

# Two types of traveling wave solutions of a KdV-like advection-dispersion equation

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## Abstract

We present a KdV-like 2-parameter equation  $u_t + (3(1 - \delta)u + (\delta + 1)\frac{u_{xx}}{u_x})u_x = \epsilon u_{xxx}$ . By using the dynamical system method, existence of different traveling wave solutions are discussed, including smooth solitary wave solution of with bell type, solitary wave solutions of valley type and peakon wave solution of valley type. Numerical integration are used to shown the different types of solutions.

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## 1 Introduction

Many nonlinear partial differential equations have been found to have a variety of traveling wave solutions. For instances, the well-known Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \quad (1)$$

has solitary wave solutions and its solitary waves are solitons [1]. Its extension, the Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \quad (2)$$

process peakon solutions[2]. Peakon solutions have a sharp peak with a discontinuous first derivative.

The KdV equation has purely linear dispersion. The KdV soliton is the balance between nonlinear steepening and linear dispersion. However, the CH equation introduces additional higher order combinations of nonlinear/nonlocal

balance. Even in the limit of vanishing linear dispersion, nonlinear dynamics still remains and peakon solutions appear. There are abundant studies on classical solitons and special peakon solitons [3–7].

An equation related to KdV in a similar way, called SIdV equation

$$u_t + \left( \frac{u_{xx}}{u} u_x \right) = \epsilon u_{xxx} \quad (3)$$

was introduced in [8]. What is interesting is that the advecting velocity is a quotient  $2u_{xx}/u$ , not the linear form  $6u$  in the KdV equation. There are special values of  $\epsilon$  at which the SIdV comes close to the KdV equation. Despite of the different advecting velocity, the SIdV equation has the same solitary wave solution as the KdV equation.

The nonlinear advecting form has been studied by Qiao and Li [9]. They derived the following equation

$$\left( -\frac{u_{xx}}{u} \right)_t = 2uu_x. \quad (4)$$

and pointed out that it has classical solitons, periodic soliton and kink solutions.

We are interesting in that whether there are special solitons solutions to other KdV-like equations with advection term  $\frac{u_{xx}}{u}$ . In this paper, we consider the following generalized SIdV equation

$$u_t + \left( 3(1 - \delta)u + (\delta + 1)\frac{u_{xx}}{u} \right) u_x = \epsilon u_{xxx} \quad (5)$$

where  $\delta$  and  $\epsilon$  are constants. It can be thought as a nonlinear wave equation for the dispersive advection of the real wave amplitude  $u$ . Clearly it is a generalization of the KdV-SidV 1-parameter family in [? ]. When  $\delta = \epsilon = 1$ , (5) turns to be the SIdV equation (4). So it interpolates between SIdV and KdV.

We look for traveling wave solutions of (5). We shall apply the bifurcation theory of dynamical systems [10] in this study.

The rest of the paper is organized as follows. Section 2 gives bifurcations conditions of (10) and different phase portraits associated with different parameters. Section 3 concerns the existence of smooth and non-smooth traveling wave solutions of (5). Section 4 is the conclusions.

## 2 Phase portraits of

We look for traveling wave solutions of (5) in the form of

$$u(x, t) = \phi(\xi), \quad \xi = x - ct, \quad (6)$$

where  $c$  is the wave speed. We only consider the situation  $c > 0$ . That means the wave traveling to the right. Substituting (6) into (5) then (5) is reduced to

$$-c\phi\phi' + 3(1 - \delta)\phi^2\phi' + (1 + \delta)\phi'\phi'' - \epsilon\phi\phi''' = 0 \tag{7}$$

where “'''” is the derivative with respect to  $\xi$ . Integrating (7) once and setting the integrating constant as  $g$ , we get

$$2(1 - \delta)\phi^3 - c\phi^2 - 2\epsilon\phi\phi'' + (1 + \delta + \epsilon)\phi'^2 - 2g \tag{8}$$

Eq. (8) is equivalent to the planar system

$$\begin{aligned} \frac{d\phi}{d\xi} &= y \\ \frac{dy}{d\xi} &= \frac{(\delta + 1 + \epsilon)y^2 + 2(1 - \delta)\phi^3 - c\phi^2 - 2g}{2\epsilon\phi} \end{aligned} \tag{9}$$

Since the phase orbits defined by the vector fields of system (9) determine all traveling wave solutions of (5), we shall investigate the bifurcations of the phase portraits of (9) in the phase plane as the parameters are changed. Here we only consider the bounded solutions.

Clearly, system (9) has a singular line  $\phi = 0$ . On the singular straight line of the phase plane  $(\phi, y)$ ,  $\phi''$  has no definition. To avoid the singularity, let  $d\xi = 2\epsilon\phi d\tau$  for  $\phi \neq 0$ . Then system (9) becomes an regular system

$$\begin{aligned} \frac{d\phi}{d\tau} &= 2\epsilon\phi y \\ \frac{dy}{d\tau} &= (\delta + 1 + \epsilon)y^2 + 2\phi^3 - 2\phi^3\delta - \phi^2c - 2g \end{aligned} \tag{10}$$

Both (9) and (10) have the following first integral

$$y^2 - 2\frac{-1 + \delta}{\delta + 1 - 2\epsilon}\phi^3 - \frac{c}{\delta + 1 - \epsilon}\phi^2 - 2\frac{g}{\delta + 1 + \epsilon} = h\phi^{\frac{\delta+1+\epsilon}{\epsilon}}, \tag{11}$$

where  $h$  is an arbitrary constant.

We now investigate the bifurcations of phase portraits of system (10). Let

$$f(\phi) = 2(1 - \delta)\phi^3 - c\phi^2 - 2g$$

Then  $f'(\phi) = 2\phi(3(1 - \delta)\phi - c)$  has two roots  $\phi_1^* = 0$  and  $\phi_2^* = \frac{c}{3(1-\delta)}$  provided  $c^2 - 12(1 - \delta) > 0$ . It follows that  $f''(\phi_1^*) = -2c$ ,  $f''(\phi_2^*) = 2c$ . Then  $f(\phi_1^*) = -2g$  is a local maximum value while  $f(\phi_2^*) = -2g - \frac{c^3}{27(1-\delta)^2}$  is a local minimum value. Without loss of generality, we assume  $0 < \delta < 1$ . Then we can see that  $\phi_1^* < \phi_2^*$ .















