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The Modified Extended Tanh Method with the Riccati Equation for Solving Nonlinear Partial Differential Equations

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Abstract

The modified extended tanh method is one of most effective algebraic method for obtaining exact solutions of nonlinear partial differential equations.In this paper,we seek exact solutions of the modified Benjamin-Bona-Mahony(MBBM)equation and the Zakharov-Kuzetsov(ZK)equation.

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1 Introduction

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as,tanh-sech method [1, 2, 3],extended tanh method [4, 5, 6], hyperbolic

function method [7], sine-cosine method [8, 9, 10], Jacobi elliptic function expansion method [11], F-expansion method [12] ,and the first integral method [13, 14].
The modified Benjamin-Bona-Mahony(MBBM)equation [18] is in the form,

$$u_t + u_x + au^2 u_x + bu_{xxt} = 0,$$

and the Zakharov-Kuzetsov(ZK)is in the form

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0.$$

The ZK equation, presented in [15], governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [16, 17]. The ZK equation, which is a more isotropic two-dimensional, was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions [15]. The aim of this paper is to find exact solutions of the MBBM equation and the ZK equation by modified extended tanh method with the Riccati equation.

2 Modified extended tanh method with the Riccati equation

Let us investigated these methods.For given a nonlinear equation

$$F(u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{xt}, ...) = 0, \tag{1}$$

when we look for its traveling wave solutions, the first step is to introduce the wave transformation $u(x, y, t) = U(\xi), \quad \xi = x + y + \lambda t$ and change Eq.(1) to an ordinary differential equation(ODE)

$$H(U, U', U'', U''', ...) = 0. \tag{2}$$

The next crucial step is to introduce a new variable $\phi = \phi(\xi)$, which is a solution of the Riccati equation

$$\frac{d\phi}{d\xi} = k + \phi^2. \tag{3}$$

The modified extended tanh method admits the use of the finite expansion:

$$u(x, y, t) = U(\xi) = \sum_{i=0}^{m} a_i \phi^i(\xi) + \sum_{i=1}^{m} b_i \phi^{-i}(\xi), \tag{4}$$

where the positive integer m is usually obtained by balancing the highest-order linear term with the nonlinear terms in Eq.(2). Substituting Eq.(3) and Eq.(4) into Eq.(2) and then setting zero all coefficients of $\phi^i(\xi)$, we can obtain a system of algebraic equations with respect to the constants $k, \lambda, a_0, ..., a_m, b_1, ..., b_m$. Then we can determine the constants $k, \lambda, a_0, ..., a_m, b_1, ..., b_m$. The Riccati equation (3) has the general solutions:
If $k < 0$ then

$$\phi(\xi) = -\sqrt{-k}\ tanh(\sqrt{-k}\xi), \tag{5}$$
$$\phi(\xi) = -\sqrt{-k}\ coth(\sqrt{-k}\xi).$$

If $k = 0$ then

$$\phi(\xi) = -\frac{1}{\xi}. \tag{6}$$

If $k > 0$ then

$$\phi(\xi) = \sqrt{k}\ tan(\sqrt{k}\xi), \tag{7}$$
$$\phi(\xi) = -\sqrt{k}\ cot(\sqrt{k}\xi).$$

Therefore,by the sign test of k can be obtained exact solutions of Eq.(1).

3 Application

3.1.The modified Benjamin-Bona-Mahony(MBBM)equation
Let us consider the MBBM equation [18]

$$u_t + u_x + au^2u_x + bu_{xxt} = 0, \tag{8}$$

where a and b are positive constants. Using the wave variable $u(x,t) = U(\xi)$, $\xi = x + \lambda t$ carries the PDE (8) into the ODE

$$\lambda U' + U' + aU^2U' + b\lambda U''' = 0, \tag{9}$$

where by integrating Eq.(16) and neglecting the constant of integration we obtain

$$(\lambda + 1)U + \frac{a}{3}U^3 + b\lambda U'' = 0. \tag{10}$$

Balancing U'' with U^3 in Eq.(10) give

$$m + 2 = 3m,$$

so that $m = 1$.
The modified extended tanh method (4) admits the use of the finite expansion

$$U(\xi) = a_0 + a_1\phi(\xi) + \frac{b_1}{\phi(\xi)}. \tag{11}$$

Substituting (11) into Eq.(10), making use of Eq.(3),collecting the coefficients of $\phi^i(\xi)\ -3 \leq i \leq 3$, we obtain:

Coefficient of ϕ^3: $\frac{1}{3}aa_1^3 + 2b\lambda a_1$.
Coefficient of ϕ^2: $aa_0a_1^2$.
Coefficient of ϕ^1: $ab_1a_1^2 + 2b\lambda ka_1 + (\lambda+1)a_1 + aa_0^2a_1$.
Coefficient of ϕ^0: $2ab_1a_0a_1 + (\lambda+1)a_0 + \frac{1}{3}aa_0^3$.
Coefficient of ϕ^{-1}: $aa_1b_1^2 + aa_0^2b_1 + (\lambda+1)b_1 + 2b\lambda b_1k$.
Coefficient of ϕ^{-2}: $aa_0b_1^2$.
Coefficient of ϕ^{-3}: $2b\lambda b_1k^2 + \frac{1}{3}ab_1^3$.

Setting these coefficients equal to zero, and solving the resulting system, by using Maple, we find the following set of solutions:
Case A:

$$k = \frac{6b - aa_1^2}{2aa_1^2b}, \quad \lambda = -\frac{aa_1^2}{6b}, \quad a_0 = b_1 = 0, \tag{12}$$

where a_1 is an arbitrary constant.
If $6b < aa_1^2$, then $k < 0$,Substituting (12) into (11) and using (5) the solution of Eq.(8) is given by:

$$u_1(x,t) = -\sqrt{\frac{aa_1^2 - 6b}{2ab}}tanh(\frac{1}{a_1}\sqrt{\frac{aa_1^2-6b}{2ab}}(x - \frac{aa_1^2}{6b}t)).$$

If $6b > aa_1^2$, then $k > 0$,Substituting (12) into (11) and using (7) the solution of Eq.(8) is given by:

$$u_2(x,t) = \sqrt{\frac{6b - aa_1^2}{2ab}}tan(\frac{1}{a_1}\sqrt{\frac{6b - aa_1^2}{2ab}}(x - \frac{aa_1^2}{6b}t)).$$

Case B:

$$k = \frac{aa_1^2 - 6b}{4aa_1^2b}, \quad \lambda = -\frac{aa_1^2}{6b}, \quad a_0 = 0, \quad b_1 = \frac{aa_1^2 - 6b}{4aa_1b}, \tag{13}$$

where a_1 is an arbitrary constant.
If $6b < aa_1^2$, then $k > 0$,Substituting (13) into (11) and using (7) the solution of Eq.(8) is given by:

$$u_3(x,t) = \sqrt{\frac{aa_1^2 - 6b}{4ab}}[tan(\frac{1}{a_1}\sqrt{\frac{aa_1^2-6b}{4ab}}(x - \frac{aa_1^2}{6b}t))$$

$$+cot(\frac{1}{a_1}\sqrt{\frac{aa_1^2-6b}{4ab}}(x - \frac{aa_1^2}{6b}t))].$$

If $6b > aa_1^2$, then $k < 0$,Substituting (13) into (11) and using (5) the solution of Eq.(8) is given by:

$$u_4(x,t) = \sqrt{\frac{6b - aa_1^2}{4ab}}[coth(\frac{1}{a_1}\sqrt{\frac{6b - aa_1^2}{4ab}}(x - \frac{aa_1^2}{6b}t))$$

$$-tanh(\frac{1}{a_1}\sqrt{\frac{6b - aa_1^2}{4ab}}(x - \frac{aa_1^2}{6b}t))].$$

Case C:

$$k = \frac{6b - aa_1^2}{8aa_1^2 b}, \quad \lambda = -\frac{aa_1^2}{6b}, \quad a_0 = 0, \quad b_1 = \frac{aa_1^2 - 6b}{8aa_1 b}, \tag{14}$$

where a_1 is an arbitrary constant.
If $6b < aa_1^2$, then $k < 0$,Substituting (14) into (11) and using (5) the solution of Eq.(8) is given by:

$$u_5(x,t) = -\sqrt{\frac{aa_1^2 - 6b}{8ab}}[tanh(\frac{1}{a_1}\sqrt{\frac{aa_1^2 - 6b}{8ab}}(x - \frac{aa_1^2}{6b}t))$$

$$+coth(\frac{1}{a_1}\sqrt{\frac{aa_1^2 - 6b}{8ab}}(x - \frac{aa_1^2}{6b}t))].$$

If $6b > aa_1^2$, then $k > 0$,Substituting (14) into (11) and using (7) the solution of Eq.(8) is given by:

$$u_6(x,t) = \sqrt{\frac{6b - aa_1^2}{8ab}}[tan(\frac{1}{a_1}\sqrt{\frac{6b - aa_1^2}{8ab}}(x - \frac{aa_1^2}{6b}t))$$

$$-cot(\frac{1}{a_1}\sqrt{\frac{6b - aa_1^2}{8ab}}(x - \frac{aa_1^2}{6b}t))].$$

3.2.The Zakharov-Kuzetsov(ZK)equation

In this section we study the ZK equation

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0. \tag{15}$$

Using the wave variable $u(x,t) = U(\xi)$, $\xi = x + y + \lambda t$ carries the PDE (15) into the ODE

$$\lambda U' + \frac{a}{2}(U^2)' + 2bU''' = 0, \tag{16}$$

where by integrating Eq.(16) and neglecting the constant of integration we obtain

$$\lambda U + \frac{a}{2}U^2 + 2bU'' = 0. \tag{17}$$

Balancing U'' with U^2 in Eq.(10) give

$$m+2=2m,$$

so that $m=2$.
The modified extended tanh method (4) admits the use of the finite expansion

$$U(\xi)=a_0+a_1\phi(\xi)+a_2\phi^2(\xi)+\frac{b_1}{\phi(\xi)}+\frac{b_2}{\phi^2(\xi)}. \tag{18}$$

Substituting (18) into Eq.(17), making use of Eq.(3),collecting the coefficients of $\phi^i(\xi)\ -4\le i\le 4$, we obtain:

Coefficient of ϕ^4: $\frac{1}{2}aa_2^2+12ba_2$.
Coefficient of ϕ^3: $4ba_1+aa_1a_2$.
Coefficient of ϕ^2: $\lambda a_2+16ba_2k+aa_0a_2+\frac{1}{2}aa_1^2$.
Coefficient of ϕ^1: $\lambda a_1+4kba_1+aa_0a_1+ab_1a_2$.
Coefficient of ϕ^0: $ab_1a_1+ab_2a_2+\frac{1}{2}aa_0^2+\lambda a_0+4bb_2+4k^2ba_2$.
Coefficient of ϕ^{-1}: $4bb_1k+aa_1b_2+ab_1a_0+\lambda b_1$.
Coefficient of ϕ^{-2}: $\lambda b_2+16bb_2k+ab_2a_0+\frac{1}{2}ab_1^2$.
Coefficient of ϕ^{-3}: $4bb_1k^2+ab_1b_2$.
Coefficient of ϕ^{-4}: $12bb_2k^2+\frac{1}{2}ab_2^2$.

Setting these coefficients equal to zero, and solving the resulting system, by using Maple, we find the following set of solutions:
Case A:

$$k=\frac{\lambda}{8b},\quad a_0=-\frac{3\lambda}{a},\quad a_1=0,\quad a_2=-\frac{24b}{a},\quad b_1=b_2=0, \tag{19}$$

where λ is an arbitrary constant.
If $\frac{\lambda}{b}<0$ then $k<0$,Substituting (19) into (18) and using (5) the solution of the ZK is given by:

$$u_1(x,y,t)=-\frac{3\lambda}{a}sech^2(\frac{1}{2}\sqrt{-\frac{\lambda}{2b}}(x+y+\lambda t)).$$

If $\frac{\lambda}{b}>0$ then $k>0$,Substituting (19) into (18) and using (7) the solution of the ZK is given by:

$$u_2(x,y,t)=-\frac{3\lambda}{a}sec^2(\frac{1}{2}\sqrt{\frac{\lambda}{2b}}(x+y+\lambda t)).$$

Case B:

$$k=-\frac{\lambda}{8b},\quad a_0=\frac{\lambda}{a},\quad a_1=0,\quad a_2=-\frac{24b}{a},\quad b_1=b_2=0, \tag{20}$$

where λ is an arbitrary constant.
If $\frac{\lambda}{b} < 0$ then $k > 0$,Substituting (20) into (18) and using (7) the solution of the ZK is given by:

$$u_3(x,y,t) = \frac{\lambda}{a}(1 + 3tan^2(\frac{1}{2}\sqrt{-\frac{\lambda}{2b}}(x + y + \lambda t))).$$

If $\frac{\lambda}{b} > 0$ then $k < 0$,Substituting (20) into (18) and using (5) the solution of the ZK is given by:

$$u_4(x,y,t) = \frac{\lambda}{a}(1 - 3tanh^2(\frac{1}{2}\sqrt{\frac{\lambda}{2b}}(x + y + \lambda t))).$$

Case C:

$$k = \frac{\lambda}{32b}, \quad a_0 = -\frac{3\lambda}{2a}, \quad a_1 = 0, \quad a_2 = -\frac{24b}{a}, \quad b_1 = 0, \quad b_2 = -\frac{3\lambda^2}{128ab}, \tag{21}$$

where λ is an arbitrary constant.
If $\frac{\lambda}{b} < 0$ then $k < 0$,Substituting (21) into (18) and using (5) the solution of the ZK is given by:

$$u_5(x,y,t) = -\frac{3\lambda}{2a}[1 - \frac{1}{2}tanh^2(\frac{1}{4}\sqrt{-\frac{\lambda}{2b}}(x+y+\lambda t)) - \frac{1}{2}coth^2(\frac{1}{4}\sqrt{-\frac{\lambda}{2b}}(x+y+\lambda t))].$$

If $\frac{\lambda}{b} > 0$ then $k > 0$,Substituting (21) into (18) and using (7) the solution of the ZK is given by:

$$u_6(x,y,t) = -\frac{3\lambda}{2a}[1 + \frac{1}{2}tan^2(\frac{1}{4}\sqrt{\frac{\lambda}{2b}}(x + y + \lambda t)) + \frac{1}{2}cot^2(\frac{1}{4}\sqrt{\frac{\lambda}{2b}}(x + y + \lambda t))].$$

Case D:

$$k = -\frac{\lambda}{32b}, \quad a_0 = -\frac{\lambda}{2a}, \quad a_1 = 0, \quad a_2 = -\frac{24b}{a}, \quad b_1 = 0, \quad b_2 = -\frac{3\lambda^2}{128ab}, \tag{22}$$

where λ is an arbitrary constant.
If $\frac{\lambda}{b} < 0$ then $k > 0$,Substituting (22) into (18) and using (7) the solution of the ZK is given by:

$$u_7(x,y,t) = -\frac{\lambda}{2a}[1 - \frac{3}{2}tan^2(\frac{1}{4}\sqrt{-\frac{\lambda}{2b}}(x+y+\lambda t)) - \frac{3}{2}cot^2(\frac{1}{4}\sqrt{-\frac{\lambda}{2b}}(x+y+\lambda t))].$$

If $\frac{\lambda}{b} > 0$ then $k < 0$,Substituting (22) into (18) and using (5) the solution of the ZK is given by:

$$u_8(x,y,t) = -\frac{\lambda}{2a}[1 + \frac{3}{2}tanh^2(\frac{1}{4}\sqrt{\frac{\lambda}{2b}}(x+y+\lambda t)) + \frac{3}{2}coth^2(\frac{1}{4}\sqrt{\frac{\lambda}{2b}}(x+y+\lambda t))].$$

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