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Important Results on Janowski Starlike Log-harmonic Mappings Of Complex Order b

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Abstract

Let $H(D)$ be a linear space of all analytic functions defined on the open unit disc D. A sense-preserving log-harmonic function is the solution of the non-linear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z,$$

where $w(z)$ is analytic, satisfies the condition $|w(z)| < 1$ for every $z \in D$ and is called the second dilatation of f. It has been shown that if f is a non-vanishing log-harmonic mapping then f can be represented by

$$f(z) = h(z)\overline{g(z)},$$

where $h(z)$ and $g(z)$ are analytic in D with $h(0) \neq 0$, $g(0) = 1([1])$. If f vanishes at $z = 0$ but it is not identically zero, then f admits the representation

$$f(z) = z\,|z|^{2\beta}\,h(z)\overline{g(z)},$$

where $Re\beta > -\frac{1}{2}$, $h(z)$ and $g(z)$ are analytic in D with $g(0) = 1$ and $h(0) \neq 0$. The class of sense-preserving log-harmonic mappins is denoted by S_{LH}. We say that f is a Janowski starlike log-harmonic mapping.If

$$1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right) = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

where $\phi(z)$ is Schwarz function. The class of Janowski starlike log-harmonic mappings is denoted by $S^*_{LH}(A, B, b)$. We also note that, if $(zh(z))$ is a starlike function, then the Janowski starlike log-harmonic mappings will be called a perturbated Janowski starlike log-harmonic mappings. And the family of such mappings will be denoted by $S^*_{PLH}(A, B, b)$. The aim of this paper is to give some distortion theorems of the class $S^*_{LH}(A, B, b)$.

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1 Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in D and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in D$.
Next, denote by $P(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + ...$$

regular in D, such that $p(z)$ is in $P(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, \quad -1 \leq B < A \leq 1 \tag{1}$$

for some function $\phi(z) \in \Omega$ and for every $z \in D$. Therefore we have $p(0) = 1$, $Rep(z) > \frac{1-A}{1-B} > 0$ whenever $p(z) \in P(A, B)$. Moreover, let $S^*(A, B)$ denote the family of functions

$$s(z) = z + a_2 z^2 + ...$$

regular in D, and such that $s(z)$ is in S^* if and only if

$$Re\left(z\frac{s'(z)}{s(z)}\right) = p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}, p(z) \in P(1, -1) \tag{2}$$

Let $S_1(z)$ and $S_2(z)$ be analytic functions in D with $S_1(0) = S_2(0)$. We say that $S_1(z)$ subordinated to $S_2(z)$ and denote by $S_1(z) \prec S_2(z)$, if $S_1(z) = S_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in D$. If $S_1(z) \prec S_2(z)$, then $S_1(D) \subset S_2(D)$([5]).
The radius of starlikeness of the class of sense-preserving log-harmonic mapping is

$$r_s = sup\left\{r|Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > 0, 0 < r < 1\right\}.$$

Finally, let $H(D)$ be the linear space of all analytic functions defined on the open unit disc D. A sense-preserving log-harmonic mapping is the solution of the non-linear elliptic partial differantial equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = w(z)\frac{f_z}{f}, \tag{3}$$

where $w(z) \in H(D)$ is the second dilatation of f such that $|w(z)| < 1$ for every $z \in D$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f = h(z)\overline{g(z)} \tag{4}$$

where $h(z)$ and $g(z)$ are analytic functions in D.
On the other hand, if f vanishes at $z = 0$ and at no other point, then f admits the representation,

$$f = z\,|z|^{2\beta}\,h(z)\overline{g(z)}, \tag{5}$$

where $Re\beta > -1/2$, $h(z)$ and $g(z)$ are analytic in D with $g(0) = 1$ and $h(0) \neq 0$. We note that the class of log-harmonic mappings is denoted by S_{LH}.
Let $f = zh(z)\overline{g(z)}$ be an element of D_{LH}. We say that f is a Janowski starlike log-harmonic mapping if

$$1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right) = p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, p(z) \in P(A, B) \tag{6}$$

where $-1 \leq B < A \leq 1$, $b \neq 0$ and complex and denote by $S^*_{LH}(A, B, b)$ the set of all starlike log-harmonic mappings. Also we denote ${}^*_{PLH}(A, B, b)$ the class of all functions in $S^*_{LH}(A, B, b)$ for which $(zh(z)) \in S^*(A, B)$ for all $z \in D$.
We note that if we give special values to b, then we obtain important subclasses of Janowski starlike log-harmonic mappings

i. For $b = 0$, we obtain the class of starlike log-harmonic mappings.

ii. For $b = 1 - \alpha$, $0 \leq \alpha < 1$, we obtain the class of starlike log-harmonic mappings of order α.

iii. For $b = e^{-i\lambda}cos\lambda$, $|\lambda| < \frac{\pi}{2}$, we obtain the class of $\lambda-$ spirallike log-harmonic mappings.

iv. For $b = (1 - \alpha)e^{-i\lambda}cos\lambda$, $0 \leq \alpha < 1$, $|\lambda| < \frac{\pi}{2}$, we obtain the class of $\lambda-$ spirallike log-harmonic mappings of order α.

2 Main Results

Theorem 2.1 *Let $f = zh(z)\overline{g(z)}$ be an element of $S^*_{PLH}(A, B, b)$. Then*

$$f = zh(z)\overline{g(z)} \in S^*_{LH}(A, B, b) \Leftrightarrow z\frac{h'(z)}{h(z)} - \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}} \prec \frac{b(A - B)z}{1 + Bz}; B \neq 0, \tag{7}$$

$$\Leftrightarrow z\frac{h'(z)}{h(z)} - \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}} \prec bAz, B = 0 \tag{8}$$

Proof: Let $f \in S^*_{LH}(A, B, b)$. Using the principle of subordination then we have

$$1 + \frac{1}{b}\left(\frac{zfz - \overline{z}f_{\overline{z}}}{f} - 1\right) = 1 + \frac{1}{b}\left(z\frac{h'(z)}{h(z)} - \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\right) =$$

$$\frac{1+A\phi(z)}{1+B\phi(z)}; B\neq 0,$$

$$1+A\phi(z); B=0,$$

$$\Leftrightarrow z\frac{h'(z)}{h(z)}-\overline{z}\overline{\frac{g'(z)}{g(z)}}=$$

$$\frac{b(A-B)\phi(z)}{1+B\phi(z)}; B\neq 0,$$

$$bA\phi(z); B=0$$

$$\Leftrightarrow z\frac{h'(z)}{h(z)}-\overline{z}\overline{\frac{g'(z)}{g(z)}}\prec$$

$$\frac{b(A-B)z}{1+Bz}; B\neq 0,$$

$$bAz; B=0.$$

Theorem 2.2 *Let* $F=z.\left|z\right|^{2\beta}.H(z).\overline{G(z)}\in S_{LH}$ *and then;*

$$1+\frac{1}{b}(\frac{zF_z-\bar{z}F_{\bar{z}}}{F}-1)=1+\frac{1}{b}(z\frac{H'(z)}{H(z)}-\bar{z}.\overline{\frac{G'(z)}{G(z)}})$$

$$\frac{1+A\phi(z)}{1+B\phi(z)}, B\neq 0;$$

$$1+A\phi(z), B=0;$$

Proof: Let $F=z.\left|z\right|^{2\beta}.H(z).\overline{G(z)}\in S_{LH}$

$$\log F=\log z.\left|z\right|^{2\beta}.H(z).\overline{G(z)}\in S_{LH}$$

$$\log F=\log z+\beta\log z+\beta\log\overline{z}+\log H(z)+\log\overline{G(z)}......\textit{(2.1)}$$

On the other hand we have;

$$F_z=F(\frac{1}{z}+\frac{\beta}{z}+\frac{H'(z)}{H(z)})......\textit{(2.2)}$$

$$F_{\overline{z}}=F(\frac{\beta}{\overline{z}}+\overline{\frac{G'(z)}{G(z)}})......\textit{(2.3)}$$

$f=z.h(z).\overline{g(z)}$ is a log-harmonic mapping;

$$1+\frac{1}{b}(\frac{zf_z-\bar{z}f_{\bar{z}}}{f}-1)=$$

$$\frac{1+A\phi(z)}{1+B\phi(z)}, B \neq 0;$$

$$1+A\phi(z), B=0;$$

Therefore; if $\beta \neq 0$; $F = z.\left|z\right|^{2\beta}.H(z).\overline{G(z)}$
If we make simple calculations at *(2.2)* and *(2.3)* ; we get the result.

Lemma 2.3 *Let* $f(z) = z\left|z\right|^{2\beta} h(z)\overline{g(z)} \in S_{LH}$. *Re*$\beta > -\frac{1}{2}$; $h(z)$ *and* $g(z)$ *are both analytic in* D, $g(0) = 1$ *and* $h(0) \neq 0$. *Then*

$$Re\frac{h(z)}{g(z)} > 0 \Leftrightarrow Re\frac{f(z)}{z\left|z\right|^{2\beta}} > 0 ...\textit{(2.4)}$$

Proof: Let $f = z\left|z\right|^{2\beta} h(z)\overline{g(z)} \in S_{LH}$

$$Re\frac{f(z)}{z\left|z\right|^{2\beta}} > 0 \Rightarrow 0 < Re\frac{\left|z\right|^{2\beta} h(z)\overline{g(z)}}{z\left|z\right|^{2\beta}} = Reh(z)\overline{g(z)}$$

$$= Re\frac{h(z)\overline{g(z)}g(z)}{g(z)} = Re\frac{h(z)\left|g(z)\right|^{2}}{g(z)} = \left|g(z)\right|^{2}.Re\frac{h(z)}{g(z)}$$

satisfied.

$$0 < \left|g(z)\right|^{2}.Re\frac{h(z)}{g(z)} \Rightarrow Re\frac{h(z)}{g(z)} > 0...\textit{(2.5)}$$

satisfied. On the contrary;

$$Re\frac{h(z)}{g(z)} > 0 \Rightarrow Re\frac{h(z)\left|g(z)\right|^{2}}{g(z)} > 0 \Leftrightarrow Re\frac{h(z)\overline{g(z)}g(z)}{g(z)} > 0$$

$$Reh(z).\overline{g(z)} > 0 \Rightarrow Re\frac{\left|z\right|^{2\beta} h(z)\overline{g(z)}}{z\left|z\right|^{2\beta}} > 0...\textit{(2.6)}$$

satisfied. If we use *(2.5)* and *(2.6)* ;we take the expression of *(2.4)*.

Lemma 2.4 *Let* $f = z.h(z)\overline{g(z)} \in S^{*}_{LH}(A,B,b)$ *ve* $\frac{h(z)}{g(z)} = p(z)$
$h(z)$, $g(z)$, $p(z)$ *are all analytic functions at* D. *And their Taylor formulas are* ; $h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$;

$$\left|a_n\right| \leq 2\sum_{k=0}^{n-1}\left|b_k\right| + \left|b_n\right|; \left|b_0\right| = 1$$

Proof: Let $f = zh(z)\overline{g(z)} \in S^*_{LH}(A,B,b)$. Then $h(z) = 1 + a_1z + a_2z^2 + ... + a_nz^n$; $g(z) = 1 + b_1z + b_2z^2 + ... + b_nz^n$; $p(z) = 1 + p_1z + p_2z^2 + ... + p_nz^n$ are like this. Here $\frac{h(z)}{g(z)} = p(z) \Rightarrow$

$$\Rightarrow (1+a_1z+a_2z^2+...+a_nz^n) = (1+p_1z+p_2z^2+...+p_nz^n).(1+b_1z+b_2z^2+...+b_nz^n)$$

satisfied. Then,
$1 + a_1z + a_2z^2 + ... + a_nz^n = 1 + (b_1 + p_1)z + (b_2 + p_1b_1 + p_2)z^2 + (b_3 + p_1b_2 + p_2b_1 + p_3)z^3 + (b_4 + p_1b_3 + p_2b_2 + p_3b_1 + p_4)z^4 + (b_5 + p_1b_4 + p_2b_3 + p_3b_2 + p_4b_1 + p_5)z^5 + ... + (b_n + p_1b_{n-1} + p_2b_{n-2} + ... + p_n)z^n + ...$*(2.7)*
we get the expression . In this expression; If we look coefficient equalities and take their absolute values;
$|a_1| = |b_1 + p_1|$
$|a_2| = |b_2 + p_1b_1 + p_2|$
$|a_3| = |b_3 + p_1b_2 + p_2b_1 + p_3|$
$|a_4| = |b_4 + p_1b_3 + p_2b_2 + p_3b_1 + p_4|$
$|a_5| = |b_5 + p_1b_4 + p_2b_3 + p_3b_2 + p_4b_1 + p_5|$
....
$|a_n| = |b_n + p_1b_{n-1} + p_2b_{n-2} + p_3b_{n-3} + ... + p_n|$
By using $p_n \leq 2$ at all of the equalities
$|a_1| \leq 2 + |b_1|$
$|a_2| \leq 2 + 2|b_1| + |b_2|$
$|a_3| \leq 2 + 2|b_1| + 2|b_2| + |b_3|$
$|a_4| \leq 2 + 2|b_1| + 2|b_2| + 2|b_3| + |b_4|$
$|a_5| \leq 2 + 2|b_1| + 2|b_2| + 2|b_3| + 2|b_4| + |b_5|$
....
$|a_n| \leq 2 + 2|b_1| + 2|b_2| + 2|b_3| + 2|b_4| + 2|b_5| + ... + |b_n|$
then we get the result.

Theorem 2.5 *Let* $f = z|z|^{2\beta} h(z)\overline{g(z)} \in S_{LH}$
$f = zh(z)\overline{g(z)} \in S^*_{LH}(A,B,b)$ *and* $\frac{f}{z.|z|^{2\beta}} = p(z)$ *If* $p(z) = 1 + \sum_{n=1}^{\infty} p_nz^n$ *, then*

$$1 + \frac{1}{b}(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1) = 1 + \frac{1}{b}z\frac{p'(z)}{p(z)}$$

Proof: Let $f = z|z|^{2\beta} h(z)\overline{g(z)} \in S_{LH}$; Using *Lemma (2.3)* $Re\frac{h(z)}{g(z)} > 0 \Leftrightarrow Re\frac{f}{z.|z|^{2\beta}}$ satisfied. Then; take $\frac{f(z)}{z.|z|^{2\beta}} = p(z)$ and from this expression
$f = z|z|^{2\beta} .p(z)$ get the result. First take logarithm of both sides ;

$$\log f = \log z + \beta \log z + \beta \log \overline{z} + \log p...\textit{(2.8)}$$

At *(2.5)* taking derivatives first to z and multiplying by z ;

$$\frac{f_z}{f}=\frac{1}{z}+\frac{\beta}{z}+\frac{p'}{p}$$

$$z\frac{f_z}{f}=1+\beta+z\frac{p'}{p}...(2.9)$$

Now at *(2.8)* take derivative to $\overline{z}$ and multiplying both sides by $\overline{z}$

$$\frac{f_z}{f}=\beta\frac{1}{\overline{z}}$$

$$\overline{z}\frac{f_z}{f}=\beta...(2.10)$$

If we substract from *(2.6)* to *(2.10)*

$$\frac{zf_z-\overline{z}f_{\overline{z}}}{f}=1+z\frac{p'}{p}...(2.11)$$

we take this.
At expression of *(2.11)* multiply both sides by $\frac{1}{b}$ and then add 1 .

Theorem 2.6 *Let* $f=zh(z)\overline{g(z)}\in S^*_{LH}(A,B,b)$. $s(z)=1+\frac{1}{b}(\frac{zf_z-\overline{z}f_{\overline{z}}}{f}-1)$ *and* $s(z)=1+\sum_{n=1}^{\infty}s_nz^n$ *;*
$|s_1|\leq\frac{2}{|b|}$,$|s_2|\leq\frac{8}{|b|}$,$|s_3|\leq\frac{26}{|b|}$, $|s_4|\leq\frac{80}{|b|}$, $|s_5|\leq\frac{202}{|b|}$

Proof: Let $f=zh(z)\overline{g(z)}\in S^*_{LH}(A,B,b)$ and from *Theorem(2.5)*;
$s(z)=1+\frac{1}{b}(\frac{zf_z-\overline{z}f_{\overline{z}}}{f}-1)=1+\frac{1}{b}z\frac{p'(z)}{p(z)}$
$\Rightarrow b.p(z)+z.p'(z)=b.p(z).s(z)$.....(2.12) satisfied.
$p(z)=1+p_1z+p_2z^2+...+p_nz^n$...(2.13)
$s(z)=1+s_1z+s_2z^2+...+s_nz^n$...(2.14)
(2.13) and *(2.14)* if we multiply them by b ;
$b.p(z).s(z)=b+b(s_1+p_1)z+b(s_2+p_1s_1+p_2)z^2+b(s_3+p_1s_2+p_2s_1+p_3)z^3+b(s_4+p_1s_3+p_2s_2+p_3s_1+p_4)z^4+...+b(s_{n-1}+p_1s_{n-2}+p_2s_{n-3}+p_3s_{n-4}+p_4s_{n-5}+...+p_{n-1})z^{n-1}+b(s_n+p_1s_{n-1}+p_2s_{n-2}+p_3s_{n-3}+p_4s_{n-4}+p_{n-1}s_1+p_n)z^n+b(s_{n+1}+p_1s_n+p_2s_{n-1}+p_3s_{n-2}+p_4s_{n-3}+p_{n-1}s_2+p_ns_1+p_{n+1})z^{n+1}+....(2.15)$

On the other hand;

$b.p(z)+z.p'(z) = b(1+p_1z+p_2z^2+p_3z^3+...+p_{n-1}z^{n-1}+p_nz^n+p_{n+1}z^{n+1}+ ...)+z(p_1+2p_2z+3p_3z^2+4p_4z^3+...+(n-1)p_{n-1}z^{n-2}+np_nz^{n-1}+(n+1)p_{n+1}z^n+ (n+2)p_{n+2}z^{n+1} + ...(2.16)$
$= b + bp_1z + bp_2z^2 + bp_3z^3 + ... + bp_{n-1}z^{n-1} + bp_nz^n + bp_{n+1}z^{n+1} + ... + p_1z + 2p_2z^2 + 3p_3z^3 + ... + (n-1)p_{n-1}z^{n-1} + np_nz^n + (n+1)p_{n+1}z^{n+1} + ...(2.17)$
(2.17) can be written;
$b.p(z) + z.p'(z) = b + (p_1 + bp_1)z + (2p_2 + bp_2)z^2 + (3p_3 + bp_3)z^3 + ... + ((n-1)p_{n-1} + bp_{n-1})z^{n-1} + (np_n + bp_n)z^n + ((n+1)p_{n+1} + bp_{n+1})z^{n+1} + ...(2.18)$
If we make an equality between (2.15) and (2.18) then;
$b(s_1 + p_{1)} = p_1 + bp_1$
$b(s_2 + s_1p_1 + p_2) = 2p_2 + bp_2$
$b(s_3 + s_2p_1 + s_1p_2 + p_3) = 3p_3 + bp_3$
$b(s_4 + s_3p_1 + s_2p_2 + s_1p_3 + p_4) = 4p_4 + bp_4$
....
$b(s_{n-1} + s_{n-2}p_1 + s_{n-3}p_2 + s_{n-4}p_3 + ... + p_{n-1}) = (n-1)p_{n-1} + bp_{n-1}$
$b(s_n + s_{n-1}p_1 + s_{n-2}p_2 + s_{n-3}p_3 + ... + s_1p_{n-1} + p_n) = np_n + bp_n$
$b(s_{n+1}+s_np_1+s_{n-1}p_2+s_{n-2}p_3+....+s_2p_{n-1}+s_1p_n+p_{n+1}) = (n+1)p_{n+1}+bp_{n+1}$
satisfed. From here using $|p_n| \leq 2$ inequality orderly; we can take the estimations for first five coefficients easily.

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