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An identity from the Al-Salam-Carlitz polynomials

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Abstract

In this paper, we derive an identity from the q-integral representations of the Al-Salam-Carlitz polynomials and the q-integral representations of the Rogers-Szegö polynomials[4].

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1 Introduction and main results

The following is the well-known Al-Salam-Carlitz polynomials $\varphi_n^{(a)}(x|q)$ [3]:

$$\varphi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (a;q)_k. \tag{1}$$

The case $a = 0$ in above leads to the Rogers-Szegö polynomials:

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{2}$$

The Rogers-Szegö polynomials play an important role in the theory of orthogonal polynomials. In this paper, by means of the q-integral representations of the Al-Salam-Carlitz polynomials [4], we derive an identity. The main result of this paper is the following theorem:

Theorem 1.1 *If $|a| < 1$, then we have*

$$\begin{aligned} &\sum_{k=0}^{\min\{m,n\}} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a;q)_k a^{m-k}}{(q;q)_{m-k}} \\ = \; &(a;q)_\infty \sum_{k=(m-n)I_{(m\geq n)}}^{\infty} \begin{bmatrix} n+k \\ m \end{bmatrix} \frac{a^k}{(q;q)_k}. \end{aligned} \tag{3}$$

where

$$I_{(m\geq n)}=\begin{cases}1, & \text{when } m\geq n,\\ 0, & \text{when } m<n.\end{cases}$$

2 notations and known results

Before the proof of the theorem, we recall some definitions, notations and known results in [1] which will be used in the proof. Throughout this paper, it is supposed that $0<q<1$. The q-shifted factorials are defined as

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k). \tag{4}$$

We also adopt the following compact notation for multiple q-shifted factorial:

$$(a_1,a_2,...,a_m;q)_n=(a_1;q)_n(a_2;q)_n...(a_m;q)_n, \tag{5}$$

where n is an integer or ∞. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{6}$$

F.H. Jackson defined the q-integral by [2]

$$\int_0^d f(t)d_qt=d(1-q)\sum_{n=0}^{\infty}f(dq^n)q^n, \tag{7}$$

and

$$\int_c^d f(t)d_qt=\int_0^d f(t)d_qt-\int_0^c f(t)d_qt. \tag{8}$$

In [4], the author gave the following q-integral representations of the Al-Salam-Carlitz polynomials

$$\varphi_n^{(a)}(x|q)=\frac{(ax,a;q)_\infty}{(1-q)(q,q/x,x;q)_\infty}\int_x^1\frac{(qt/x,qt;q)_\infty t^n}{(at;q)_\infty}d_qt. \tag{9}$$

provided that no zero factors in the denominator.

If $a=0$ in (9), we have the q-integral representations of the Rogers-Szegö polynomials

$$h_n(x|q)=\frac{1}{(1-q)(q,q/x,x;q)_\infty}\int_x^1(qt/x,qt;q)_\infty t^nd_qt. \tag{10}$$

3 the proof of the theorem

Using the q-integral representations of the Al-Salam-Carlitz polynomials and the q-integral representations of the Rogers-Szegö polynomials, we give the following proof.

Proof. Recall the following formula[1]

$$\frac{1}{(at;q)_\infty} = \sum_{k=0}^{\infty} \frac{(at)^k}{(q;q)_k}, \quad |at| < 1. \tag{11}$$

First, multiplying both sides of (11) by $(qt/x, qt; q)_\infty t^n$ and then taking the q-integral with respect to t from x to 1 gives

$$\int_x^1 \frac{(qt/x, qt; q)_\infty t^n}{(at;q)_\infty} d_q t = \sum_{k=0}^{\infty} \frac{a^k}{(q;q)_k} \int_x^1 (qt/x, qt; q)_\infty t^{k+n} d_q t. \tag{12}$$

Using (9) and (10) into (12) , we obtain

$$\frac{\varphi_n^{(a)}(x|q)}{(ax;q)_\infty} = (a;q)_\infty \sum_{k=0}^{\infty} \frac{a^k h_{n+k}(x|q)}{(q;q)_k}. \tag{13}$$

Substituting

$$\frac{1}{(ax;q)_\infty} = \sum_{k=0}^{\infty} \frac{(ax)^k}{(q;q)_k}, \quad |ax| < 1. \tag{14}$$

into (13) and comparing the coefficients of x^m gives (3).

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