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Characterization of ternary semigroups in terms of $(\in, \in \vee q_k)$ ideals

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Abstract

In this paper, we present the concepts of $(\in, \in \vee q_k)$-fuzzy ideals in ternary semigroups, which is a generalization of the $(\in, \in \vee q)$ fuzzy ideals of a ternary semigroups. In this regard, we define $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals and prove some basic results using these definitions. Special concentration is paid to $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals. Furthermore, we characterize regular ternary semigroups in terms of these notions.

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1 Introduction

In 1932, Lehmer introduced the concept of a ternary semigroups [22]. A nonempty set X is called a ternary semigroup if there exists a ternary operation $X \times X \times X \to X$; written as $(x_1, x_2, x_3) \longrightarrow x_1x_2x_3$ satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in X$,

$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$.

The algebraic structures of ternary semigroups were studied by several authors, for example, Sioson studied ideals in ternary semigroups [30], Santiago studied regular ternary semigroups [27], Dixit and Dewan presented quasi-ideals and bi-ideals in ternary semigroups [8], Kar and Maity investigated congruences of ternary semigroups [19] and Iampan studied minimal and maximal lateral ideals of ternary semigroups [12]. Further detail on ternary semigroups, we refer the reader to [8,9,10,11,13,18,22,25,30].

In 1965, Zadeh [31] defined fuzzy sets in order to study mathematical vague situations. Many researchers who are involved in studying, applying, refining and teaching fuzzy sets have successfully applied this theory in many different fields. The idea of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [3], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [2,4,5] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. Keeping this in view, Davvaz in [7] introduced the concept of $(\in, \in \vee q)$-fuzzy sub-nearrings (R-subgroups, ideals) of a nearring and investigated some of their interesting properties. Jun and Song [14] discussed general forms of fuzzy interior ideals in semigroups. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [17] and gave some properties of fuzzy bi-ideals in terms of $(\in, \in \vee q)$-fuzzy bi-ideals. Khan et al. [21] characterized ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy interior ideals and gave some generalized forms of interior ideals in ordered semigroups. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra, for example see [1,15,16,21,28,29].

In this paper, we study the concepts of $(\in, \in \vee q_k)$-fuzzy ideals in ternary semigroups, which is a generalization of the $(\in, \in \vee q)$ fuzzy ideals of a ternary semigroups. We investigate some basic results and properties.We also characterize regular ternary semigroups in terms of these notions.

2 Preliminary Notes

A ternary semigroup X is a non-empty set whose elements are closed under the ternary operation [ ] of multiplication and satisfy the associative law defined as [22]:

$[[abc]de] = [a[bcd]e] = [ab[cde]]$ for all $a, b, c, d, e \in X$.

For simplicity we shall write $[abc]$ as abc. For nonempty subsets A, B and C of X, let

$[ABC] := \{[abc] | a \in A, b \in B \text{ and } c \in C\}$.

An element a of a ternary semigroup X is called *regular* [11] if there exist elements $x, y \in X$ such that $a = axaya$. A ternary semigroup X is regular if every element of S is regular. A non-empty subset A of a ternary semigroup X is called *left* (resp. *right, lateral*) *ideal* of X if $X^2A \subseteq A$ (resp. $AX^2 \subseteq A$, $XAX \subseteq A$)[12]. A non-empty subset A of a ternary semigroup X is called *ideal* of X if it is left, right and lateral ideal of X. A non-empty subset B of a ternary semigroup X is called a *ternary subsemigroup* [8] if $B^3 \subseteq B$. A subsemigroup B of a ternary subsemigroup X is called a *bi-ideal* of X if $BXBXB \subseteq B$ [8]. A subset Q of a ternary semigroup X is called a *quasi-ideal* [8] of X if $X^2Q \cap XQX \cap QX^2 \subseteq Q$.

Now, we review some fuzzy logic concepts.

A function f from a non-empty set X to the unit interval $[0, 1]$ of real numbers is called a fuzzy subset of X, that is $f : X \longrightarrow [0, 1]$. For fuzzy subsets f, g of X, $f \leq g$ means that for all $a \in X$, $f(a) \leq g(a)$. The symbols $f \wedge g \wedge h$ and $f \vee g \vee h$ will mean the following fuzzy subsets of X :

$$(f \wedge g \wedge h)(a) = f(a) \wedge g(a) \wedge h(a),$$

$$(f \vee g \vee h)(a) = f(a) \vee g(a) \vee h(a)$$

for all $a \in X$, where $\wedge$ denotes min or infimum and $\vee$ denotes max or supremum.

Definition 2.1 *[26] A fuzzy subset f of a ternary semigroup X is a fuzzy ternary subsemigroup of X if $f(abc) \geq f(a) \wedge f(b) \wedge f(b)$ for all $a, b, c \in X$.*

Definition 2.2 *[26] A fuzzy subset f of a ternary semigroup X is a fuzzy left (resp. lateral, right) ideal of X if $f(abc) \geq f(c)$ (resp. $f(abc) \geq f(b), f(abc) \geq f(a)$) for all $a, b, c \in X$.*

Definition 2.3 *[25] A fuzzy subset f of a ternary semigroup X is a fuzzy bi-ideal of X if*

(i) $f(abc) \geq f(a) \wedge f(b) \wedge f(b)$ and

(ii) $f(abcde) \geq f(a) \wedge f(c) \wedge f(e)$ for all $a, b, c, d, e \in X$.

For fuzzy subsets f, g, h of a ternary semigroup X, the fuzzy product $f \circ g \circ h$ is defined as

$$(f \circ g \circ h)(a) = \begin{cases} \bigvee_{a=xyz} \{f(x) \wedge g(y) \wedge h(z)\} \text{ for } x, y, z \in X. \\ 0 \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

The fuzzy subset "0" and "$\mathcal{X}$" of X are defined as follows:

$$0 : X \longrightarrow [0,1] | x \longmapsto 0(x) = 0,$$

$$\mathcal{X} : X \longrightarrow [0,1] | x \longmapsto \mathcal{X}(x) = 1,$$

for all $x \in X$.

Definition 2.4 *[26] A fuzzy subset f of a ternary semigroup X is a fuzzy quasi-ideal of X if*

$$f(a) \geq \min \{(f \circ \mathcal{X} \circ \mathcal{X})(a), (\mathcal{X} \circ f \circ \mathcal{X})(a), (\mathcal{X} \circ \mathcal{X} \circ f)(a)\}.$$

Let X be a ternary semigroup and f a fuzzy subset of X, then the set of the form

$$f(y) = \begin{cases} t \neq 0 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$$

is called a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set f in a set X, Pu and Liu [24] introduced the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

For any fuzzy set f in a set X, we say that a fuzzy point x_t is

(i) contained in f, denoted by $x_t \in f$, [24] if $f(x) \geq t$.

(ii) quasi-coincident with f, denoted by $x_t q f$, [24] if $f(x) + t > 1$.

For a fuzzy point (x,t) and a fuzzy set f in X, we say that

(iii) $x_t \in \vee q f$ if $x_t \in f$ or $x_t q f$.

(iv) $x_t \bar{\alpha} f$ if $x_t \alpha f$ does not hold for $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

3 $(\in, \in \vee q_k)-$fuzzy ideals

In what follows let X denote a ternary semigroup and k an arbitrary element of $(0, 1]$ unless otherwise specified.

Generalizing the concept of $x_t q f$, Jun [14,16] defined $x_t q_k f$, where $k \in [0, 1)$ as $x_t q_k f$ if $f(x) + t + k > 1$. In this section we define the concepts of $(\in, \in \vee q_k)-$fuzzy ideal and $(\in, \in \vee q_k)-$fuzzy bi-ideal of a ternary semigroup X and study some of their basic properties.

Definition 3.1 *A fuzzy subset f of X is called $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup of X, if for all $a, b, c \in X$ and $t, r, s \in (0, 1]$,*

$$a_t \in f, b_r \in f, c_s \in f \Longrightarrow (abc)_{t \wedge r \wedge s} \in \vee q_k f.$$

Example 3.2 (26) *Consider the set $Z_5^- = \{0, -1, -2, -3, -4\}$. Then $(Z_5^-, \cdot)$ is a ternary semigroup where ternary multiplication "$\cdot$" is defined as follows:*

$\cdot$	0	−1	−2	−3	−4	$\cdot$	0	−1	−2	−3	−4
0	0	0	0	0	0	0	0	0	0	0	0
−1	0	1	2	3	4	1	0	−1	−2	−3	−4
−2	0	2	4	1	3	2	0	−2	−4	−1	−3
−3	0	3	1	4	2	3	0	−3	−1	−4	−2
−4	0	4	3	2	1	4	0	−4	−3	−2	−1

Define a fuzzy subset f of Z_5^- as

$$f(0) = 0.9, f(-1) = f(-2) = 0.8, f(-3) = f(-4) = 0.7,$$

then f is an $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup of Z_5^- for any $k \in (0, 1]$.

Definition 3.3 *A fuzzy subset f of X is called $(\in, \in \vee q_k)-$fuzzy left (resp. lateral, right) ideal of X, if*

$$x_t \in f(resp. y_r \in f z_s \in f) \Longrightarrow (xyz)_t \in \vee q_k f forall x, y, z \in X and t, r, s \in (0, 1].$$

f is called an $(\in, \in \vee q_k)-$fuzzy ideal of X, if it is $(\in, \in \vee q_k)-$fuzzy left ideal, $(\in, \in \vee q_k)-$fuzzy lateral ideal and $(\in, \in \vee q_k)-$fuzzy right ideal of X.

Definition 3.4 *An $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup f of X is called $(\in, \in \vee q_k)-$fuzzy bi-ideal of X, if*

$$a_t \in f, c_r \in f, e_s \in f \Longrightarrow (abcde)_{t \wedge r \wedge s} \in \vee q_k f forall a, b, c, d, e \in X and t, r, s \in (0, 1].$$

Example 3.5 *Consider $X = \{0, a, b\}$. Define the multiplication on X as*

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then $(X, \cdot)$ is a ternary semigroup. Define a fuzzy subset f of X as

$$f(0) = 0.9, f(a) = 0.8, f(b) = 0.7,$$

then f is $(\in, \in \vee q_k)-$fuzzy bi-ideal of X for $k = 0.4$.

Definition 3.6 *A fuzzy subset f of X is called $(\in, \in \vee q_k)-$fuzzy quasi ideal of X, if it satisfies*

$$f(x) \geq \min\left\{(f \circ \mathcal{X} \circ \mathcal{X})(x), (\mathcal{X} \circ \mathsf{f} \circ \mathcal{X})(x), (\mathcal{X} \circ \mathcal{X} \circ \mathsf{f})(x)\ , \frac{1-k}{2}\right\}$$

where $\mathcal{X}$ is the fuzzy subset of X mapping every element of X on 1.

Example 3.7 *For Z_5^- given in Example 6, define a fuzzy subset f as*

$$f(0) = 0.8, f(-1) = 0.4, f(-2) = 0.5, f(-3) = 0.8, f(-4) = 0.3,$$

then f is $(\in, \in \vee q_k)-$fuzzy quasi ideal of Z_5^- for $k = 0.6$.

Proposition 3.8 *If B be a ternary subsemigroup of a ternary semigroup X then f defined by*

$$f(x) = \begin{cases} \geq \frac{1-k}{2} \text{ if } x \in B \\ 0 \text{ otherwise} \end{cases}$$

is

(i) $(q, \in \vee q_k)-$fuzzy ternary subsemigroup of X.

(ii) $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup of X.

Proof 3.9 *(i) Let $x, y, z \in X$ and $t, r, s \in (0, 1]$ be such that $x_t, y_r, z_s q\lambda$ then $f(x)+t > 1, f(y)+r > 1$ and $f(z)+s > 1$ which shows that $f(x), f(y), f(z) > 0$. Thus $x, y, z \in B$. Since B is ternary subsemigroup so $xyz \in B \Longrightarrow f(xyz) \geq \frac{1-k}{2}$.*

If $t \wedge r \wedge s \leq \frac{1-k}{2}$ then

$$f(xyz) \geq t \wedge r \wedge s andso (xyz)_{t \wedge r \wedge s} \in f.$$

If $t \wedge r \wedge s > \frac{1-k}{2}$ then

$$f(xyz) + t \wedge r \wedge s + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1 andso (xyz)_{t \wedge r \wedge s} q_k f.$$

Hence $(xyz)_{t \wedge r \wedge s} \in \vee q_k f$.

(ii) Let $x, y, z \in X$ and $t, r, s \in (0, 1]$ be such that $x_t, y_r, z_s \in f$ then $f(x) \geq t > 0, f(y) \geq r > 0$ and $f(z) \geq s > 0$. Thus $x, y, z \in B$. Since B is ternary subsemigroup so $xyz \in B \Longrightarrow f(xyz) \geq \frac{1-k}{2}$. If $t \wedge r \wedge s \leq \frac{1-k}{2}$ then

$$f(xyz) \geq t \wedge r \wedge s andso (xyz)_{t \wedge r \wedge s} \in f.$$

If $t \wedge r \wedge s > \frac{1-k}{2}$ then

$$f(xyz) + t \wedge r \wedge s + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1 andso (xyz)_{t \wedge r \wedge s} q_k f.$$

Hence $(xyz)_{t \wedge r \wedge s} \in \vee q_k f$.

Theorem 3.10 *Let f be a fuzzy subset of X then f is $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup of X if and only if*

$$f(xyz) \geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\} forall x, y, z \in X and k \in (0, 1].$$

Proof 3.11 $\Longrightarrow$. *Let f be $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup of X. Let us suppose on the contrary that there exist $x, y, z \in S$ such that*

$$f(xyz) < \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}.$$

Choose $t \in (0, 1]$ such that

$$f(xyz) < t \leq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}$$

then $x_t, y_t, z_t \in f$ but $f(xyz) < t \Longrightarrow (xyz)_t \overline{\in} f$. Also

$$f(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1 \Longrightarrow (xyz)_{t \wedge t \wedge t} = (xyz)_t \overline{q_k} f.$$

Thus $(xyz)_t \overline{\in \vee q_k}$, a contradiction. Hence

$$f(xyz) \geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\} forall x, y, z \in X.$$

$\Longleftarrow$. *Conversely assume that $f(xyz) \geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}$ for all $x, y, z \in X$. Let $x_t, , y_r, z_s \in f$ for some $t, r, s \in (0, 1]$ then $f(x) \geq t$, $f(y) \geq r$ and $f(z) \geq s$, so*

$$f(xyz) \geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\} \geq \min\left\{t, r, s, \frac{1-k}{2}\right\}$$

If $t \wedge r \wedge s \leq \frac{1-k}{2}$ then

$$f(xyz) \geq t \wedge r \wedge s so that (xyz)_{t \wedge r \wedge s} \in f.$$

If $t \wedge r \wedge s > \frac{1-k}{2}$ then

$$f(xyz) + t \wedge r \wedge s + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1 so we get (xyz)_{t \wedge r \wedge s} q_k f.$$

Hence $(xyz)_{t \wedge r \wedge s} \in \vee q_k f$.

Definition 3.12 *Let f be a fuzzy subset of X, define $U(f; t) = \{x \in S | f(x) \geq t\}$. We call $U(f; t)$ an upper level cut or upper level set.*

Theorem 3.13 *A non-empty subset f of X is an $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup of X if and only if $U(f;t)(\neq \emptyset)$ is a ternary subsemigroup of X for all $t \in (0, \frac{1-k}{2}]$.*

Proof 3.14 $\Longrightarrow$ *. Let f be an $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup of X. Let $x, y, z \in U(f;t)$ for some $t \in (0, \frac{1-k}{2}]$ then $f(x) \geq t, f(y) \geq t$ and $f(z) \geq t$. Since f is an $(\in, \in \vee q_k)-$fuzzy ternary subsemigroup so*

$$f(xyz) \geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t$$

and so $xyz \in U(f;t)$. Consequently $U(f;t)$ is a ternary subsemigroup of X. $\Longleftarrow$ *. Let $U(f;t)$ be a ternary subsemigroup of X for all $t \in (0, \frac{1-k}{2}]$. Suppose that there exist $x, y, z \in X$ such that*

$$f(xyz) < \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}.$$

Choosing $t \in (0, \frac{1-k}{2}]$ such that

$$f(xyz) < t \leq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}.$$

Then $x, y, z \in U(f;t)$ but $xyz \notin U(f;t)$ which contradicts our supposition. Hence

$$f(xyz) \geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}$$

and thus f is $\in, \in \vee q_k)-$fuzzy ternary subsemigroup.

Theorem 3.15 *Let L be a left (resp. lateral, right) ideal of X and f be a fuzzy subset defined as*

$$f(x) = \begin{cases} \geq \frac{1-k}{2} if x \in L \\ 0 otherwise \end{cases}$$

then

(i) f is $(q, \in \vee q_k)-$fuzzy left (resp. lateral, right) ideal of X.

(ii) f is $(\in, \in \vee q_k)-$fuzzy left (resp. lateral, right) ideal of X.

Proof 3.16 *Proof is similar to Proposition* 3.8.

Theorem 3.17 *A fuzzy subset f of X is an $(\in, \in \vee q_k)-$fuzzy left (resp. lateral, right) ideal of X if and only if*

$$f(xyz) \geq \min\left\{f(z), \tfrac{1-k}{2}\right\}$$

$$\left(resp. f(xyz) \geq \min\left\{f(y), \tfrac{1-k}{2}\right\}, f(xyz) \geq \min\left\{f(x), \tfrac{1-k}{2}\right\}\right).$$

for all $x, y, z \in X$ and $k \in (0, 1]$

Proof 3.18 *Proof is similar to Theorem 3.10.*

By above Theorem, we have the following corollary.

Corollary 3.19 *A fuzzy subset f of X is an $(\in, \in \vee q_k)-$fuzzy ideal of X if and only if it satisfies the following*

(i) $f(xyz) \geq \min\left\{f(z), \frac{1-k}{2}\right\}$,

(ii) $f(xyz) \geq \min\left\{f(y), \frac{1-k}{2}\right\}$,

(iii) $f(xyz) \geq \min\left\{f(x), \frac{1-k}{2}\right\}$ *for all $x, y, z \in X$ and $k \in (0, 1]$.*

Theorem 3.20 *A non-empty subset f of X is an $(\in, \in \vee q_k)-$fuzzy left (resp. lateral, right) ideal of X if and only if $U(f;t)(\neq \emptyset)$ is a left (resp. lateral, right) ideal of X for all $t \in (0, \frac{1-k}{2}]$.*

Proof 3.21 *Proof is similar to Theorem 3.13.*

Example 3.22 *Let $Z^- = X$ be the set of all negative integers, then Z^- is a ternary semigroup. Let $A = 3X$ then*

$$X^2A \subseteq A, AX^2 \subseteq A and XAX \subseteq A.$$

Hence A is an ideal of X. Define $f : X \to [0, 1]$ by

$$f(x) = \begin{cases} \geq t if x \in A \\ 0 otherwise \end{cases}$$

for any $t \in (0, \frac{1-k}{2})$ then $U(f;t) = \{x \in X | f(x) \geq t\} = \{3X\}$. Since $3X$ is an deal of X so by Theorem 3.20 f is an $(\in, \in \vee q_k)-$fuzzy ideal of X.

Theorem 3.23 *Let B be a bi-ideal of X and f be a fuzzy subset defined as*

$$f(x) = \begin{cases} \geq \frac{1-k}{2} if x \in B \\ 0 otherwise \end{cases}$$

then

(i) f *is* $(q, \in \vee q_k)-$*fuzzy bi-ideal of* X.

(ii) f *is* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X.

Proof 3.24 *Proof is similar to proposition*3.8.

Theorem 3.25 *A fuzzy subset* f *of* X *is* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X *if and only if it satisfies the following*

(i) $f(abc) \geq \min\left\{f(a), f(b), f(c), \frac{1-k}{2}\right\}$ *for all* $a, b, c \in X$ *and* $k \in (0, 1]$.

(ii) $f(abcde) \geq \min\left\{f(a), f(c), f(e), \frac{1-k}{2}\right\}$ *for all* $a, b, c, d, e \in X$ *and* $k \in (0, 1]$.

Proof 3.26 *Proof is similar to Theorem 3.10.*

Theorem 3.27 *A non-empty subset* f *of* X *is an* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X *if and only if* $U(f;t)(\neq \emptyset)$ *is a bi-ideal of* X *for all* $t \in (0, \frac{1-k}{2}]$.

Proof 3.28 *Proof is similar to Theorem 3.13.*

Example 3.29 *Let* $Z^- = X$ *be the set of all negative integers, then* Z^- *is a ternary semigroup. Let* $B = 5X$ *then*

$$BXBXB = 5XX5XX5X = 125X \subseteq 5X,$$

hence B *is a bi-ideal of* X. *Define* $f : X \to [0, 1]$ *by*

$$f(x) = \begin{cases} \geq t if x \in B \\ 0 otherwise \end{cases}$$

for any $t \in (0, \frac{1-k}{2})$ *then* $U(f;t) = \{x \in X | f(x) \geq t\} = \{5X\}$. *Since* $5X$ *is a bi-ideal of* X *so by Theorem 3.27* f *is* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X.

Theorem 3.30 *Let* f *be* $(\in, \in \vee q_k)-$*fuzzy quasi ideal of* X, *then the set* $f_\circ = \{x \in X | f(x) > 0\}$ *is a quasi-ideal of* X.

Proof 3.31 *In order to prove that* $f_\circ$ *is a quasi-ideal of* X, *we need to show that* $X^2 f_\circ \cap X f_\circ X \cap f_\circ X^2 \subseteq f_\circ$. *Let* $a \in X^2 f_\circ \cap X f_\circ X \cap f_\circ X^2$. *This implies that* $a \in X^2 f_\circ$, $a \in X f_\circ X$ *and* $a \in f_\circ X^2$. *Thus there exist* $x_1, y_1, x_2, y_2, x_3, y_3$ *in* X *and* a_1, a_2, a_3 *in* $f_\circ$ *such that* $a = x_1 y_1 a_1$, $a = x_2 a_2 y_2$ *and* $a = a_3 x_3 y_3$ *thus* $f(a_1) > 0$, $f(a_2) > 0$ *and* $f(a_3) > 0$. *Since*

$$\begin{aligned}(f \circ \mathcal{X} \circ \mathcal{X})(a) &= \bigvee_{a = p p_1 q_1} \{f(p) \wedge \mathcal{X}(p_1) \wedge \mathcal{X}(q_1)\} \\ &\geq f(a_3) \wedge \mathcal{X}(x_3) \wedge \mathcal{X}(y_3) \\ &= f(a_3)\end{aligned}$$

Similarly $(\mathcal{X}\circ f\circ\mathcal{X})(\mathsf{x}) \geq f(a_2)$ *and* $(\mathcal{X}\circ\mathcal{X}\circ f)(\mathsf{x}) \geq f(a_1)$. *Thus*

$$\begin{aligned} f(a) &\geq \min\left\{(\mathcal{X}\circ\mathcal{X}\circ\mathsf{f})(a), (\mathcal{X}\circ\mathsf{f}\circ\mathcal{X})(a), (\mathsf{f}\circ\mathcal{X}\circ\mathcal{X})(a), \frac{1-k}{2}\right\} \\ &\geq \min\left\{f(a_1), \mathsf{f}(a_2), f(a_3), \frac{1-k}{2}\right\} \\ &> 0 \;\; \textit{because} \;\; f(a_3) > 0, \mathsf{f}(a_2) > 0 \;\; \textit{and} \;\; f(a_1) > 0 \end{aligned}$$

Thus $a \in f_\circ$. *Hence* $f_\circ$ *is a quasi-ideal of* X.

Lemma 3.32 *A non-empty subset* Q *of* X *is a quasi-ideal of* X *if and only if* C_Q *is an* $(\in, \in \vee q_k)-$*fuzzy quasi-ideal of* X.

Proof 3.33 $\Longrightarrow$. *Suppose* Q *is a quasi-ideal of* X *and* C_Q *is the characteristic function of* Q. *if* $x \notin Q$ *then* $x \notin X^2Q$ *or* $x \notin XQX$ *or* $x \notin QX^2$. *Thus* $(\mathcal{X}\circ\mathcal{X}\circ\mathsf{C_Q})(\mathsf{x}) = 0$ *or* $(\mathcal{X}\circ\mathsf{C_Q}\circ\mathcal{X})(\mathsf{x}) = 0$ *or* $(\mathsf{C_Q}\circ\mathcal{X}\circ\mathcal{X})(\mathsf{x}) = 0$ *and so* $\min\left\{(\mathcal{X}\circ\mathcal{X}\circ C_Q)(x), (\mathcal{X}\circ\mathsf{C_Q}\circ\mathcal{X})(\mathsf{x}), (\mathsf{C_Q}\circ\mathcal{X}\circ\mathcal{X})(\mathsf{x}), \frac{1-k}{2}\right\} = 0 = C_Q(\mathsf{x})$. *If* $x \in Q$ *then* $C_Q(\mathsf{x}) = 1 \geq \min\left\{(\mathcal{X}\circ\mathcal{X}\circ C_Q)(x), (\mathcal{X}\circ\mathsf{C_Q}\circ\mathcal{X})(\mathsf{x}), (\mathsf{C_Q}\circ\mathcal{X}\circ\mathcal{X})(\mathsf{x}), \frac{1-k}{2}\right\}$. *Hence* C_Q *is* $(\in, \in \vee q_k)-$*fuzzy* .
$\Longleftarrow$. *Assume that* C_Q *is* $(\in, \in \vee q_k)-$*fuzzy quasi-ideal of* X. *Then by Theorem (3.30)* Q *is quasi-ideal of* X.

Lemma 3.34 *Every* $(\in, \in \vee q_k)-$*fuzzy left (resp. lateral, right) ideal of* X *is* $(\in, \in \vee q_k)-$*fuzzy quasi-ideal of* X.

Proof 3.35 *Let* $a \in X$ *and let* f *be* $(\in, \in \vee q_k)-$*fuzzy left ideal then*

$$(\mathcal{X}\circ\mathcal{X}\circ\mathsf{f})(\mathsf{a}) = \bigvee_{a=p_1p_2p_3} \{\mathcal{X}(p_1), \mathcal{X}(p_2), f(p_3)\} = \bigvee_{a=p_1p_2p_3} f(p_3)$$

This implies that

$$\begin{aligned} (\mathcal{X}\circ\mathcal{X}\circ\mathsf{f})(a)\wedge\frac{1-k}{2} &= \left(\bigvee_{a=p_1p_2p_3} f(p_3)\right) \wedge \frac{1-k}{2} \\ &= \bigvee_{a=p_1p_2p_3} \left(f(p_3) \wedge \frac{1-k}{2}\right) \\ &\leq \bigvee_{a=p_1p_2p_3} f(p_1p_2p_3) \;\; \textit{because } f \textit{ is } (\in, \in \vee q_k) - \textit{fuzzy left ideal of } X. \\ &= f(a) \end{aligned}$$

Thus $(\mathcal{X}\circ\mathcal{X}\circ f)(a)\wedge\frac{1-k}{2} \leq f(a)$. *Hence*

$$f(a) \geq (\mathcal{X}\circ\mathcal{X}\circ\mathsf{f})(a)\wedge\frac{1-k}{2} \geq \min\left\{(f\circ\mathcal{X}\circ\mathcal{X})(x), (\mathcal{X}\circ\mathsf{f}\circ\mathcal{X})(x), (\mathcal{X}\circ\mathcal{X}\circ\mathsf{f})(x)\;, \frac{1-k}{2}\right\}.$$

Thus f *is* $(\in, \in \vee q_k)-$*fuzzy quasi-ideal of* X.

Theorem 3.36 *Every* $(\in, \in \vee q_k)-$*fuzzy quasi-ideal of* X *is* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X.

Proof 3.37 *Suppose that* f *is* $(\in, \in \vee q_k)-$*fuzzy quasi-ideal of* X. *Now*

$$\begin{aligned} f(xyz) &\geq (f \circ \mathcal{X} \circ \mathcal{X})(xyz) \wedge (\mathcal{X} \circ \mathsf{f} \circ \mathcal{X})(xyz) \wedge (\mathcal{X} \circ \mathcal{X} \circ \mathsf{f})(xyz) \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{xyz=abc} f(a) \wedge \mathcal{X}(b) \wedge \mathcal{X}(c) \right] \wedge \left[\bigvee_{xyz=pqr} \mathcal{X}(p) \wedge f(q) \wedge \mathcal{X}(r) \right] \\ &\wedge \left[\bigvee_{xyz=p_1q_1r_1} \mathcal{X}(p_1) \wedge \mathcal{X}(q_1) \wedge f(r_1) \right] \wedge \frac{1-k}{2} \\ &\geq [f(x) \wedge \mathcal{X}(y) \wedge \mathcal{X}(z)] \wedge [\mathcal{X}(x) \wedge f(y) \wedge \mathcal{X}(z)] \wedge [\mathcal{X}(x) \wedge \mathcal{X}(y) \wedge f(z)] \wedge \frac{1-k}{2} \\ &= f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \end{aligned}$$

so $f(xyz) \geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}$. *In the same way it can be proved that* $f(abcde) \geq \min\left\{f(a), f(c), f(e)\frac{1-k}{2}\right\}$. *Thus* f *is* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X.

Theorem 3.38 *Every* $(\in, \in \vee q_k)-$*fuzzy left (resp. lateral, right) ideal of* X *is* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X.

Proof 3.39 *Let* f *be* $(\in, \in \vee q_k)-$*fuzzy left (resp. lateral, right) ideal of* X *then*

$$f(abc) \geq \min\left\{f(c), \frac{1-k}{2}\right\} forall a, b, c \in X.$$

Suppose that there exists $x, y, z \in X$ *such that*

$$f(xyz) < \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}.$$

Choose $t \in (0, 1]$ *such that*

$$f(xyz) < t \leq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}$$

then $z_t \in f$ *but* $(xyz)_t \overline{\in} f$. *Also*

$$f(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1,$$

which implies $(xyz)_t \overline{q_k} f$. *Thus* $(xyz)_t \overline{\in \vee q_k} f$, *which contradicts the hypothesis. Thus*

$$f(xyz) \geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}.$$

Similarly let there exist $a, b, c, d, e \in X$ *such that*

$$f(abcde) < \min\left\{f(a), f(c), f(e), \frac{1-k}{2}\right\}.$$

Choose $t \in (0, 1]$ *such that*

$$f(a(bcd)e) < t \leq \min\left\{f(a), f(c), f(e), \frac{1-k}{2}\right\}$$

then $e_t \in f$ *but* $(a(bcd)e)_t \overline{\in} f$. *Also*

$$f(abcde) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$$

for all $a, b, c, d, e \in X$, *thus* $(abcde)_t \overline{q_k} f$. *Hence* $(abcde)_t \overline{\in \vee q_k} f$. *A contradiction, hence*

$$f(abcde) \geq \min\left\{f(a), f(c), f(e), \frac{1-k}{2}\right\}.$$

Consequently f *is* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X.

4 Regular Ternary Semigroups

In this section we characterize regular ternary semigroups by the properties of their $(\in, \in \vee q_k)-$fuzzy ideals and $(\in, \in \vee q_k)-$fuzzy bi-ideals.

Definition 4.1 *Let* f, f_1 *and* f_2 *be fuzzy subsets of* X. *We define* $f_k, f \wedge_k f_1 \wedge_k f_2, f \vee_k f_1 \vee_k f_2, f \circ_k f_1 \circ_k f_2$ *as follows.*

$f_k(x) = f(x) \wedge \frac{1-k}{2}$

$(f \wedge_k f_1 \wedge_k f_2)(x) = (f \wedge f_1 \wedge f_2)(x) \wedge \frac{1-k}{2}$

$(f \vee_k f_1 \vee_k f_2)(x) = (f \vee f_1 \vee f_2)(x) \wedge \frac{1-k}{2}$

$(f \circ_k f_1 \circ_k f_2)(x) = (f \circ f_1 \circ f_2)(x) \wedge \frac{1-k}{2}$ *for all* $x \in X$.

Lemma 4.2 *[28] Let* f, f_1 *be fuzzy subsets of a semigroup* S. *Then the following hold.*

(1) $(f \wedge_k f_1) = (f_k \wedge f_{1k})$

(2) $(f \vee_k f_1) = (f_k \vee f_{1k})$

(3) $(f \circ_k f_1) = (f_k \circ f_{1k})$.

Lemma 4.3 *Let* f, f_1 *and* f_2 *be fuzzy subsets of* X. *Then the following hold.*

(i) $(f \wedge_k f_1 \wedge_k f_2) = (f_k \wedge f_{1k} \wedge f_{2k})$

(ii) $(f \vee_k f_1 \vee_k f_2) = (f_k \vee f_{1k} \vee f_{2k})$

(iii) $(f \circ_k f_1 \circ_k f_2) = (f_k \circ f_{1k} \circ f_{2k})$.

Proof 4.4 *Proof is a consequence of Lemma* 4.2.

Lemma 4.5 *[28] Let A, B be non-empty subsets of a semigroup S. Then the following hold.*

(1) $(C_A \wedge_k C_B) = (C_{A\cap B})_k$

(2) $(C_A \vee_k C_B) = (C_{A\cup B})_k$

(3) $(C_A \circ_k C_B) = (C_{AB})_k$.

Lemma 4.6 *Let A, B and D be non-empty subsets of a ternary semigroup X. Then the following hold.*

(1) $(C_A \wedge_k C_B \wedge_k C_D) = (C_{A\cap B\cap D})_k$

(2) $(C_A \vee_k C_B \vee_k C_D) = (C_{A\cup B\cup D})_k$

(3) $(C_A \circ_k C_B \circ_k C_D) = (C_{ABD})_k$.

Proof 4.7 *Follows from Lemma* 4.5.

Lemma 4.8 *Let L be a non-empty subset of a ternary semigroup X then L is left (resp. lateral, right) ideal of X if and only if $(C_L)_k$ is $(\in, \in \vee q_k)-$fuzzy left (resp. lateral, right) ideal of X.*

Proof 4.9 $\Longrightarrow$ *. Let L be a left ideal of a ternary semigroup X then by Theorem 3.15 $(C_L)_k$ is $(\in, \in \vee q_k)-$fuzzy left ideal of X.*
$\Longleftarrow$ *. Suppose $(C_L)_k$ be $(\in, \in \vee q_k)-$fuzzy left ideal of X. Let $z \in L$ then $(C_L)_k(z) = \frac{1-k}{2}$. So $z_{\frac{1-k}{2}} \in (C_L)_k$. Since $(C_L)_k$ is $(\in, \in \vee q_k)-$fuzzy left ideal of X so, we have,*

$$(C_L)_k(xyz) \geq \frac{1-k}{2} \text{ or } (C_L)_k(xyz) + \frac{1-k}{2} + k > 1 \text{ for all } x, y \in X.$$

If $(C_L)_k(xyz) + \frac{1-k}{2} + k > 1$ then $(C_L)_k(xyz) > \frac{1-k}{2}$, thus $(C_L)_k(xyz) \geq \frac{1-k}{2}$ which gives $(C_L)_k(xyz) = \frac{1-k}{2}$ and hence $xyz \in L$ for all $x, y \in X$ and $z \in L$. Consequently L is a left ideal of X.

Lemma 4.10 *Let B be a non-empty subset of a ternary semigroup X then B is bi-ideal of X if and only if $(C_B)_k$ is $(\in, \in \vee q_k)-$fuzzy bi-ideal of X.*

Proof 4.11 *Similar to Theorem 4.8.*

Proposition 4.12 *Let f be an $(\in, \in \vee q_k)-$fuzzy left (resp. lateral, right) ideal of X then f_k is a fuzzy left (resp. lateral, right) ideal of X.*

Proof 4.13 *Let f be an $(\in, \in \vee q_k)-$fuzzy left ideal of X then for all $a, b, c \in X$ we have*

$$f(abc) \geq \min\left\{f(c), \frac{1-k}{2}\right\}.$$

Which implies that

$$f(abc) \wedge \frac{1-k}{2} \geq \min\left\{f(c), \frac{1-k}{2}\right\} \Longrightarrow f_k(abc) \geq f_k(c),$$

so f_k is fuzzy left ideal of X.

Theorem 4.14 *[8] A ternary semigroup X is regular if and only if for every left ideal L, lateral ideal T and right ideal R of X we have $R \cap T \cap L = RTL$.*

Theorem 4.15 *For a ternary semigroup X the following are equivalent.*

(i) X is regular.

(ii) $(f \wedge_k f_1 \wedge_k f_2) = (f \circ_k f_1 \circ_k f_2)$ for every $(\in, \in \vee q_k)-$fuzzy right ideal f, $(\in, \in \vee q_k)-$fuzzy lateral ideal f_1 and $(\in, \in \vee q_k)-$fuzzy left ideal f_2 of X.

Proof 4.16 *(i) $\Longrightarrow$ (ii) Let X be a regular ternary semigroup and let f be $(\in, \in \vee q_k)-$fuzzy right ideal, f_1 be $(\in, \in \vee q_k)-$fuzzy lateral ideal and f_2 be $(\in, \in \vee q_k)-$fuzzy left ideal of X . Now*

$$\begin{aligned}
(f \circ_k f_1 \circ_k f_2)(a) &= (f \circ f_1 \circ f_2)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a=bcd} f(b) \wedge f_1(c) \wedge f_2(d)\right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=bcd} \left\{f(b) \wedge f_1(c) \wedge f_2(d) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{a=bcd} \left\{\left(f(b) \wedge \frac{1-k}{2}\right) \wedge \left(f_1(c) \wedge \frac{1-k}{2}\right) \wedge \left(f_2(d) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}\right\} \\
&\leq \bigvee_{a=bcd} \left\{f(bcd) \wedge f_1(bcd) \wedge f_2(bcd) \wedge \frac{1-k}{2}\right\} \\
&= f(a) \wedge f_1(a) \wedge f_2(a) \wedge \frac{1-k}{2} \\
&= (f_k \wedge f_{1k} \wedge f_{2k})(a),
\end{aligned}$$

so $(f \circ_k f_1 \circ_k f_2) \leq (f \wedge_k f_1 \wedge_k f_2)$.

On the other hand since X *is regular so for* $a \in X$ *there exist* $x, y \in X$ *such that* $a = axaya = axayaxaya$. *Thus*

$$\begin{aligned}(f \circ_k f_1 \circ_k f_2)(a) &= (f \circ f_1 \circ f_2)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a=bcd} f(b) \wedge f_1(c) \wedge f_2(d)\right) \wedge \frac{1-k}{2} \\ &= \bigvee_{a=bcd} \left\{f(b) \wedge f_1(c) \wedge f_2(d) \wedge \frac{1-k}{2}\right\} \\ &\geq f(axa) \wedge f_1(yax) \wedge f_2(aya) \wedge \frac{1-k}{2} \\ &\geq \left(f(a) \wedge \frac{1-k}{2}\right) \wedge \left(f_1(a) \wedge \frac{1-k}{2}\right) \wedge \left(f_2(a) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2} \\ &= f(a) \wedge \frac{1-k}{2} \wedge f_1(a) \wedge \frac{1-k}{2} \wedge f_2(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \\ &= f(a) \wedge f_1(a) \wedge f_2(a) \wedge \frac{1-k}{2} \\ &= (f_k \wedge f_{1k} \wedge f_{2k})(a).\end{aligned}$$

Hence $(f \circ_k f_1 \circ_k f_2) = (f \wedge_k f_1 \wedge_k f_2)(a)$.

$(ii) \Longrightarrow (i)$. *Suppose that* R, T *and* L *be right, lateral and left ideals of* X *respectively then by Lemma 4.8* $(C_R)_k, (C_T)_k$ *and* $(C_L)_k$ *are* $(\in, \in \vee q_k)-$*fuzzy right ideal,* $(\in, \in \vee q_k)-$*fuzzy lateral ideal and* $(\in, \in \vee q_k)-$*fuzzy left ideal of* X *respectively. Thus we have*

$$\begin{aligned}(C_{RTL})_k &= C_R \circ_k C_T \circ_k C_L \text{ by Lemma 4.6} \\ &= C_R \wedge_k C_T \wedge_k C_L \text{ by (ii) above} \\ &= (C_{R\cap T\cap L})_k \text{ by Lemma 4.6}\end{aligned}$$

Thus $RTL = R \cap T \cap L$, *and thus* X *is regular.*

Theorem 4.17 *The following assertions for a ternary semigroup* X *are equivalent.*

(i) X *is regular.*

(ii) $f_k = f \circ_k \mathcal{X} \circ_k f \circ_k \mathcal{X} \circ_k f$ *for every* $(\in, \in \vee q_k)-$*fuzzy bi-ideal* f *of* X.

Proof 4.18 $(i) \Longrightarrow (ii)$ *Let* X *be regular ternary semigroup and* f *be* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X. *Since* X *is regular so for* $a \in X$ *there exist* $x, y \in X$

such that $a = axaya$. *Now*

$$\begin{aligned}
(f \circ_k \mathcal{X} \circ_k f \circ_k \mathcal{X} \circ_k f)(a) &= \left(\bigvee_{a=bcd} (f \circ_k \mathcal{X} \circ_k f)(b) \wedge \mathcal{X}(c) \wedge f(d)\right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=bcd} \left((f \circ_k \mathcal{X} \circ_k f)(b) \wedge \mathcal{X}(c) \wedge f(d) \wedge \frac{1-k}{2}\right) \\
&= \bigvee_{a=bcd} \left\{\left(\bigvee_{b=xyz} f(x) \wedge \mathcal{X}(y) \wedge f(z) \wedge \frac{1-k}{2}\right) \wedge \mathcal{X}(c) \wedge f(d) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{a=(xyz)cd} f(x) \wedge \mathcal{X}(y) \wedge f(z) \wedge \mathcal{X}(c) \wedge f(d) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge \mathcal{X}(x) \wedge f(a) \wedge \mathcal{X}(y) \wedge f(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge 1 \wedge f(a) \wedge 1 \wedge f(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge \frac{1-k}{2} \\
&= f_k(a)
\end{aligned}$$

So $(f \circ_k \mathcal{X} \circ_k f \circ_k \mathcal{X} \circ_k f) \geq f_k$. *On the other hand*

$$\begin{aligned}
(f \circ_k \mathcal{X} \circ_k f \circ_k \mathcal{X} \circ_k f)(a) &= \left(\bigvee_{a=bcd} (f \circ_k \mathcal{X} \circ_k f)(b) \wedge \mathcal{X}(c) \wedge f(d)\right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=bcd} \left((f \circ_k \mathcal{X} \circ_k f)(b) \wedge \mathcal{X}(c) \wedge f(d) \wedge \frac{1-k}{2}\right) \\
&= \bigvee_{a=bcd} \left\{\left(\bigvee_{b=xyz} f(x) \wedge \mathcal{X}(y) \wedge f(z) \wedge \frac{1-k}{2}\right) \wedge \mathcal{X}(c) \wedge f(d) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{a=(xyz)cd} \left(f(x) \wedge \mathcal{X}(y) \wedge f(z) \wedge \mathcal{X}(c) \wedge f(d) \wedge \frac{1-k}{2}\right) \\
&= \bigvee_{a=(xyz)cd} \left(f(x) \wedge f(z) \wedge f(d) \wedge \frac{1-k}{2}\right) \\
&\leq \bigvee_{a=(xyz)cd} f(xyzcd) \wedge \frac{1-k}{2} \ \textit{ as } f \textit{ is } (\in, \in \vee q_k) - \textit{fuzzy bi-ideal of } X \\
&= f(a) \wedge \frac{1-k}{2} \\
&= f_k(a)
\end{aligned}$$

Hence we get $(f \circ_k C_{\mathcal{X}} \circ_k f \circ_k \mathcal{X} \circ_k f) \leq f_k$. *Thus*

$$(f \circ_k \mathcal{X} \circ_k f \circ_k \mathcal{X} \circ_k f) = f_k.$$

$(ii) \implies (i)$ *Let* B *be a bi-ideal of* X *then by Lemma 4.10* $(C_B)_k$ *is* $(\in, \in \vee q_k)-$*fuzzy bi-ideal of* X. *Let* $a \in B$ *then* $(C_B)_k(a) = \frac{1-k}{2}$. *Now by* (ii) *above*

$$\begin{aligned}
(C_B)_k(a) &= (C_B \circ_k \mathcal{X} \circ_k C_B \circ_k \mathcal{X} \circ_k C_B)(a) \\
&= \bigvee_{a=bfg} (C_B \circ_k \mathcal{X} \circ_k C_B)(b) \wedge \mathcal{X}(f) \wedge C_B(g) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=bfg} \left\{ \bigvee_{b=cde} C_B(c) \wedge \mathcal{X}(d) \wedge C_B(e) \wedge \frac{1-k}{2} \right\} \wedge \mathcal{X}(f) \wedge C_B(g) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=(cde)fg} C_B(c) \wedge \mathcal{X}(d) \wedge C_B(e) \wedge \mathcal{X}(f) \wedge C_B(g) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=(cde)fg} C_B(c) \wedge C_B(e) \wedge C_B(g) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=(cde)fg} \left(C_B(c) \wedge \frac{1-k}{2}\right) \wedge \left(C_B(e) \wedge \frac{1-k}{2}\right) \wedge \left(C_B(g) \wedge \frac{1-k}{2}\right) \\
&= \bigvee_{a=(cde)fg} (C_B(c))_k \wedge (C_B(e))_k \wedge (C_B(g))_k \\
&= \frac{1-k}{2} \text{ as } (C_B)_k(a) = \frac{1-k}{2}
\end{aligned}$$

Which is only possible if $c, e, g \in B$. *Thus* $cdefg \in BXBXB$ *and hence* $a \in BXBXB$ *so* $B \subseteq BXBXB$ *but* $BXBXB \subseteq B$ *so* X *is regular.*

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