

Characteristic Ideals and Fuzzy Characteristic Ideals of Γ -Semigroups

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Abstract

In this paper the correspondence between the set of all automorphisms of a Γ -semigroup S and those of its operator semigroups have been established. The notions of characteristic ideal and fuzzy characteristic ideal of a semigroup have also been extended to the general setting of Γ -semigroups. Then a bijection between the set of all fuzzy characteristic ideals of a Γ -semigroup S and that of its left operator semigroup has been obtained. This is used to obtain a similar bijection for characteristic ideals.

Mathematics Subject Classification: 08A72, 20M12, 3F55

Keywords: Γ -semigroup, Fuzzy ideal, Fuzzy characteristic ideal, Operator semigroups.

1 Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation[9]. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. In 1981 M.K. Sen[18] introduced the notion of Γ -semigroup as a generalization of semigroup and ternary semigroup. We call this Γ -semigroup a both sided Γ -semigroup. In 1986 M.K. Sen and N.K. Saha[21] modified the definition of Sen's Γ -semigroup. This newly defined Γ -semigroup is known as one sided Γ -semigroup. Γ -semigroups have been analyzed by a lot of mathematicians, for instance by Chattopadhyay[2, 3], Dutta and Adhikari[6], Hila[7, 8], Chinram[4], Saha[16], Sen et al.[19, 20], Seth[22]. T.K. Dutta and N.C. Adhikari[1, 6] mostly worked on both sided Γ -semigroups. They defined operator semigroups of such type of Γ -semigroups and established many results and found out many correspondences. In this paper we have considered both sided Γ -semigroups.

After the introduction of fuzzy sets by Zadeh[26], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroups was defined by Rosenfeld[15] and its structure was investigated. Das characterized fuzzy subgroups by their level subgroups in [5]. Nobuaki Kuroki[13] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subsemigroup was also introduced by Kuroki[12, 13, 14]. In [12], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Others who worked on fuzzy semigroup theory, such as X.Y. Xie[24, 25], Y.B. Jun[10, 11], are mentioned in the bibliography.

In 2007, Uckun Mustafa, Ali Mehmet and Jun Young Bae[23] introduced the notions of intuitionistic fuzzy ideals in Γ -semigroups. Motivated by Kuroki [12, 13, 14], Mustafa et all.[23], S.K. Sardar and S.K. Majumder[17] have initiated the study of Γ -semigroups in terms of fuzzy sets. The purpose of this paper is as stated in the abstract. The organization of the paper is as follows.

In Section 2 we recall some preliminaries for their use in the sequel. In Section 3 we introduce the notions of automorphism, characteristic ideal and fuzzy characteristic ideal of a Γ -semigroup. We then obtain level subset criterion and characteristic function criterion. In Section 4 we introduce the notion of automorphism of operator semigroups of a Γ -semigroup and also the notion

of corresponding automorphism. We then obtain among other results a bijection between the set of all fuzzy characteristic ideals of a Γ -semigroup and that of its left operator semigroup.

2 Preliminaries

We recall the following which will be required in the sequel.

Definition 2.1 [6] *Let S and Γ be two non-empty sets. S is called a Γ -semigroup if there exist mappings from $S \times \Gamma \times S \rightarrow S$, written as $(a, \alpha, b) \rightarrow a\alpha b$, and $\Gamma \times S \times \Gamma \rightarrow \Gamma$, written as $(\alpha, a, \beta) \rightarrow \alpha a \beta$ satisfying the following associative laws $(a\alpha b)\beta c = a(\alpha b\beta)c = \alpha a(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma) \forall a, b, c \in S$ and $\forall \alpha, \beta, \gamma \in \Gamma$.*

Example 1 Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. Then S is a Γ -semigroup where $a\alpha b$ and $\alpha a \beta$ ($a, b \in S, \alpha, \beta \in \Gamma$) denote usual multiplication of integers.

Definition 2.2 [6] *Let S be a Γ -semigroup. By a left(right) ideal of S we mean a non-empty subset A of S such that $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$). By a two sided ideal or simply an ideal, we mean a non-empty subset A of S which is both a left and a right ideal of S .*

Definition 2.3 [6] *Let S be a Γ -semigroup. Let us define a relation ρ on $S \times \Gamma$ as follows: $(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Then ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α) . Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then L is a semigroup with respect to the multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. This semigroup L is called the left operator semigroup of the Γ -semigroup S . Dually the right operator semigroup R of Γ -semigroup S is defined where the multiplication is defined by $[\alpha, a][\beta, b] = [\alpha a \beta, b]$.*

If there exists an element $[e, \delta] \in L$ ($[\gamma, a] \in R$) such that $e\delta s = s$ (resp. $s\gamma a = s$) for all $s \in S$ then $[e, \delta]$ (resp. $[\gamma, a]$) is called the left (resp. right) unity of S .

Definition 2.4 [26] *A function μ from a non-empty set S to the unit interval $[0, 1]$ is called a fuzzy subset of S .*

Definition 2.5 [17] *A non-empty fuzzy subset μ of a Γ -semigroup S is called a fuzzy left ideal(right ideal) of S if $\mu(x\gamma y) \geq \mu(y)$ (resp. $\mu(x\gamma y) \geq \mu(x)$) $\forall x, y \in S, \forall \gamma \in \Gamma$. A non-empty fuzzy subset μ of a Γ -semigroup S is called a fuzzy ideal of S if μ is a fuzzy left ideal and a fuzzy right ideal of S .*

Definition 2.6 [17] Let μ be a fuzzy subset of a set S . Then for $t \in [0, 1]$, the set $\mu_t = \{x \in S : \mu(x) \geq t\}$ is called the t -level subset or simply the level subset of μ .

Theorem 2.7 [17] A non-empty fuzzy subset μ of a Γ -semigroup S is a fuzzy ideal of S if and only if μ_t is an ideal of S for all $t \in [0, 1]$, provided μ_t is non-empty.

Theorem 2.8 [17] Let A be a non-empty subset of a Γ -semigroup S . Then A is an ideal of S if and only if its characteristic function χ_A is a fuzzy ideal of S .

3 Characteristic Ideals and Fuzzy Characteristic Ideals

Unless otherwise stated throughout this paper S stands for a both sided Γ -semigroup with unities.

Definition 3.1 Suppose S is a Γ -semigroup. Then a bijection $f : S \rightarrow S$ is said to be an automorphism if, $\forall x, y \in S$ and $\forall \alpha \in \Gamma$,

- (1) $f(x\alpha y) = f(x)\alpha f(y)$,
- (2) $f(e) = e$, if S has the left unity $[e, \delta]$,
- (3) $f(a) = a$, if S has the right unity $[\gamma, a]$.

Definition 3.2 An ideal A of a Γ -semigroup S is called a characteristic ideal of S if $f(A) = A \forall f \in \text{Aut}(S)$ ¹.

Definition 3.3 A fuzzy ideal μ of a Γ -semigroup S is called a fuzzy characteristic ideal of S if $\mu(f(x)) = \mu(x) \forall x \in S$ and $\forall f \in \text{Aut}(S)$.

Example 2 Let S be the set of all 2×3 matrices and Γ be the set of all 3×2 matrices over the ring of integers. Then S is a Γ -semigroup with respect to usual matrix multiplication. Let μ be a fuzzy subset of S defined as follows:

$$\mu(x) = \begin{cases} 0.3 & \text{if } x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ 0.2 & \text{otherwise} \end{cases}$$

Then μ is a fuzzy characteristic ideal of S .

¹ $\text{Aut}(S)$ denote the set of all automorphisms on S .

Theorem 3.4 *A non-empty fuzzy subset μ of S is a fuzzy characteristic ideal of S if and only if μ_t is a characteristic ideal of S for all $t \in [0, 1]$, provided μ_t is non-empty.*

Proof: Let μ be a fuzzy characteristic ideal of S and $t \in [0, 1]$ be such that $\mu_t \neq \phi$. Then by Theorem 2.7, μ_t is an ideal of S . Let $f \in \text{Aut}(S)$ and $x \in \mu_t$. Then $\mu(x) \geq t$ and $\mu(f(x)) = \mu(x) \geq t$. Hence, $f(x) \in \mu_t$. This implies that $f(\mu_t) \subset \mu_t$. To obtain the reverse inclusion, let $x \in \mu_t$ and $y \in S$ such that $f(y) = x$. Then, $\mu(y) = \mu(f(y)) = \mu(x) \geq t$, and so $y \in \mu_t$. Consequently, $f(y) \in f(\mu_t)$, whence $x \in f(\mu_t)$. Hence $\mu_t \subset f(\mu_t)$. Thus we deduce that $f(\mu_t) = \mu_t$. Therefore, μ_t is a characteristic ideal of S .

Conversely, let μ_t be a characteristic ideal of S for all $t \in [0, 1]$. Let $f \in \text{Aut}(S)$, $x \in S$ and $\mu(x) := t$. Then $x \in \mu_t$. Since, by hypothesis, $f(\mu_t) = \mu_t$, we see that $f(x) \in \mu_t$, and hence $\mu(f(x)) \geq t$. Let $s = \mu(f(x))$. Then, $f(x) \in \mu_s = f(\mu_s)$. Since f is one-one, we deduce that $x \in \mu_s$. This implies that $\mu(x) \geq s$. Hence $t \geq s$. Thus we obtain $\mu(f(x)) = t = \mu(x)$. Hence, μ is a fuzzy characteristic ideal of S .

Theorem 3.5 *A non-empty subset A of S is a characteristic ideal of S if and only if its characteristic function χ_A is a fuzzy characteristic ideal of S .*

Proof: Let A be a characteristic ideal of S . Let $x \in S$. If $x \in A$ then $\chi_A(x) = 1$. Then for all $f \in \text{Aut}(S)$, $f(x) \in f(A) = A$, whence $\chi_A(f(x)) = 1$. If $x \notin A$ then $\chi_A(x) = 0$. Then for all $f \in \text{Aut}(S)$, $f(x) \notin f(A) = A$, whence $\chi_A(f(x)) = 0$. Thus we see that $\chi_A(f(x)) = \chi_A(x) \forall x \in S$ and $\forall f \in \text{Aut}(S)$.

Conversely, let us suppose that χ_A be a fuzzy characteristic ideal of S . Then, by Definition 3.3 and Theorem 2.8, A is an ideal of S . Now, let $f \in \text{Aut}(S)$ and $a \in A$. Then $\chi_A(a) = 1$ and $\chi_A(f(a)) = \chi_A(a)$. Consequently, $f(a) \in A$. Thus we obtain $f(A) \subseteq A$ for all $f \in \text{Aut}(S)$. Since $f \in \text{Aut}(S)$, $f^{-1} \in \text{Aut}(S)$, $f^{-1}(A) \subseteq A \forall f \in \text{Aut}(S)$. Hence $A \subseteq f(A) \forall f \in \text{Aut}(S)$. Consequently, the converse follows.

4 Corresponding Characteristic Ideals and Corresponding Fuzzy Characteristic Ideals

Many results of semigroups could be extended to Γ -semigroups directly and via (left, right) operator semigroups[1] of a Γ -semigroup. In this section the correspondences between the set of all automorphisms of a Γ -semigroup S and those of its operator semigroups have been established. In order to make operator semigroups of a Γ -semigroup work in the context of fuzzy sets as it worked in the study of Γ -semigroups[1, 6], we obtain various relationships between characteristic ideals, fuzzy characteristic ideals of a Γ -semigroup and

that of its operator semigroups. Here, among other results we obtain a bijection between the set of all (fuzzy) characteristic ideals of a Γ -semigroup and that of its operator semigroups.

Definition 4.1 *Let S be a Γ -semigroup and L be the a left operator semigroup of S . Then a bijection $f : L \rightarrow L$ is said to be an automorphism if for all $[x, \alpha], [y, \beta] \in L$,*

- (1) $f([x, \alpha][y, \beta]) = f([x, \alpha])f([y, \beta])$,
- (2) $f([e, \delta]) = [e, \delta]$, if $[e, \delta]$ is the left unity of S ,
- (3) $f([a, \delta]) = [a, \delta]$, if $[\gamma, a]$ is the right unity of S .

For convenience of the readers, we note that for a Γ -semigroup S and its left, right operator semigroups L, R respectively four mappings namely $()^+, ()^{+'}, ()^+, ()^{+'}$ occur. They are defined as follows: For $I \subseteq R, I^* = \{s \in S, [\alpha, s] \in I \forall \alpha \in \Gamma\}$; for $P \subseteq S, P^{+'} = \{[\alpha, x] \in P : s\alpha x \in P \forall s \in S\}$; for $J \subseteq L, J^+ = \{s \in S, [s, \alpha] \in J \forall \alpha \in \Gamma\}$; for $Q \subseteq S, Q^{+'} = \{[x, \alpha] \in Q : x\alpha s \in Q \forall s \in S\}$.

Now we recall the following propositions which will be required in the sequel.

Proposition 4.2 [6] *Let S be a Γ -semigroup with unities and L be its left operator semigroup and A is an ideal of L . Then A^+ is an ideal of S .*

Proposition 4.3 [6] *Let S be a Γ -semigroup with unities and L be its left operator semigroup and B is an ideal of S . Then $B^{+'}$ is an ideal of L .*

Definition 4.4 [17] *Let R and L be respectively the right and left operator semigroups of a Γ -semigroup S . Then for a fuzzy subset μ of R we define a fuzzy subset μ^* of S by $\mu^*(x) = \inf_{\alpha \in \Gamma} \mu([\alpha, x])$, where $x \in S$. For a fuzzy subset η of S we define a fuzzy subset $\eta^{+'}$ of R by $\eta^{+'}([\alpha, x]) = \inf_{s \in S} \eta(s\alpha x)$, where $[\alpha, x] \in R$. For a fuzzy subset δ of L , we define a fuzzy subset δ^+ of S by $\delta^+(x) = \inf_{\alpha \in \Gamma} \delta([x, \alpha])$, where $x \in S$. For a fuzzy subset ν of S we define a fuzzy subset $\nu^{+'}$ of L by $\nu^{+'}([x, \alpha]) = \inf_{s \in S} \nu(x\alpha s)$, where $[x, \alpha] \in L$.*

Proposition 4.5 [17] *Let S be a Γ -semigroup with unities and L be its left operator semigroup and δ is a fuzzy ideal of L . Then δ^+ is a fuzzy ideal of S .*

Proposition 4.6 [17] *Let S be a Γ -semigroup with unities and L be its left operator semigroup and ν is a fuzzy ideal of S . Then $\nu^{+'}$ is a fuzzy ideal of L .*

Lemma 4.7 [17] *Let S be a Γ -semigroup and L be its left operator semigroup. Suppose I is an ideal of S , then (1) $(\lambda_I)^{+'} = \lambda_{I^{+'}}$. Suppose I is an ideal of L then, (2) $(\lambda_I)^+ = \lambda_{I^+}$, where λ_I is the characteristic function of I .*

Theorem 4.8 [6] *Let S be a Γ -semigroup with unities and L be its left operator semigroup. Then there exists an inclusion preserving bijection $A \mapsto A^{+'}$ between the set of all ideals of S and set of all ideals of L .*

Theorem 4.9 [17] *Let S be a Γ -semigroup with unities and L be its left operator semigroup. Then there exists an inclusion preserving bijection $\mu \mapsto \mu^{+'}$ between the set of all fuzzy ideals of S and set of all fuzzy ideals of L .*

Definition 4.10 *Let S be a Γ -semigroup and L be its left operator semigroup. Then for $f \in \text{Aut}(S)$, we define $f^{+'} : L \rightarrow L$ by $f^{+'}([x, \alpha]) = [f(x), \alpha]$.*

Suppose $[x, \alpha] = [y, \beta]$ then $x\alpha s = y\beta s \forall s \in S \Rightarrow f(x\alpha s) = f(y\beta s) \forall s \in S$. Now for $t \in S$, $f(x)\alpha t = f(x)\alpha f(t)$ (since f is onto $\exists t' \in S$ such that $f(t') = t$) $= f(x\alpha t') = f(y\beta t') = f(y)\beta f(t') = f(y)\beta t$. Hence $[f(x), \alpha] = [f(y), \beta] \Rightarrow f^{+'}([x, \alpha]) = f^{+'}([y, \beta])$. Thus $f^{+'}$ is well defined.

Proposition 4.11 *Let S be a Γ -semigroup and L be its left operator semigroup. Let $f \in \text{Aut}(S)$. Then $f^{+'} \in \text{Aut}(L)$.*

Proof: Let $f \in \text{Aut}(S)$ and $[x, \alpha], [y, \beta] \in L$. Then

$$\begin{aligned} f^{+'}([x, \alpha][y, \beta]) &= f^{+'}([x\alpha y, \beta]) = [f(x\alpha y), \beta] = [f(x)\alpha f(y), \beta] \\ &= [f(x), \alpha][f(y), \beta] = f^{+'}([x, \alpha])f^{+'}([y, \beta]). \end{aligned}$$

Hence $f^{+'}$ is an endomorphism of L .

Let $[x, \alpha], [y, \beta] \in L$ such that $f^{+'}([x, \alpha]) = f^{+'}([y, \beta])$. Then $[f(x), \alpha] = [f(y), \beta] \Rightarrow f(x)\alpha s = f(y)\beta s \forall s \in S \Rightarrow$ for $t \in S$, $f(x)\alpha f(t) = f(y)\beta f(t) \Rightarrow f(x\alpha t) = f(y\beta t) \Rightarrow x\alpha t = y\beta t$ (since f is one-one) $\Rightarrow [x, \alpha] = [y, \beta]$. Hence $f^{+'}$ is one-one.

Let $[x, \alpha] \in L$. Then there exist x' such that $f(x') = x$. Hence $f^{+'}([x', \alpha]) = [f(x'), \alpha] = [x, \alpha]$. Consequently, $f^{+'}$ is onto. Suppose L has the left unity $[e, \delta]$. Then for any $\alpha \in \Gamma$, $f^{+'}([e, \alpha]) = [f(e), \alpha] = [e, \alpha]$ (cf. Definition 3.1). Again if S has the right unity $[\gamma, a]$ then $f^{+'}([a, \alpha]) = [f(a), \alpha] = [a, \alpha]$ (cf. Definition 3.1). Hence by Definition 4.1, $f^{+'} \in \text{Aut}(L)$.

We have the following observation.

$L \times S \rightarrow S, ([a, \alpha], s) \mapsto [a, \alpha]s$, is an action of the semigroup L on S . Similarly $S \times R \rightarrow S, (s, [\alpha, a]) \mapsto s[\alpha, a]$, is an action of the semigroup R on S .

We use this observation to formulate the following definition and also to prove the subsequent propositions.

Definition 4.12 *Let S be a Γ -semigroup with right unity $[\gamma, a]$ and L be its left operator semigroup. Then for $f \in \text{Aut}(L)$, we define $f^+ : S \rightarrow S$ by $f^+(s) = f([s, \gamma])a$.*

Proposition 4.13 *Let S be a Γ -semigroup with right unity $[\gamma, a]$ and L be its left operator semigroup. Let $f \in \text{Aut}(L)$. Then $f^+ \in \text{Aut}(S)$.*

Proof: Let $s, t \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} f^+(sat) &= f([sat, \gamma])a = f([s, \alpha][t, \gamma])a \\ &= f([s, \alpha])f([t, \gamma])a = f([s\gamma a, \alpha])f([t, \gamma])a = f([s, \gamma][a, \alpha])f([t, \gamma])a \\ &= f([s, \gamma])f([a, \alpha])f([t, \gamma])a = f([s, \gamma][a, \alpha])f([t, \gamma])a \text{ (cf. Definition 4.1)} \\ &= f([s, \gamma])a\alpha f([t, \gamma])a = f^+(s)\alpha f^+(t). \end{aligned}$$

Hence f^+ is an endomorphism of S .

Let for $s, t \in S$, $f^+(s) = f^+(t)$. Then

$$\begin{aligned} f([s, \gamma])a &= f([t, \gamma])a \\ \Rightarrow [f([s, \gamma])a, \gamma] &= [f([t, \gamma])a, \gamma] \\ \Rightarrow f([s, \gamma])[a, \gamma] &= f([t, \gamma])[a, \gamma] \\ \Rightarrow f([s, \gamma])f([a, \gamma]) &= f([t, \gamma])f([a, \gamma]) \text{ (cf. Definition 4.1)} \\ \Rightarrow f([s\gamma a, \gamma]) &= f([t\gamma a, \gamma]) \\ \Rightarrow [s\gamma a, \gamma] &= [t\gamma a, \gamma] \text{ (since } f \text{ is one-one)} \\ \Rightarrow [s, \gamma] &= [t, \gamma] \\ \Rightarrow s\gamma x &= t\gamma x \forall x \in S \\ \Rightarrow s\gamma a &= t\gamma a \\ \Rightarrow s &= t. \end{aligned}$$

Hence f^+ is one-one.

Let $t \in S$. Then since $f : L \rightarrow L$ is onto, $\exists [x, \alpha] \in L$ such that $f([x, \alpha]) = [t, \gamma]$. Then $f^+(x\alpha a) = f([x\alpha a, \gamma])a = f([x, \alpha][a, \gamma])a = f([x, \alpha])f([a, \gamma])a = [t, \gamma][a, \gamma]a = [t, \gamma]a\gamma a = [t, \gamma]a = t\gamma a = t$. Hence f^+ is onto.

Again if $[e, \delta]$ is the left unity of S then $f^+(e) = f([e, \gamma])a = [e, \gamma]a = e\gamma a = e$ and $f^+(a) = f([a, \gamma])a = [a, \gamma]a = a\gamma a = a$. Consequently, $f^+ \in \text{Aut}(S)$.

Proposition 4.14 *Let S be a Γ -semigroup with left unity $[e, \delta]$, right unity $[\gamma, a]$ and L be its left operator semigroup. Let $f \in \text{Aut}(S)$. Then $(f^{+'})^+ = f$.*

Proof: By Proposition 4.11, $f^{+'} \in \text{Aut}(L)$ whence by Proposition 4.13, $(f^{+'})^+ \in \text{Aut}(S)$. Let $x \in S$. Then $(f^{+'})^+(x) = f^{+'}([x, \gamma])a = [f(x), \gamma]a = f(x)\gamma a = f(x)$. Hence $(f^{+'})^+ = f$.

Proposition 4.15 *Let S be a Γ -semigroup with left unity $[e, \delta]$, right unity $[\gamma, a]$ and L be its left operator semigroup. Let $f \in \text{Aut}(L)$. Then $(f^+)^{+'} = f$.*

Proof: By Proposition 4.13, $f^+ \in \text{Aut}(S)$ whence by Proposition 4.11, $(f^+)^{+'} \in \text{Aut}(L)$. Let $[x, \alpha] \in L$. Then

$$\begin{aligned} (f^+)^{+'}([x, \alpha]) &= [f^+(x), \alpha] = [f([x, \gamma])a, \alpha] \\ &= f([x, \gamma])[a, \alpha] = f([x, \gamma])f([a, \alpha]) \text{ (cf. Definition 4.1)} \\ &= f([x, \gamma][a, \alpha]) = f([x\gamma a, \alpha]) = f([x, \alpha]). \end{aligned}$$

Hence $(f^+)^{+'} = f$.

Theorem 4.16 *Let S be a Γ -semigroup with unities and L be its left operator semigroup. Then there exists a bijection between the set of all automorphisms of S and set of all automorphisms of L .*

Proof: Let us consider the mapping $\phi : \text{Aut}(S) \rightarrow \text{Aut}(L)$, defined by $\phi(f) = f^{+'}$, where $f^{+'}$ is defined in Definition 4.10. Let $f, g \in \text{Aut}(S)$ such that $\phi(f) = \phi(g)$. Then $f^{+'} = g^{+'} \Rightarrow f^{+'}([x, \alpha]) = g^{+'}([x, \alpha]) \forall [x, \alpha] \in L \Rightarrow [f(x), \alpha] = [g(x), \alpha] \Rightarrow f(x)\alpha s = g(x)\alpha s \forall s, x \in S, \forall \alpha \in \Gamma$. In particular, $f(x)\gamma a = g(x)\gamma a \Rightarrow f(x) = g(x) \forall x \in S \Rightarrow f = g$. Hence ϕ is one-one.

Let $f \in \text{Aut}(L)$. Then by Proposition 4.13, $f^+ \in \text{Aut}(S)$. Now $\phi(f^+) = (f^+)^{+'} = f$ (cf. Proposition 4.15). Consequently, ϕ is onto. Hence ϕ is a bijection.

Proposition 4.17 *Let S be a Γ -semigroup with unities and L be its left operator semigroup and δ be a fuzzy characteristic ideal of L . Then δ^+ is a fuzzy characteristic ideal of S , where δ^+ is defined in Definition 4.4.*

Proof: By Proposition 4.5, δ^+ is a fuzzy ideal of the Γ -semigroup S . Let $a \in S$ and $f \in \text{Aut}(S)$. Then by Proposition 4.11, $f^{+'} \in \text{Aut}(L)$. Hence by using Definitions 4.4 and 4.10 we obtain $\delta^+(f(a)) = \inf_{\alpha \in \Gamma} \delta([f(a), \alpha]) = \inf_{\alpha \in \Gamma} \delta(f^{+'}[a, \alpha]) = \inf_{\alpha \in \Gamma} \delta([a, \alpha]) = \delta^+(a)$. Hence δ^+ is a fuzzy characteristic ideal of S .

Proposition 4.18 *Let S be a Γ -semigroup with unities and L be its left operator semigroup and δ be a fuzzy characteristic ideal of S . Then $\delta^{+'}$ is a fuzzy characteristic ideal of L .*

Proof: By Proposition 4.6, $\delta^{+'}$ is a fuzzy ideal of L . Let $[a, \alpha] \in L$ and $g \in \text{Aut}(L)$. Then by Theorem 4.16, $\exists f \in \text{Aut}(S)$ such that $f^{+'} = g$. Now

$$\begin{aligned} \delta^{+'}(g([a, \alpha])) &= \delta^{+'}(f^{+'}([a, \alpha])) \\ &= \delta^{+'}([f(a), \alpha]) \\ &= \inf_{s \in S} \delta(f(a)\alpha s) \\ &= \inf_{t \in S} \delta(f(a)\alpha f(t)) \text{ (since } f \text{ is a bijection)} \\ &= \inf_{t \in S} \delta(f(a\alpha t)) \\ &= \inf_{t \in S} \delta(a\alpha t) \text{ (by using Definition 3.3 since } \delta \text{ is a fuzzy} \\ &\text{characteristic ideal of } S \text{ and } f \in \text{Aut}(S)) \\ &= \delta^{+'}([a, \alpha]). \end{aligned}$$

Hence $\delta^{+'}$ is a fuzzy characteristic ideal of L .

Theorem 4.19 *Let S be a Γ -semigroup with unities and L be its left operator semigroup. Then there exists a bijection between the set of all fuzzy characteristic ideals of S and set of all fuzzy characteristic ideals of L .*

Proof: Let ϕ be the mapping from the set of all fuzzy characteristic ideals of S to that of L . Let μ be a fuzzy characteristic ideal of S . Let us define $\phi(\mu) = \mu^{+'}$. Then by Proposition 4.18, $\phi(\mu)$ is a fuzzy characteristic ideal of L . Let δ be a fuzzy characteristic ideal of L . Then by Proposition 4.17, $\delta^{+'}$ is a fuzzy characteristic ideal of S . Now by Theorem 5.13[17], $(\delta^{+'})^{+'} = \delta$, i.e., $\phi(\delta^{+'}) = \delta$. Hence ϕ is onto. Again if for fuzzy characteristic ideals μ and ν of S , $\phi(\mu) = \phi(\nu)$ then $\mu^{+'} = \nu^{+'} \Rightarrow (\mu^{+'})^{+'} = (\nu^{+'})^{+'} \Rightarrow \mu = \nu$ (by Theorem 5.13[17]). Hence ϕ is one-one. Hence the proof.

To draw an end to the paper we use the above results on correspondence between the fuzzy characteristic ideals of a Γ -semigroup S and that of its operator semigroups and obtain similar correspondence for characteristic ideals.

Proposition 4.20 *Let S be a Γ -semigroup with left unity $[e, \delta]$, right unity $[\gamma, a]$ and L its left operator semigroup. Let I be a characteristic ideal of L . Then $I^{+'}$ is a characteristic ideal of S .*

Proof: Let $f \in \text{Aut}(S)$. Then by Proposition 4.11, $f^{+'} \in \text{Aut}(L)$. Hence $f^{+'}(I) = I$. Let $f(x) \in f(I^{+'})$, where $x \in I^{+'}$. Then $[x, \alpha] \in I \forall \alpha \in \Gamma$. Hence

$f^+([x, \alpha]) \in f^+(I) \forall \alpha \in \Gamma \Rightarrow [f(x), \alpha] \in I \forall \alpha \in \Gamma \Rightarrow f(x) \in I^+$. Thus $f(I^+) \subseteq I^+$. Hence $f^{-1}(I^+) \subseteq I^+$ (since $f \in \text{Aut}(S) \Rightarrow f^{-1} \in \text{Aut}(S) \Rightarrow I^+ \subseteq f(I^+)$). Hence $f(I^+) = I^+$. Consequently, I^+ is a characteristic ideal of S .

Alternative proof: Since I is a characteristic ideal of L , χ_I is a fuzzy characteristic ideal of L . Then by Proposition 4.17, $(\chi_I)^+$ is a fuzzy characteristic ideal of S . Hence by Lemma 4.7, χ_{I^+} is a fuzzy characteristic ideal of S whence by Theorem 3.5, I^+ is characteristic ideal of S .

Corollary 4.21 (With the same notation as above) $f(I^+) = (f^+(I))^+$ for all $f \in \text{Aut}(S)$ and for all characteristic ideal I of L .

Proposition 4.22 Let S be a Γ -semigroup with left unity $[e, \delta]$, right unity $[\gamma, a]$ and L be its left operator semigroup. Let I be a characteristic ideal of S . Then $I^{+'}$ is a characteristic ideal of L .

Proof: Let I be a characteristic ideal of S . Then χ_I is a fuzzy characteristic ideal of S (cf. Theorem 3.5). Then by Proposition 4.18, $(\chi_I)^{+'}$ is a fuzzy characteristic ideal of L . By Lemma 4.7, $\chi_{I^{+'}}$ is a fuzzy characteristic ideal of L . Hence $I^{+'}$ is a characteristic ideal of L (cf. Theorem 3.5).

Corollary 4.23 (With the same notation as above) $f(I^{+'}) = (f^+(I))^{+'}$ for all $f \in \text{Aut}(L)$ and for all characteristic ideal I of S .

Theorem 4.24 Let S be a Γ -semigroup with unities and L be its left operator semigroup. Then there exists an inclusion preserving bijection between the set of all characteristic ideals of S and set of all characteristic ideals of L via the mapping $I \mapsto I^{+'}$.

Proof: Let us denote the mapping by ψ . Let I and J be two characteristic ideals of S such that $\psi(I) = \psi(J)$. Then $I^{+'} = J^{+'} \Rightarrow (I^{+'})^+ = (J^{+'})^+ \Rightarrow I = J$ (cf. Theorem 5.18[17]). So, ψ is one-one.

Let I be a characteristic ideal of L , then by Proposition 4.20, I^+ is a characteristic ideal of S . Also $(I^+)^{+'} = I$. Thus $\psi(I^+) = (I^+)^{+'} = I$. Hence ψ is onto. From Theorem 5.18[17], it follows that ψ is inclusion preserving.

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Received: January, 2012