}
	10^{-3}	98.2	2.7×10^{-5}
	10^{-2}	98.7	1.8×10^{-3}
	0	65.3	1.7×10^{-7}
$J = 6$	10^{-4}	98.8	3.0×10^{-5}
	10^{-3}	97.9	3.1×10^{-5}
	10^{-2}	98.7	1.8×10^{-3}

Table 6: Sparsity and MaxError for $m = 2$ for Example 4(the second method).

	Thresholdparameter(ε)	Sparsity(S_ε)(%)	MaxError
	0	0	5.5×10^{-2}
$J = 5$	10^{-4}	96.4	7.5×10^{-2}
	10^{-3}	98.7	8.1×10^{-2}
	10^{-2}	98.7	7.4×10^{-2}

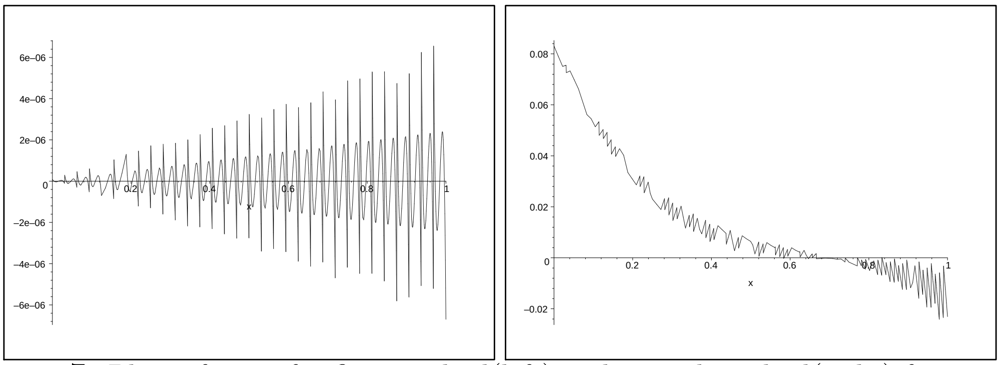

Figure 7: Plots of error for first method(left) and second method(right) for $m = 2$ and $J = 5$ for Example 4.

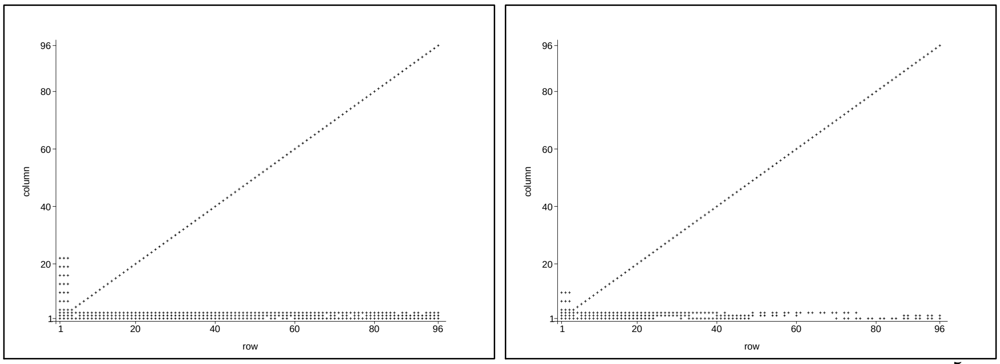

Figure 8: Plots of spars matrix(first method) after thresholding with $\varepsilon = 10^{-5}$(left) $\varepsilon = 10^{-4}$(right) for Example 4.

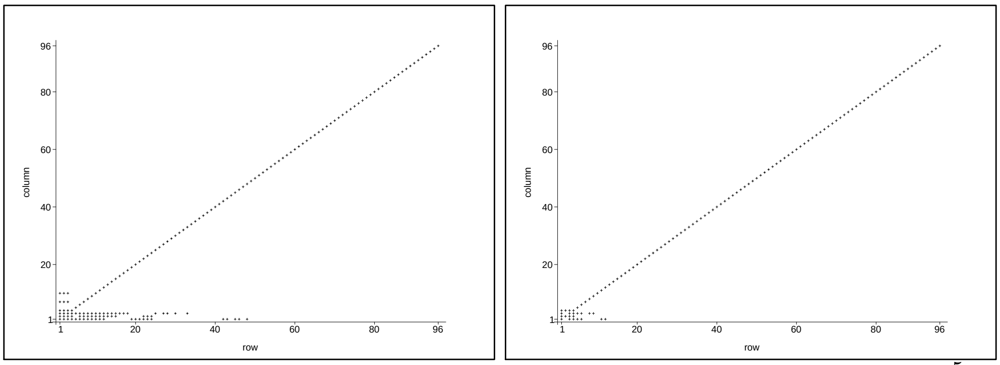

Figure 9: Plots of spars matrix(first method) after thresholding with $\varepsilon = 10^{-5}$(left) $\varepsilon = 10^{-4}$(right) for Example 4.

reduced. Several test examples are used to observe the efficiency and applicability of the new method.

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