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LC Algebras and LC Frames

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Abstract

We investigate the properties of LC (resp. LD) algebras and α-LC (resp. β-LC) frames. We show that each α-LC frame induces a complete LC algebra. Moreover, each β-LC frame induces a complete LD algebra.

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1 Introduction

Urquhart [13] showed that the dual space of a bounded lattice is a doubly ordered topological space. This viewpoint develops many representation theorems for various algebraic structures [1,5,6]. On the other hand, Bělohlávek [2-4] developed the notion of lattice structures with $R \in L^{X\times Y}$ on a complete residuated lattice L. Lattice structures are important mathematical tools for data analysis and knowledge processing [2-4,11]. In [10], using doubly fuzzy preordered sets, we defined l-stable and r-stable fuzzy sets and showed that the family of l-stable fuzzy sets is a bounded lattice.

In this paper, we investigate the properties of LC (resp. LD) algebras and α-LC (resp. β-LC) frames as a sense in [5,6]. We show that every α-LC frame induces an complete LC algebra. Moreover, each β-LC frame induces a complete LD algebra.

2 Preliminaries

Definition 2.1 [8,9,12] A triple $(L, \leq, \odot)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L, \leq, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(L, \odot, 1)$ is a commutative monoid;

(L3) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Example 2.2 [8,9,12] (1) Each frame $(L, \leq, \wedge)$ is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a complete residuated lattice.

(3) Define a binary operation $\odot$ on $[0,1]$ by $x \odot y = \max\{0, x+y-1\}$. Then $([0,1], \leq, \odot)$ is a complete residuated lattice.

Let $(L, \leq, \odot)$ be a complete residuated lattice. A order reversing map $^* : L \to L$ defined by $a^* = a \to 0$ is called a *strong negation* if $(a^*)^* = a$ for each $a \in L$.

In this paper, we assume $(L, \leq, \odot, ^*)$ is a complete residuated lattice with a strong negation *.

Definition 2.3 [8,9,12] Let X be a set. A function $e_X : X \times X \to L$ is called *fuzzy preorder* on X if it satisfies the following conditions:

(E1) $e_X(x,x) = 1$ for all $x \in X$,

(E2) $e_X(x,y) \odot e_X(y,z) \leq e_X(x,z)$, for all $x, y, z \in X$,

The pair (X, e_X) is a *fuzzy preorder set*.

Let e^1_X, e^2_X be fuzzy preorder on X. A structure (X, e^1_X, e^2_X) is called a doubly fuzzy preordered set. If for all $x, y \in X$, $e^1_X(x,y) = e^2_X(x,y) = 1$ implies $x = y$, (X, e^1_X, e^2_X) is called a doubly fuzzy ordered set.

Let (X, e^1_X, e^2_X) and (Y, e^1_Y, e^2_Y) be doubly fuzzy preordered sets. A function $f : X \to Y$ is a doubly order preserving map if $e^i_X(x,y) \leq e^i_Y(f(x), f(y))$ for all $x, y \in X$ and $i \in \{1,2\}$.

Lemma 2.4 [8,9,12] *For each $x, y, z, x_i, y_i \in L$, we define $x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$. Then the following properties hold.*

(1) *If $y \leq z$, $(x \odot y) \leq (x \odot z)$ and $x \to y \leq x \to z$ and $z \to x \leq y \to x$.*

(2) *$x \odot y \leq x \wedge y$ and $x \odot (x \to y) \leq y$.*

(3) *$x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.*

(4) *$(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.*

(5) *$x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$*

(6) *$(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$.*

(7) *$\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.*

(8) *$(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.*

(9) $1 \to x = x$.
(10) $x \leq y$ *iff* $x \to y = 1$.
(11) $(x \to y) \odot (y \to z) \leq x \to z$.
(12) $(x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \odot x_2 \to y_1 \odot y_2)$.

Example 2.5 (1) We define a map $e_L : L \times L \to L$ $e_L(x,y) = x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$ and $e_L^{-1}(x,y) = e_L(y,x)$. Then (L, e_L, e_L^{-1}) is a doubly fuzzy ordered set from Lemma 2.4 (10-11).

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(f,g) = \bigwedge_{x\in X}(f(x) \to g(x))$. Then (L^X, e_{L^X}) is a fuzzy preordered set.

(3) If (X, e_X) is a fuzzy preordered set and we define a function $e_X^{-1}(x,y) = e_X(y,x)$, then (X, e_X^{-1}) is a fuzzy preordered set.

Definition 2.6 [10] Let e_X^1, e_X^2 be fuzzy preorder on X.
(1) $A \in L^X$ is e_X^1-extensional iff $A(x) \odot e_X^1(x,y) \leq A(y)$.
(2) $B \in L^X$ is e_X^2-extensional iff $B(x) \odot e_X^2(x,y) \leq B(y)$.
The family of e_X^1-extensional (resp. e_X^2-extensional) fuzzy sets will be denoted by $E_1(L^X)$ (resp. $E_2(L^X)$).

Definition 2.7 [10] Let (X, e_X^1, e_X^2) be a doubly fuzzy preorderd set. We define maps $l, r : L^X \to L^X$ as

$$l(A)(x) = \bigwedge_{y\in X} (e_X^1(x,y) \to A^*(y)),$$

$$r(A)(x) = \bigwedge_{y\in X} (e_X^2(x,y) \to A^*(y)).$$

A fuzzy set $A \in L^X$ is called l-stable (resp. l-stable) iff $lr(A) = A$ (resp. rl(A)=A). The family of all l-stable (resp. r-stable) fuzzy sets will be denoted by $L(L^X)$ (resp. $R(L^X)$).

Theorem 2.8 [10] *Let (X, e_X^1, e_X^2) be a doubly fuzzy preorderd set. We have the following properties.*
(1) $l(A) \in E_1(L^X)$ *and* $l(A) \leq A^*$.
(2) $r(A) \in E_2(L^X)$ *and* $r(A) \leq A^*$.
(3) *If* $A \in E_1(L^X)$, *then* $A \leq lr(A)$.
(4) *If* $A \in E_2(L^X)$, *then* $A \leq rl(A)$.
(5) *If* A *is* $(e_X^2)^{-1}$*-extensional, then* $lr(A) = l(A^*) \leq A$.
(6) *If* A *is* $(e_X^1)^{-1}$*-extensional, then* $rl(A) = r(A^*) \leq A$.
(7) *If* $A \in E_1(L^X)$, *then* $r(A) \in R(L^X)$.
(8) *If* $A \in E_2(L^X)$, *then* $l(A) \in L(L^X)$.
(9) *If* $A \in L(L^X)$, *then* $r(A) \in R(L^X)$.
(10) *If* $A \in R(L^X)$, *then* $l(A) \in L(L^X)$.
(11) *If* $A, B \in L(L^X)$, *then* $r(A) \wedge r(B) \in R(L^X)$.

3 LC algebra and LC Frames

Definition 3.1 Let $(X, \wedge, \vee, 0, 1)$ be a bounded lattice.

(1) The structure $(X, \wedge, \vee, 0, 1, \gamma)$ is called an *LC algebra* if a map $\gamma : X \to X$ satisfies the following conditions:

(C1) $\gamma(\gamma(a)) = a$,

(C2) $\gamma(a \vee b) = \gamma(a) \vee \gamma(b)$.

If $(X, \wedge, \vee, 0, 1)$ is a complete lattice, $(X, \wedge, \vee, 0, 1, \gamma)$ is called a *complete LC algebra*.

(2) The structure $(X, \wedge, \vee, 0, 1, \eta)$ is called an *LD algebra* if a map $\eta : X \to X$ satisfies the following conditions:

(D1) $a \leq \eta(\eta(a))$,

(D2) $a \leq \eta(b)$ iff $b \leq \eta(a)$.

If $(X, \wedge, \vee, 0, 1)$ is a complete lattice, $(X, \wedge, \vee, 0, 1, \eta)$ is called a *complete LD algebra*.

Lemma 3.2 Let $(X, \wedge, \vee, 0, 1, \gamma)$ be an LC algebra. Then the following properties hold:

(1) $\gamma(0) = 0$ and $\gamma(1) = 1$.

(2) If $a \leq b$, then $\gamma(a) \leq \gamma(b)$.

(3) $\gamma(a \wedge b) = \gamma(a) \wedge \gamma(b)$.

Proof. (1) It follows from:

$$\gamma(0) = 0 \vee \gamma(0) = \gamma(\gamma(0)) \vee \gamma(0) = \gamma(\gamma(0) \vee 0) = \gamma(\gamma(0)) = 0,$$

$$1 = 1 \vee \gamma(1) = \gamma(\gamma(1)) \vee \gamma(1) = \gamma(\gamma(1) \vee 1) = \gamma(1).$$

(2) Let $a \leq b$ be given. Then $a \vee b = b$. Thus $\gamma(a \vee b) = \gamma(a) \vee \gamma(b) = \gamma(b)$. Hence $\gamma(a) \leq \gamma(b)$.

(3) By (2), $\gamma(a \wedge b) \leq \gamma(a), \gamma(a \wedge b) \leq \gamma(b)$. Then $\gamma(a \wedge b) \leq \gamma(a) \wedge \gamma(b)$. Since $\gamma(a) \wedge \gamma(b) \leq \gamma(a)$, $\gamma(\gamma(a) \wedge \gamma(b)) \leq \gamma(\gamma(a)) = a$ and $\gamma(\gamma(a) \wedge \gamma(b)) \leq \gamma(\gamma(b)) = b$. Thus $\gamma(\gamma(a) \wedge \gamma(b)) \leq a \wedge b$ implies $\gamma(a) \wedge \gamma(b) \leq \gamma(a \wedge b)$.

Lemma 3.3 Let $(X, \wedge, \vee, 0, 1, \eta)$ be an LD algebra. Then the following properties hold:

(1) $a = \eta(\eta(a))$, for all $a \in X$.

(2) If $a \leq b$, then $\eta(a) \geq \eta(b)$.

(3) $\eta(a \wedge b) = \eta(a) \vee \eta(b)$ and $\eta(a \vee b) = \eta(a) \wedge \eta(b)$.

(4) $\eta(0) = 1$ and $\eta(1) = 0$.

Proof. (1) Since $\eta(a) \leq \eta(a)$, by (D2), $a \leq \eta(\eta(a))$.

(2) Let $a \leq b = \eta(\eta(b))$ be given. By (D2), $\eta(a) \geq \eta(b)$.

(3) By (2), $\eta(a \wedge b) \geq \eta(a), \eta(a \wedge b) \geq \eta(b)$. Then $\eta(a \wedge b) \geq \eta(a) \vee \eta(b)$. Since $\eta(a) \vee \eta(b) \geq \eta(a)$, $\eta(a) \vee \eta(b) \geq \eta(b)$ and $\eta(\eta(a) \vee \eta(b)) \leq \eta(\eta(a)) = a$, $\eta(\eta(a) \vee \eta(b)) \leq \eta(\eta(b)) = b$, we have $\eta(\eta(a) \vee \eta(b)) \leq a \wedge b$ implies $\eta(a) \vee \eta(b) \leq \eta(a \wedge b)$. Hence $\eta(a \wedge b) = \eta(a) \vee \eta(b)$. Similarly, $\eta(a \vee b) = \eta(a) \wedge \eta(b)$.

(4) It follows from:

$$\eta(0) = \eta(0 \wedge \eta(1)) = \eta(0) \vee \eta(\eta(1)) = \eta(0) \vee 1$$

$$0 \vee \eta(1) = \eta(\eta(0)) \vee \eta(1) = \eta(\eta(0) \wedge 1) = 0.$$

Remark 3.4 Let $(X, \wedge, \vee, 0, 1, \gamma)$ be an LC algebra. By Lemma 3.2, we regard $\gamma : X \to X$ as a lattice isomorphism.

Definition 3.5 Let (X, e^1_X, e^2_X) be a doubly fuzzy preordered set. A structure $(X, e^1_X, e^2_X, \alpha)$ is called an *α-LC-frame* with a map $\alpha : X \to X$ satisfying the following conditions:

(A1) $e^1_X(x, y) \leq e^1_X(\alpha(x), \alpha(y))$,

(A2) $e^2_X(x, y) \leq e^2_X(\alpha(x), \alpha(y))$,

(A3) $\alpha(\alpha(x)) = x$.

Remark 3.6 Let $(X, e^1_X, e^2_X, \alpha)$ be an α-LC-frame. Then

$$e^i_X(x, y) \leq e^i_X(\alpha(x), \alpha(y)) \leq e^i_X(\alpha(\alpha(x)), \alpha(\alpha(y))) = e^i_X(x, y).$$

Hence $e^i_X(x, y) = e^i_X(\alpha(x), \alpha(y))$, for $i = 1, 2$. Furthermore, $\alpha(x) = \alpha(y)$ implies $x = \alpha(\alpha(x)) = \alpha(\alpha(y)) = y$. Thus α is injective. By (A3), α is surjective. Hence α is a bijective function. Furthermore, α, α^{-1} are doubly order preserving maps, then α is a doubly order isomorphism.

Theorem 3.7 *Let $(X, e^1_X, e^2_X, \alpha)$ be an α-LC-frame. Define a map $\alpha^{\to} : L^X \to L^X$ as*

$$\alpha^{\to}(A)(y) = \bigvee_{x \in \alpha^{-1}(\{y\})} A(x)$$

Then we have the following properties.

(1) $\alpha^{\to}(A)(\alpha(x)) = A(x), \alpha^{\to}(A)(x) = A(\alpha(x))$.

(2) $\alpha^{\to}(\alpha^{\to}(A)) = A$.

(3) $\alpha^{\to}(A) \leq \alpha^{\to}(B)$ *for* $A \leq B$,

(4) $\alpha^{\to}(A \wedge B) = \alpha^{\to}(A) \wedge \alpha^{\to}(B)$.

(5) $\alpha^{\to}(A \vee B) = \alpha^{\to}(A) \vee \alpha^{\to}(B)$.

(6) $l(\alpha^{\to}(A)) = \alpha^{\to}(l(A))$.

(7) $r(\alpha^{\to}(A)) = \alpha^{\to}(r(A))$.

(8) *If A is l-stable, then so is $\alpha^{\to}(A)$.*

Proof. (1) Since α is bijective,

$$\alpha^{\rightarrow}(A)(\alpha(x)) = \bigvee_{w\in\alpha^{-1}(\{\alpha(x)\})} A(w) = A(x),$$

$$\alpha^{\rightarrow}(A)(\alpha(\alpha(x))) = \alpha^{\rightarrow}(A)(x) = A(\alpha(x))..$$

(2) Since $\alpha(\alpha(y)) = y$ and α is bijective,

$$\alpha^{\rightarrow}(\alpha^{\rightarrow}(A))(y) = \alpha^{\rightarrow}(A)(\alpha(y)) = A(\alpha(\alpha(y))) = A(y).$$

(3) It is easy.

(4) By (3), since $\alpha^{\rightarrow}(A \wedge B) \leq \alpha^{\rightarrow}(A)$ and $\alpha^{\rightarrow}(A \wedge B) \leq \alpha^{\rightarrow}(B)$, then $\alpha^{\rightarrow}(A \wedge B) \leq \alpha^{\rightarrow}(A) \wedge \alpha^{\rightarrow}(B)$.

Furthermore, $\alpha^{\rightarrow}(\alpha^{\rightarrow}(A) \wedge \alpha^{\rightarrow}(B)) \leq \alpha^{\rightarrow}(\alpha^{\rightarrow}(A)) \wedge \alpha^{\rightarrow}(\alpha^{\rightarrow}(B)) = A \wedge B$. Thus $\alpha^{\rightarrow}(A) \wedge \alpha^{\rightarrow}(B) = \alpha^{\rightarrow}(\alpha^{\rightarrow}(\alpha^{\rightarrow}(A) \wedge \alpha^{\rightarrow}(B)) \leq \alpha^{\rightarrow}(A \wedge B)$.

(5) It is similarly proved as in (4).

(6)

$$\begin{array}{ll} \alpha^{\rightarrow}(l(A))(y) & = l(A)(\alpha(y)) \\ & = \bigwedge_{z\in X}(e^1_X(\alpha(y), z) \rightarrow A^*(z)) \\ & = \bigwedge_{z\in X}(e^1_X(\alpha(y), \alpha(z)) \rightarrow A^*(\alpha(z)) \\ & = \bigwedge_{z\in X}(e^1_X(y, z) \rightarrow A^*(\alpha(z)) = l(\alpha^{\rightarrow}(A))(y). \end{array}$$

(7) It is similarly proved as in (6).

(8) It follows from $lr(\alpha^{\rightarrow}(A)) = l\alpha^{\rightarrow}(r(A)) = \alpha^{\rightarrow}(lr(A)) = \alpha^{\rightarrow}(A)$.

Lemma 3.8 *Let (X, e^1_X, e^2_X) be a doubly fuzzy preordered set.*

(1) *If A_i are l-stable, then $\bigwedge_{i\in\Gamma} A_i$ is l-stable.*

(2) *If A_i are l-stable, then $\bigwedge_{i\in\Gamma} rA_i$ is r-stable.*

Proof. (1) Since $A_i = lr(A_i) \in E_1(L^X)$ and $(\bigwedge A_i(x)) \odot e^1_X(x, y) \leq \bigwedge A_i(y)$, then $\bigwedge A_i \in E_1(L^X)$. By Theorem 2.8 (3), $\bigwedge A_i \leq lr(\bigwedge A_i)$.

Suppose there exists $x \in X$ such that

$$lr(\bigwedge A_i)(x) \not\leq \bigwedge A_i(x).$$

Then there exists $j \in \Gamma$ such that

$$lr(\bigwedge A_i)(x) \not\leq A_j(x).$$

By the definition of $lr(A_j) = A_j$, there exists $j \in \Gamma$ such that

$$lr(\bigwedge A_i)(x) \not\leq e_1(x, y) \rightarrow r(A_j)^*(y)$$

On the other hand, since $r(A_i) \leq r(\bigwedge A_j)$, then

$$\begin{aligned} lr(\bigwedge A_i)(x) &\leq e_1(x,y) \to r(\bigwedge A_i)^*(y) \\ &\leq e_1(x,y) \to r(A_j)^*(y). \end{aligned}$$

It is a contradiction. Hence $lr(\bigwedge A_i) \leq \bigwedge A_i$.

(2) Since $r(A_i) \in E_2(L^X)$, then $\bigwedge r(A_i) \in E_2(L^X)$. Thus $\bigwedge r(A_i) \leq rl(\bigwedge r(A_i))$.

Suppose there exists $x \in X$ such that

$$rl(\bigwedge r(A_i))(x) \not\leq \bigwedge r(A_i)(x).$$

Then there exists $j \in \Gamma$ such that

$$rl(\bigwedge r(A_i))(x) \not\leq r(A_j)(x)$$

By the definition of $r(A_j)$, there exists $j \in \Gamma$ such that

$$rl(\bigwedge r(A_i))(x) \not\leq e_2(x,y) \to A_j^*(y).$$

On the other hand, since $\bigwedge r(A_i) \leq r(A_j)$, $l(\bigwedge r(A_i)) \geq A_j$. Thus,

$$\begin{aligned} rl(\bigwedge r(A_i))(x) &\leq e_2(x,y) \to l(\bigwedge r(A_i))^*(y) \\ &\leq e_2(x,y) \to A_j^*(y). \end{aligned}$$

It is a contradiction. Hence $rl(\bigwedge r(A_i)) \leq \bigwedge r(A_i)$.

Definition 3.9 Let (X, e_X^1, e_X^2) be a doubly fuzzy preordered set. A structure (X, e_X^1, e_X^2, β) is called a *β-LC-frame* with a map $\beta : X \to X$ satisfying the following conditions:

(B1) $e_X^1(x,y) \leq e_X^2(\beta(x), \beta(y))$,

(B2) $e_X^2(x,y) \leq e_X^1(\beta(x), \beta(y))$,

(B3) $\beta(\beta(x)) = x$.

Remark 3.10 Let (X, e_X^1, e_X^2, β) be an β-LC-frame. Then

$$e_X^i(x,y) \leq e_X^j(\beta(x), \beta(y)) \leq e_X^i(\beta(\beta(x)), \beta(\beta(y))) = e_X^i(x,y).$$

Hence $e_X^i(x,y) = e_X^j(\beta(x), \beta(y))$, for $i \neq j \in \{1, 2\}$. Furthermore, $\beta(x) = \beta(y)$ implies $x = \beta(\beta(x)) = \beta(\beta(y)) = y$. Thus β is injective. By (B3), β is surjective. Hence β is a bijective function. Furthermore, $\beta : (X, e_X^1, e_X^2) \to (X, e_X^2, e_X^1)$ and β^{-1} are doubly order preserving maps, then β is a doubly order isomorphism.

Theorem 3.11 *Let (X, e^1_X, e^2_X, β) be a β-LC-frame. Define a map $\beta^{\rightarrow} : L^X \rightarrow L^X$ as*

$$\beta^{\rightarrow}(A)(y) = \bigvee_{x \in \beta^{-1}(\{y\})} A(x)$$

Then we have the following properties.

(1) $\beta^{\rightarrow}(A)(\beta(x)) = A(x), \beta^{\rightarrow}(A)(x) = A(\beta(x))$.

(2) $\beta^{\rightarrow}(\beta^{\rightarrow}(A)) = A$.

(3) $\beta^{\rightarrow}(A) \leq \beta^{\rightarrow}(B)$ *for* $A \leq B$.

(4) $\beta^{\rightarrow}(A \wedge B) = \beta^{\rightarrow}(A) \wedge \beta^{\rightarrow}(B)$.

(5) $\beta^{\rightarrow}(A \vee B) = \beta^{\rightarrow}(A) \vee \beta^{\rightarrow}(B)$.

(6) $r(\beta^{\rightarrow}(A)) = \beta^{\rightarrow}(l(A))$.

(7) $l(\beta^{\rightarrow}(A)) = \beta^{\rightarrow}(r(A))$.

(8) *If A is r-stable (resp. l-stable), then $\beta^{\rightarrow}(A)$ is l-stable (resp. r-stable).*

Proof. (1-5) are similarly proved as in Theorem 3.7 (1-5).

(6)

$$\begin{aligned}
\beta^{\rightarrow}(l(A))(y) &== l(A)(\beta(y)) \\
&= \textstyle\bigwedge_{z \in X}(e^1_X(\beta(y), z) \rightarrow A(z)) \\
&= \textstyle\bigwedge_{z \in X}(e^1_X(\beta(y), \beta(z)) \rightarrow A(\beta(z)) \\
&= \textstyle\bigwedge_{z \in X}(e^2_X(y, z) \rightarrow A(\beta(z)) = r(\beta^{\rightarrow}(A))(y).
\end{aligned}$$

(7) It is similarly proved as in (6).

(8) It follows from $lr(\beta^{\rightarrow}(A)) = l\beta^{\rightarrow}(l(A)) = \beta^{\rightarrow}(rl(A)) = \beta^{\rightarrow}(A)$.

Theorem 3.12 *Let (X, e^1_X, e^2_X, β) be a β-LC-frame. For $A \in L(L^X)$, we define a map $\eta^{\rightarrow} : L(L^X) \rightarrow L(L^X)$ as $\eta^{\rightarrow}(A)(x) = r(A)(\beta(x))$.*

(1) $\eta^{\rightarrow}(\eta^{\rightarrow}(A))(x) = r(\eta^{\rightarrow}(A))(\beta(x)) = A(x)$ *for all* $x \in X$.

(2) $lr(\eta^{\rightarrow}(A)) = \eta^{\rightarrow}(A)$.

(3) $A \leq \eta^{\rightarrow}(B)$ *iff* $B \leq \eta^{\rightarrow}(A)$, *for each* $A, B \in L(L^X)$.

(4) *If* $A \leq B$, *for each* $A, B \in L(L^X)$, *then* $\eta^{\rightarrow}(A) \geq \eta^{\rightarrow}(B)$.

(5) $\eta^{\rightarrow}(A \sqcup B) = \eta^{\rightarrow}(A) \wedge \eta^{\rightarrow}(B)$ *and* $\eta^{\rightarrow}(A \wedge B) = \eta^{\rightarrow}(A) \sqcup \eta^{\rightarrow}(B)$, *for each* $A, B \in L(L^X)$.

Proof. (1) We have $\eta^{\rightarrow}(\eta^{\rightarrow}(A))(x) = r(\eta^{\rightarrow}(A))(\beta(x))$. Suppose there exists $x \in X$ such that $r\eta^{\rightarrow}(A)(\beta(x)) \not\leq A(x)$. Since $A = lr(A)$, there exists $w \in X$ such that

$$r\eta^{\rightarrow}(A)(\beta(x)) \not\leq e^1_X(x, w) \rightarrow (rA)^*(w).$$

Since $e^1_X(x, w) \rightarrow (rA)^*(w) = e^2_X(\beta(x), \beta(w)) \rightarrow (\eta^{\rightarrow}(A))^*(\beta(w))$,

$$r\eta^{\rightarrow}(A)(\beta(x)) \leq e^2_X(\beta(x), \beta(w)) \rightarrow (\eta^{\rightarrow}(A))^*(\beta(w)).$$

It is a contradiction.

Suppose there exist $x \in X$ such that $A(x) \not\leq r\eta^{\rightarrow}(A)(\beta(x))$. By the definition of $r\eta^{\rightarrow}(A)$, there exists $w \in X$ such that

$$A(x) \not\leq e_X^2(\beta(x), w) \rightarrow (\eta^{\rightarrow}(A))^*(w) = e_X^1(x, \beta(w)) \rightarrow (rA)^*(\beta(w))).$$

Thus, $A(x) \not\leq lrA(x)$. It is a contradiction.

(2) Suppose there exist $x \in X$ such that $lr\eta^{\rightarrow}(A)(x) \not\leq \eta^{\rightarrow}(A)(x) = rA(\beta(x))$. By the definition of $r(A)$, there exists $w \in X$ such that

$$lr\eta^{\rightarrow}(A)(x) \not\leq e_X^2(\beta(x), w) \rightarrow A^*(w).$$

Since $(r\eta^{\rightarrow}(A))^*(\beta(w)) = (r\eta^{\rightarrow}(A)(\beta(w)))^* = (\eta^{\rightarrow}(\eta^{\rightarrow}(A))(x))^* = A^*(x)$,

$$\begin{aligned} lr\eta^{\rightarrow}(A)(x) &\leq e_X^1(x, \beta(w)) \rightarrow (r\eta^{\rightarrow}(A))^*(\beta(w)) \\ &\leq e_X^1(x, \beta(w)) \rightarrow (r\eta^{\rightarrow}(A))^*(\beta(w)) \\ &\leq e_X^2(\beta(x), w) \rightarrow A^*(w) \end{aligned}$$

It is a contradiction.

(3) Let $A \leq \eta^{\rightarrow}(B)$. Suppose there exist $x \in X$ such that $B(x) \not\leq \eta^{\rightarrow}(A)(x) = rA(\beta(x))$. By the definition of $r(A)$, there exists $w \in X$ such that

$$B(x) \not\leq e_X^2(\beta(x), w) \rightarrow A^*(w) = e_X^1(x, \beta(w)) \rightarrow A^*(w).$$

Since $A \leq \eta^{\rightarrow}(B)$,

$$B(x) \not\leq e_X^1(x, \beta(w)) \rightarrow \eta^{\rightarrow}(B)^*(w) = e_X^1(x, \beta(w)) \rightarrow (rB(\beta(w)))^*.$$

Hence $B(x) \not\leq lr(B)(x)$. It is a contradiction.

(4) It follows from $r(A) \geq r(B)$ for $A \leq B$.

(5) Since $A \sqcup B = l(r(A) \wedge r(B))$ and $r(A) \wedge r(B) \in R(L^X)$,

$$\begin{aligned} \eta^{\rightarrow}(A \sqcup B)(x) &= r(A \sqcup B)(\beta(x)) = rl(r(A) \wedge r(B))(\beta(x)) \\ &= (r(A) \wedge r(B))(\beta(x)) = (\eta^{\rightarrow}(A) \wedge \eta^{\rightarrow}(B))(x). \end{aligned}$$

$$\begin{aligned} \eta^{\rightarrow}(A \wedge B) &= \eta^{\rightarrow}\Big(\eta^{\rightarrow}(\eta^{\rightarrow}(A)) \wedge \eta^{\rightarrow}(\eta^{\rightarrow}(B))\Big) \\ &= \eta^{\rightarrow}\Big(\eta^{\rightarrow}(\eta^{\rightarrow}(A) \sqcup \eta^{\rightarrow}(B))\Big) \\ &= \eta^{\rightarrow}(A) \sqcup \eta^{\rightarrow}(B). \end{aligned}$$

Theorem 3.13 *Let (X, e_X^1, e_X^2) be a doubly fuzzy preordered set. We define*

$$\bigwedge A_i, \bigsqcup A_i = l(\bigwedge rA_i), \quad A_i \in L(L^X).$$

Then:

(1) *$(L(L^X), \bigwedge, \bigsqcup, \overline{0}, \overline{1})$ is a complete lattice.*

(2) *If $(X, e_X^1, e_X^2, \alpha)$ is an α-LC frame, then $(L(L^X), \bigwedge, \bigsqcup, \overline{0}, \overline{1}, \alpha^{\rightarrow})$ is a complete LC-algebra.*

(3) *If (X, e_X^1, e_X^2, β) is a β-LC frame, then $(L(L^X), \bigwedge, \bigsqcup, \overline{0}, \overline{1}, \eta^{\rightarrow})$ is a complete LD-algebra.*

Proof. (1) It follows from $A_i \leq B$ for all i iff $r(B) \leq r(A_i)$ for all i iff $r(B) \leq \bigwedge r(A_i)$ iff $r(B) = r(B) \wedge \bigwedge r(A_i)$ iff $lr(B) = l(r(B) \wedge \bigwedge r(A_i)) = B \sqcup l(\bigwedge r(A_i))$ iff $B = B \sqcup (\bigsqcup A_i)$ iff $\bigsqcup A_i \leq B$.

(2) $\alpha^{\rightarrow} : L(L^X) \to L(L^X)$ is well-defined from Theorem 3.7 (8). Furthermore, we have $\alpha^{\rightarrow}(\alpha^{\rightarrow}(A)) = A$ and

$$\begin{aligned}\alpha^{\rightarrow}(A \sqcup B) &= \alpha^{\rightarrow}(l(r(A) \wedge r(B)) = l(\alpha^{\rightarrow}(r(A) \wedge r(B)) \\ &= l(\alpha^{\rightarrow}(r(A)) \wedge \alpha^{\rightarrow}(r(B))) = l(r(\alpha^{\rightarrow}(A)) \wedge r(\alpha^{\rightarrow}(B))) \\ &= \alpha^{\rightarrow}(A) \sqcup \alpha^{\rightarrow}(B)\end{aligned}$$

(3) $\eta^{\rightarrow} : L(L^X) \to L(L^X)$ is well-defined from Theorem 3.12 (2). Furthermore, we have $\eta^{\rightarrow}(\eta^{\rightarrow}(A)) = A$ and $A \leq \eta^{\rightarrow}(B)$ iff $B \leq \eta^{\rightarrow}(A)$ for all $A, B \in L(L^X)$.

Example 3.14 Let $X = \{0, x, y, z, 1\}$ be a set and $(L = [0, 1], \odot)$ with $x \odot y = \max\{0, x + y - 1\}$. Let $(X, \wedge, \vee, 0, 1)$ be a bounded lattice as follows:

$\wedge$	0	x	y	z	1
0	0	0	0	0	0
x	0	x	0	0	x
y	0	0	y	0	y
z	0	0	0	z	z
1	0	x	y	z	1

$\vee$	0	x	y	z	1
0	0	x	y	z	1
x	x	x	1	1	1
y	y	1	y	1	1
z	z	1	1	z	1
1	1	1	1	1	1

(1) Define $\alpha : X \to X$ as

$$\alpha(0) = 0, \alpha(1) = 1, \alpha(x) = y, \alpha(y) = x, \alpha(z) = z.$$

We define $e_i : X \times X \to L$ as

e_1	0	x	y	z	1
0	1	0.5	0.5	0.4	0.6
x	0.6	1	0.8	0.6	0.7
y	0.6	0.8	1	0.6	0.7
z	0.4	0.4	0.4	1	0.3
1	0.3	0.8	0.8	0.5	1

e_2	0	x	y	z	1
0	1	0.6	0.6	0.5	0.5
x	0.7	1	0.4	0.7	0.6
y	0.7	0.4	1	0.7	0.6
z	0.5	0.5	0.5	1	0.3
1	0.4	0.6	0.6	0.4	1

Since $e_i(x, y) = e_i(\alpha(x), \alpha(x))$ for $i = 1, 2$, (X, e_1, e_2, α) is an α-frame. For $A = (A(0), A(x), A(y), A(z), A(1))^t = (0.5, 0.6, 0.4, 0.7, 0.5)^t$, we have

$$l(\alpha^{\rightarrow}(A)) = (0.5, 0.6, 0.4, 0.3, 0.5)^t = \alpha^{\rightarrow}(l(A)).$$

(2) Define $\beta : X \to X$ as

$$\beta(0) = 0, \beta(1) = 1, \beta(x) = x, \beta(y) = z, \beta(z) = y.$$

We define $e_i : X \times X \to L$ as

e_1	0	x	y	z	1
0	1	0.5	0.5	0.4	0.6
x	0.6	1	0.8	0.6	0.7
y	0.6	0.8	1	0.6	0.7
z	0.4	0.4	0.4	1	0.3
1	0.3	0.8	0.8	0.5	1

e_2	0	x	y	z	1
0	1	0.5	0.4	0.5	0.6
x	0.6	1	0.6	0.8	0.7
y	0.4	0.4	1	0.4	0.3
z	0.6	0.8	0.6	1	0.7
1	0.3	0.8	0.5	0.8	1

Since $e_1(x, y) = e_2(\beta(x), \beta(y))$ and $e_2(x, y) = e_1(\beta(x), \beta(y))$, (X, e_1, e_2, β) is a β-frame. We denote $A = (A(0), A(x), A(y), A(z), A(1))^t = (0.5, 0.6, 0.4, 0.7, 0.5)^t$, we have $A \in L(L^X)$. We obtain

$$\eta^{\rightarrow}(A) = rA(\beta(x)) = rA(0.5, 0.6, 0.7, 0.4, 0.5)^t = (0.5, 0.4, 0.3, 0.6, 0.5)^t \in L(L^X).$$

For $B = (0.4, 0.4, 0.3, 0.5, 0.4)^t \in L(L^X)$, we have

$$A \leq \eta^{\rightarrow}(B) = (0.6, 0.6, 0.5, 0.7, 0.6)^t \text{ iff } B \leq \eta^{\rightarrow}(A) = (0.5, 0.4, 0.3, 0.6, 0.5)^t.$$

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