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GENERAL HELICES AND BERTRAND CURVES IN RIEMANNIAN SPACE FORM

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Abstract

M.Barros gave the definition of general helices in space form in his paper [4]. In this paper, some characterizations for general helices in space forms are given. Moreover, we show that curvatures of a general helix which it has Bertrand couple holds the equation $\lambda\kappa + \frac{\tau}{\sqrt{c}} = 1$.

Keywords: General helix, Bertrand curve.

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1. Introduction

Helices in E^3 are curves whose tangents make a constant angle with a fixed straight line. In 1802, Lancret proved that the necessary and sufficient condition

for a curve to be a helix is that the ratio of its curvature be constant. M.Barros, in the studies, showed that their exists a relation such as $\tau = b\kappa + a$ for a general helices in 3-dimensional real space-form, where a and b are constant.
Now, let us consider the curve

$$\gamma : I \subset R \to M\,,$$

where M is a 3-dimensional real space-form with sectional curvature c. Let us the denote the tangent vector field of γ as $\gamma'(t) = V(t)$, unit tangent vector field as $T = T(t)$ and the velocity as $v(t) = \|V(t)\| = < V(t), V(t) >^{1/2}$. Then, the Frenet-Serret formula of γ are

$$\nabla_T T = \kappa N$$
$$\nabla_T N = -\kappa T + \tau B$$
$$\nabla_T B = -\tau N$$

where ∇ is Levi-Civita connection in M, $\kappa > 0$ and τ are the curvature and torsion of the curve, respectively.

The variation of γ in M is the find as

$$\Gamma = \Gamma(t,z) : I \times (-\varepsilon, \varepsilon) \to M,\ \Gamma(t,0) = \gamma(t) \tag{1}$$

Note that the vector field $\frac{\partial \Gamma}{\partial z}|_{z=0} = Z(t)$ is a variation vector field. From now on through out the paper and will use the notations $v = v(t,z),\ T = T(t,z),\ V = V(t,z)$, where t is arbitrary and s is the arc parameter of γ.

If $\frac{\partial v}{\partial z}|_{z=0} = \frac{\partial \kappa^2}{\partial z}|_{z=0} = \frac{\partial \tau^2}{\partial z}|_{z=0} = 0$, then the vector field $Z(s)$ along $\gamma(s)$ is called Killing vector field. Note that

$$\frac{\partial v}{\partial z}|_{z=0} = < \nabla_T Z, T > v, \tag{2}$$

$$\frac{\partial \kappa^2}{\partial z}|_{z=0} = 2\kappa < \nabla_T^2 Z, N > -4\kappa^2 < \nabla_T Z, T > +2c\kappa < Z, N >, \tag{3}$$

$$\begin{aligned}\frac{\partial \tau^2}{\partial z}|_{z=0} = \frac{2\tau}{\kappa} < \nabla_T^3 Z, B > - \frac{2\kappa'\tau}{\kappa^2} < \nabla_T^2 Z + cZ, B > \\ + \frac{2\tau(\mathrm{c}+\kappa^2)}{\kappa} < \nabla_T Z, B > -2\tau^2 < \nabla_T Z, T >.\end{aligned} \tag{4}$$

If the angle between the Killing vector field Z which as the find along the curve γ and the tangent of the curve is nonzero constant of each point then the curve γ is called a general helix [4]. Also in [1], it was defined a new type of curves called LC helix when the angle between tangent of this curve and LC parallel vector field in space form is constant.

2. General helix in space form

Theorem 2.1 Let 3-dimensional manifold M be a space form with a constant sectional curvature c and $\gamma = \gamma(s): I \to M$ be a general helix given by an arc parameter. Then we have $\tau = b\kappa + a$, where κ and τ are curvature and torsion of the curve, respectively, and a and b are constants with the condition $a^2 = c$ [4].

Lemma 2.1 Let γ be a regular curve in 3-dimensional Riemannian space. Then, γ satisfies the following equation.

$$\nabla_T^3 T - (2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau})\nabla_T^2 T + \left[-\frac{\kappa''}{\kappa} + \frac{\kappa'}{\kappa}\frac{\tau'}{\tau} + 2(\frac{\kappa'}{\kappa})^2 + \kappa^2 + \tau^2 \right]\nabla_T T + \kappa\tau(\frac{\kappa}{\tau})' T = 0 \tag{5}$$

[3].

Theorem 2.2 Let M be a 3-dimensional real space-form, and γ be a regular curve on M. In this case, γ is a general helix if and only if the equation

$$\nabla_T^3 T + \omega_1 \nabla_T^2 T + \omega_2 \nabla_T T + \omega_3 T = 0 \tag{6}$$

Holds for γ, where $\omega_1 = -\frac{\kappa'}{\kappa}(\frac{3b\kappa + 2a}{b\kappa + a})$, $\omega_2 = \left[-\frac{\kappa''}{\kappa} + (\frac{\kappa'}{\kappa})^2 \left[\frac{3b\kappa + 2a}{b\kappa + a} \right] + \kappa^2 + \tau^2 \right]$ and $\omega_3 = \frac{a\kappa\kappa'}{b\kappa + a}$.

Proof Let γ be a general helix. Since γ is a regular curve, it satisfies the equation in (5). Moreover, since γ is a general helix, then we have $\tau = b\kappa + a$. Since

$$-(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}) = -\frac{\kappa'}{\kappa}(\frac{3b\kappa + 2a}{b\kappa + a}), \tag{7}$$

$$-2\frac{\kappa''}{\kappa} + \frac{\kappa'}{\kappa}\frac{\tau'}{\tau} + 2(\frac{\kappa'}{\kappa})^2 + \kappa^2 + \tau^2 = -\frac{\kappa''}{\kappa} + \left[\frac{3b\kappa + 2a}{b\kappa + a} \right] + \kappa^2 + \tau^2, \tag{8}$$

$$\kappa\tau(\frac{\kappa}{\tau})' = \frac{a\kappa\kappa'}{b\kappa + a}, \tag{9}$$

Then setting these equalities in (5), we obtain equation (6).

Assume that the regular curve γ satisfies the equality in (6). Since γ is a regular curve, it also satisfies the equality in (5). Substracting (6) from (5), we get

$$X\nabla_T^2 T + Y\nabla_T T + ZT = 0, \tag{10}$$

where

$$X = \frac{\kappa'}{\kappa}(\frac{3b\kappa + 2a}{b\kappa + a}) - (2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}), \tag{11}$$

$$Y = \frac{\kappa'}{\kappa}\frac{\tau'}{\tau} + 2(\frac{\kappa'}{\kappa})^2 - (\frac{\kappa'}{\kappa})^2 - \left[\frac{3b\kappa + 2a}{b\kappa + a} \right], \tag{12}$$

$$Z = \kappa\tau(\frac{\kappa}{\tau})' - \frac{a\kappa\kappa'}{b\kappa + a}. \tag{13}$$

Setting the equations

$$\nabla_T T = \kappa N, \tag{14}$$

$$\nabla_T^2 T = \kappa^2 T + \kappa' N + \kappa\tau B, \tag{15}$$

In (10), we obtain

$$X(-\kappa^2 T + \kappa' N + \kappa\tau B) + Y\kappa N + ZT = 0 \tag{16}$$

And

$$(Z - \kappa^2 X)T + (\kappa' X + Y\kappa)N + X\kappa\tau B = 0. \tag{17}$$

Since $\{T, N, B\}$ are linearly independent, we have

$$Z - \kappa^2 X = 0, \tag{18}$$

$$\kappa' X + Y\kappa = 0, \tag{19}$$

$$X\kappa\tau = 0. \tag{20}$$

Hence,

$$\frac{\tau'}{\tau} = \frac{\kappa'}{\kappa} - \frac{\kappa'}{\kappa}\left(\frac{a}{b\kappa + a}\right). \tag{21}$$

Solution of this equation leads to

$$\tau = b\kappa + a. \tag{22}$$

Thus, the regular curve γ is a helix.

Theorem 2.3 Let M be a Riemannian space form, γ be a regular curve on M. If γ is a cyclic helix (i.e. $\kappa = \text{constant}$, $\tau = \text{constant}$) then

$$\nabla_T^3 T + (\kappa^2 + \tau^2)\nabla_T T = 0, \ \kappa = \text{constant}, \ \tau = \text{constant}. \tag{23}$$

Proof The proof is straight forward from Theorem 2.2.

A curve $\gamma : I \subset R \to M$ with $\kappa \neq 0$ is called a Bertrand curve if there exists a curve $\beta : I \subset R \to M$ such that the principal normal lines of γ and β at $s \in I$ are equal. In this case, β is called a Bertrand couple of γ [2].

Theorem 2.4 Let M be 3-dimensional space form with sectional curvature c and γ be a regular curve on M. In this case, γ has Bertrand couple if and only if below the equation holds:

$$\lambda\kappa + \lambda \cot\theta\tau = 1,$$

where κ and τ are curvature and torsion of the curve γ, λ is distance from γ to Bertrand couple, and θ is angle between tangent vector of γ and tangent vector of Bertrand couple of γ [3].

Theorem 2.5 Let M be 3-dimensional space form with sectional curvature c and γ be a regular curve on M. γ is a general helix which it has Bertrand couple and, tangent vector of Bertrand couple of γ and axes of γ is not orthonormal if and only if curvature and torsion of γ are constant.

Proof Let curve β on M be a Bertrand couple of the general helix γ. Let angle between T and T^*, which is tangent vector of β, be θ. Thus, we have

$$T^* = \cos\theta T + \sin\theta N, \tag{24}$$

and

$$Z = \cos\varphi T + \sin\varphi N\,, \tag{25}$$

where φ is the angle between T and Z. In this case, we get

$$\lambda\kappa + \mu\tau = 1,\ \mu = \lambda\cot\theta,\ \lambda,\mu \in R \tag{26}$$

$$\tau = b\kappa + a,\ b = \cot\varphi,\ a^2 = c. \tag{27}$$

Thus, from the equations (26) and (27), we have

$$\kappa = \frac{1-\lambda a\cot\theta}{\lambda(1+\cot\theta\cot\varphi)},$$

$$\tau = \frac{\lambda a + \cot\varphi}{\lambda(1+\cot\theta\cot\varphi)}.$$

Since $1+\cot\theta\cot\varphi \neq 0$, κ and τ are constant.

Now, we suppose that κ and τ be constant. In this case

$$Z = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2+\tau^2}}$$

is the Killing vector field of the curve γ. Therefore, $<Z,T> = \dfrac{-\kappa}{\sqrt{\kappa^2+\tau^2}}$. On the other hand, $\cos\theta = \dfrac{-\kappa}{\sqrt{\kappa^2+\tau^2}}$ is a constant. Furthermore, we choose

$$\beta(s) = \gamma(s) + \frac{1}{\kappa}N(s).$$

Hence, we obtain

$$\kappa^* N^* = -\kappa N\,,$$

where κ^* is curvature of β and N^* is principal normal vector of β. Thus, β is Bertrand couple of the curve γ.

Theorem 2.6 Let M be 3-dimensional space form with sectional curvature c and γ be a regular curve on M. γ is a general helix which it has Bertrand couple and, tangent vector of Bertrand couple of γ and axes of γ is orthonormal if and only if the equation $\lambda\kappa + \dfrac{\tau}{\sqrt{c}} = 1$ is hold, where the constant λ is distance between γ and Bertrand couple of γ.

Proof Let γ be general helix which it has Bertrand couple and, tangent vector of Bertrand couple of γ and axes of γ be orthonormal. Since $\varphi = \dfrac{\pi}{2} + \theta$,

$$T^* = \cos\theta T + \sin\theta N,$$

$$Z = -\sin\theta T + \cos\theta N.$$

Assume that the curve β on M is a Bertrand couple of the general helix γ. Then,

$$\beta(s) = \gamma(s) + \lambda N(s).$$

If we derivative above equation along to γ, we get

$$T^* = \frac{ds}{ds^*}(1-\lambda\kappa)T + \frac{ds}{ds^*}\lambda\tau N,$$

where $\mu = \lambda\cot\theta$ and $\lambda = d(\alpha,\beta) = \text{constant}$.
So, we obtain

$$\frac{\cos\theta}{1-\lambda\kappa} = \frac{\sin\theta}{\lambda\tau},$$

and

$$\kappa\sin\theta + \tau\cos\theta = \frac{\sin\theta}{\lambda}.$$

If we derivative the Killing vector field Z along to γ, we have

$$\nabla_T Z = -\frac{\sin\theta}{\lambda}N, \tag{28}$$

$$\nabla_T^2 Z = -\frac{\sin\theta}{\lambda}(-\kappa T + \tau B), \tag{29}$$

$$\nabla_T^3 Z = -\frac{\sin\theta}{\lambda}\left[-\kappa T - (\kappa^2+\tau^2)N + \tau' B\right]. \tag{30}$$

From equations (28), (29) and (30), we can easily verify that $\frac{\partial v}{\partial z}|_{z=0} = 0$, $\frac{\partial \kappa^2}{\partial z}|_{z=0} = 0$. Since Z is a Killing vector field, for $\frac{\partial \tau^2}{\partial z}|_{z=0} = 0$, then we obtain

$$\cos\theta = \frac{1}{\sqrt{1+c\lambda^2}},$$

and

$$\cot\theta = \frac{1}{\lambda\sqrt{c}}.$$

Because of the Bertrand couple condition, we get $\lambda\kappa + \lambda\cot\theta\tau = 1$. That is,

$$\lambda\kappa + \frac{\tau}{\sqrt{c}} = 1.$$

Now, we consider the equation $\lambda\kappa + \frac{\tau}{\sqrt{c}} = 1$, for the curve γ. From theorem 2.5, the curve γ is a Bertrand curve.
Assume that

$$Z = \frac{-\lambda\sqrt{c}}{\sqrt{1+c\lambda^2}}T + \frac{1}{\sqrt{1+c\lambda^2}}B.$$

In this case, $\frac{\partial v}{\partial z}|_{z=0} = 0$, $\frac{\partial \kappa^2}{\partial z}|_{z=0} = 0$ and $\frac{\partial \tau^2}{\partial z}|_{z=0} = 0$. On the other hand, the curve γ is general helix and $1+\cos\varphi\cos\theta = 0$. This completes the proof.

REFERENCES

[1] A.Şenol and Y.Yaylı, LC helices in space forms, *Chaos, Solitons & Fractals*,Vol: 42-4 (2009) pp 2115-2119.
[2] H.H. Hacisalihoglu, Diferensiyel Geometri, *Ankara University Faculty of Science Press*, (1993).
[3] H. Kocayigit, Biharmonic curves in Lorentz 3-manifolds and contact geometry, Doctoral thesis, *Ankara University, Graduate School of Natural and Applied Sciences* (2004).
[4] M. Barros, General helices and a theorem of Lancret, *Proceedings of the American Mathematical Society*, Vol:125, Number 5 (1997) pp 1503-1509.