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A Short Proof for k-Gon Partitions

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Abstract

A k-gon partition is a non-decreasing sequence of k positive integers such that the last element is less than the sum of the others. By considering non-k-gon partitions, we derive the multivariable generating function for k-gon partitions, as given by Andrews, Paule and Riese.

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1 Introduction

The k-gon partitions were introduced by Andrews, Paule and Riese as a partition counterpart to Hermite's problem (see [2] and references there). They are sequences $a_1 \leq a_2 \leq \cdots \leq a_k$ of positive integers such that $a_k < a_1 + \cdots + a_{k-1}$. Based on MacMahon's Partition Analysis, Andrews, Paule and Riese [3] derived the following multivariable generating function for k-gon partitions.

Theorem 1.1. *Let T_k be the set of k-gon partitions. Then*

$$\sum_{(a_1,\ldots,a_k)\in T_k} x_1^{a_1}\cdots x_k^{a_k} = \frac{x_1x_2\cdots x_k}{(1-x_1\cdots x_k)(1-x_2\cdots x_k)\cdots(1-x_k)}$$
$$-\frac{x_1x_2\cdots x_{k-1}x_k^{k-1}}{(1-x_k)(1-x_1\cdots x_{k-1}x_k^{k-1})(1-x_2\cdots x_{k-1}x_k^{k-2})\cdots(1-x_{k-1}x_k)}. \quad (1)$$

Hirschhorn [4] proved this formula by substitutions. Xin [5] reduced the calculation of Andrews, Paule and Riese using substitutions and exclusion formula ([5, Equation (6.1)]). In this paper, we provide a simple proof for Theorem 1.1.

2 Proof of Theorem 1.1

Denote by P_k the set of all partitions with k parts, i.e., non-decreasing sequences of k positive integers. Then $P_k \setminus T_k$ consists of sequences $a_1 \leq \cdots \leq a_k$ satisfying $a_k \geq a_1 + \cdots + a_{k-1}$. Setting $d = a_k - (a_1 + \cdots + a_{k-1})$, then we have

$$\begin{aligned}
&\sum_{(a_1,\ldots,a_k)\in P_k\setminus T_k} x_1^{a_1}\cdots x_k^{a_k} \\
&= \sum_{(a_1,\ldots,a_{k-1})\in P_{k-1}} (x_1x_k)^{a_1}\cdots(x_{k-1}x_k)^{a_{k-1}} \times \sum_{d\geq 0} x_k^d \\
&= \frac{1}{1-x_k}\sum_{(a_1,\ldots,a_{k-1})\in P_{k-1}} (x_1x_k)^{a_1}\cdots(x_{k-1}x_k)^{a_{k-1}}. \qquad (2)
\end{aligned}$$

Now using the same substitutions $b_1 = a_2 - a_1, \ldots, b_k = a_k - a_{k-1}$ as given by Hirschhorn and Xin, we easily derive that

$$\begin{aligned}
&\sum_{(a_1,\ldots,a_k)\in P_k} x_1^{a_1}\cdots x_k^{a_k} \\
&= \sum_{a_1\geq 0, b_i\geq 0} (x_1\cdots x_k)^{a_1}(x_2\cdots x_k)^{b_1}(x_3\cdots x_k)^{b_2}\cdots(x_k)^{b_{k-1}} \\
&= \frac{x_1\cdots x_k}{1-x_1\cdots x_k}\frac{1}{1-x_2\cdots x_k}\cdots\frac{1}{1-x_k}. \qquad (3)
\end{aligned}$$

Combining (2) and (3), we immediately obtain Equation (1). ∎

Especially, setting $x_i = q$ in Equation (1) leads to

Corollary 2.1. *Let $T_k(n)$ be the set of k-gon partitions of n, i.e., $T_k(n) = \{(a_1, \ldots, a_k) \in T_k : a_1 + \cdots + a_k = n\}$. Then*

$$\sum_{n\geq k} |T_k(n)|q^n = \frac{q^k}{(1-q)\cdots(1-q^k)} - \frac{1}{1-q}\frac{q^{2k-2}}{(1-q^2)(1-q^4)\cdots(1-q^{2(k-1)})}. \qquad (4)$$

This special case can also be derived by using the following bijection. Let $P_k(n) = \{(a_1, \ldots, a_k) \in P_k : a_1 + \cdots + a_k = n\}$, and

$$E_k(n) = \{(e_1, \ldots, e_{k-1}, e_k) : 1 \leq e_1 \leq \cdots \leq e_{k-1}, e_k \geq 0, \text{ and } 2e_1+\cdots+2e_{k-1}+e_k = n\}.$$

We have the obvious bijection

$$\begin{aligned}
\phi: \quad P_k(n)\setminus T_k(n) &\to E_k(n) \\
(a_1,\ldots,a_k) &\to (a_1,\ldots,a_{k-1},a_k-(a_1+\cdots+a_{k-1})).
\end{aligned}$$

Then Equation (4) follows from the well-known generating functions

$$\frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)}$$

for partitions with k parts (see, for example, [1]).

References

[1] G. E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and Its Applications, Vol. 2, G.-C. Rota ed., Addison-Wesley, Reading, 1976.

[2] G. E. Andrews, P. Paule and Axel Riese, MacMahon's Partition Analysis: the Omega package, European J. Combin. 22 (2001), 887–904.

[3] G. E. Andrews, P. Paule and Axel Riese, MacMahon's Partition Analysis IX: k-Gon Partitions, Bull. Austral. Math. Soc. 64 (2001), 231–242.

[4] M. D. Hirschhorn, k-gon partitions, Bull. Austral. Math. Soc. 66 (1) (2002), 149–150.

[5] G. C. Xin, A fast algorithm for MacMahon's partition analysis, Electron. J. Combin. 11 (2004), R58.

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