Mathematica Aeterna, Vol. 2, 2012, no. 4, 309 - 320

Comparison of solutions of mKdV equation by using the first integral method and $(\frac{G'}{G})$-expansion method

N.Taghizadeh

taghizadeh@guilan.ac.ir
Department of Mathematics,
Faculty of Mathematical Sciences,
University of Guilan,
P.O.Box 1914, Rasht, Iran

M.Najand

mona_najand@yahoo.com
Department of Mathematics,
Faculty of Mathematical Sciences,
University of Guilan,
P.O.Box 1914, Rasht, Iran

Abstract

The first integral method and the $(\frac{G'}{G})$-expansion method are two efficient methods for obtaining exact solutions of some nonlinear partial differential equations.
In this paper, we first describe the first integral method and the $(\frac{G'}{G})$-expansion method. Then we solve the mKdV equation with both methods and compare the solutions.

Mathematics Subject Classification: 35Q55; 35Q80; 35G25.

Keywords: The first integral method, The $(\frac{G'}{G})$-expansion method, mKdV equation.

1 Introduction

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, fluid dynamics. In order to better make efforts to seek more exact solutions to them. Several powerful methods have been proposed to obtain exact solutions of nonlinear evolution equations, such as inverse scattering method [1], Backlund transformation method [2, 3], Darboux transformation method[4, 5], Hirota's bilinear method [11, 12], F-expansion method[13-16], and so on.

The first integral method was first proposed by Feng [17] in solving Burgers-KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method is widely used by many such as in $[18-20]$ and by the reference therein.

Very recently, Wang et al.[21] introduced a new method called the $(\frac{G'}{G})$-expansion method to look for traveling wave solutions of nonlinear evolution equations. The$(\frac{G'}{G})$-expansion method is based on the assumptions that the traveling wave solutions can be expressed by a polynomial in $(\frac{G'}{G})$, and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation(LODE).

The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given nonlinear evolution equations. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. By using the$(\frac{G'}{G})$-expansion method, Wang et al. success fully obtain more traveling wave solutions of four nonlinear evolution equations.

The aim of this paper is to compare between the first integral method and the $(\frac{G'}{G})$-expansion method.

2 The first integral method(FIM)

Consider the nonlinear partial differential equation in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0, \tag{1}$$

where $u = u(x,t)$ is the solution of nonlinear partial differential equation Eq.(1). We use the transformations,

$$u(x,t) = f(\xi), \tag{2}$$

where $\xi = x - vt$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -v\frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), \quad .... \tag{3}$$

Using Eq.(3) to transfer the nonlinear partial differential equation Eq.(1) to nonlinear ordinary differential equation

$$G(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, ...) = 0. \tag{4}$$

Next, we introduce a new independent variable

$$X(\xi) = f(\xi), \qquad Y = \frac{\partial f(\xi)}{\partial \xi}, \tag{5}$$

which leads a system of nonlinear ordinary differential equations

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \tag{6}$$

$$\frac{\partial Y(\xi)}{\partial \xi} = F_1(X(\xi), Y(\xi)).$$

By the qualitative theory of ordinary differential equations [22], if we can find the integrals to Eq.(6) under the same conditions, then the general solutions to Eq.(6) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to Eq.(6) which reduces Eq.(4) to a first order integrable ordinary differential equation. An exact solution to Eq.(1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem. Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in $C[w, z]$; and $P(w, z)$ is irreducible in $C[w, z]$; If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C[w, z]$ such that

$$Q(w, z) = P(w, z)G(w, z).$$

3 The $(\frac{G'}{G})$-expansion method

Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{7}$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $(\frac{G'}{G})$-expansion method.

Step 1. Combining the independent variables x and t into one variable $\xi = x - vt$, we suppose that

$$u(x, t) = u(\xi), \quad \xi = x - vt, \tag{8}$$

the traveling wave variable (8) permits us reducing Eq.(7) to an ordinary differential equation(ODE) for $u = u(\xi)$

$$P(u, -vu', u', v^2u'', -vu'', u'', ...) = 0, \tag{9}$$

Step 2. Suppose that the solution of ODE (9) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m(\frac{G'}{G})^m + ..., \tag{10}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{11}$$

$\alpha_0, \alpha_1, ..., \alpha_m, \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (9).

Step 3. By substituting (10) into (9) and using second order LODE (11), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of Eq.(9) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_0, \alpha_1, ..., \alpha_m, \lambda$ and μ.

Step4. Assuming that the constants $\alpha_0, \alpha_1, ..., \alpha_m, \lambda$ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (11) have been well known for us, then substituting $\alpha_0, \alpha_1, ..., \alpha_m, v$ and the general solutions of Eq.(11) into (10) we have more traveling wave solutions of the nonlinear evolution equation (8).

4 Exact solutions of mKdV equation by using the first integral method

In this section, we study the mKdV equation

$$u_t - u^2u_x + u_{xxx} = 0. \tag{12}$$

We use the traveling wave transformation

$$u(x, t) = u(\xi), \quad \xi = x - vt. \tag{13}$$

By the travelling variable (13) permits us reducing Eq.(12) to an ODE for $u = u(\xi)$

$$-vu' - u^2u' + u''' = 0. \tag{14}$$

Integrating (14) with respect to ξ, then we have

$$R + vu + \frac{1}{3}u^3 = u'' \tag{15}$$

Using(5) and (6), we can get

$$\begin{aligned} \dot{X}(\xi) &= Y(\xi), & (16)\\ \dot{Y}(\xi) &= \frac{1}{3}X^3(\xi) + vX(\xi) + R. & (17) \end{aligned}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (16)-(17), and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \tag{18}$$

where $a_i(X)(i = 0, 1, ..., m)$, are polynomials of X and $a_m(X) \neq 0$. Equation (18) is called the first integral to (16) and (17). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{19}$$

In this example, Suppose that $m = 1$, by comparing with the coefficients of $Y^i(i = 2, 1, 0)$ on both sides of (19), we have

$$\begin{aligned} \dot{a_1}(X) &= h(X)a_1(X), & (20)\\ \dot{a_0}(X) &= g(X)a_1(X) + h(X)a_0(X), & (21) \end{aligned}$$

$$a_1(X)[\frac{1}{3}X^3(\xi) + vX(\xi) + R] = g(X)a_0(X). \tag{22}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (20) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_0X + A_1$, then we find $a_0(X)$,

$$a_0(X) = \frac{1}{2}A_0X^2 + A_1X + A_2, \tag{23}$$

where A_2 is arbitrary integration constant.
Substituting $a_0(X)$ and $g(X)$ into (22) and setting all the coefficients of powers

X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = \frac{\sqrt{6}}{3}, \quad A_1 = 0, \quad A_2 = \frac{\sqrt{6}}{2}v, \quad R = 0, \tag{24}$$

$$A_0 = -\frac{\sqrt{6}}{3}, \quad A_1 = 0, \quad A_2 = -\frac{\sqrt{6}}{2}v, \quad R = 0, \tag{25}$$

where v is arbitrary constant.
Using the conditions (24) in (18), we obtain

$$Y_1(\xi) = -\frac{\sqrt{6}}{6}X^2(\xi) - \frac{\sqrt{6}}{2}v. \tag{26}$$

Combining (26) with (16), we obtain the exact solution to equations (16) and (17):
When $v < 0$

$$u_{1,1}(\xi) = \sqrt{3v}\ \tanh(\frac{\sqrt{2v}}{2}\xi + \xi_0), \tag{27}$$

where $\xi = x - vt$ and ξ_0 is an arbitrary constant.
When $v > 0$

$$u_{1,2}(\xi) = -\sqrt{3v}\ \tan(\frac{\sqrt{2v}}{2}\xi + \xi_0), \tag{28}$$

where $\xi = x - vt$ and ξ_0 is an arbitrary constant.
Similarly, in the case of (25), from (18), we obtain

$$Y_2(\xi) = \frac{\sqrt{6}}{6}X^2(\xi) + \frac{\sqrt{6}}{2}v, \tag{29}$$

and then the exact solution of mKdV equation can be written as:
When $v < 0$

$$u_{2,1}(\xi) = -\sqrt{3v}\ \tanh(\frac{\sqrt{2v}}{2}\xi + \xi_0), \tag{30}$$

where $\xi = x - vt$ and ξ_0 is an arbitrary constant.
When $v > 0$

$$u_{2,2}(\xi) = \sqrt{3v}\ \tan(\frac{\sqrt{2v}}{2}\xi + \xi_0), \tag{31}$$

where $\xi = x - vt$ and ξ_0 is an arbitrary constant.

5 Exact solutions of mKdV equation by using the $(\frac{G'}{G})$-expansion method

We use the traveling wave transformation

$$u(x,t) = u(\xi), \quad \xi = x - vt. \tag{32}$$

By the travelling variable (32) permits us reducing Eq.(12) to an ODE for $u = u(\xi)$

$$-vu' - u^2u' + u''' = 0. \tag{33}$$

Integrating (33) with respect to ξ, then we have

$$R + vu + \frac{1}{3}u^3 = u'' \tag{34}$$

where R is integration constant.
Suppose that the solution of ODE (34) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m(\frac{G'}{G})^m + ..., \tag{35}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{36}$$

By using (35) and (36) it is easily derived that

$$u^3 = \alpha_m^3(\frac{G'}{G}) + ..., \tag{37}$$

$$u' = -m\alpha_m^3(\frac{G'}{G})^{m+1} + ..., \tag{38}$$

$$u'' = m(m+1)\alpha_m^3(\frac{G'}{G})^{m+2} + ... \tag{39}$$

Considering the homogeneous balance between u'' and u^3 in Eq.(34), we required that

$$m + 2 = 3m$$

then $m = 1$, so we can write (35) as

$$u(\xi) = \alpha_1(\frac{G'}{G}) + \alpha_0, \quad \alpha_1 \neq 0 \tag{40}$$

and therefore

$$u^3 = \alpha_1^3(\frac{G'}{G})^3 + 3\alpha_0\alpha_1^2(\frac{G'}{G})^2 + 3\alpha_0^2\alpha_1(\frac{G'}{G}) + \alpha_0^3, \tag{41}$$

$$u'' = 2\alpha_1(\frac{G'}{G})^3 + 3\alpha_1\lambda(\frac{G'}{G})^2 + \alpha_1(\lambda^2 + 2\mu)(\frac{G'}{G}) + \alpha_1\mu\lambda. \tag{42}$$

By substituting (40), (41) and (42) into ODE.(34) and collecting all terms with the same power of $(\frac{G'}{G})$ together, the left-hand side of ODE.(34) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for $\alpha_0, \alpha_1, v, \mu$ and λ as follows:

$$0 \ : \ R + \alpha_0 v + \frac{1}{3}\alpha_0^3 - \alpha_1\mu\lambda = 0, \tag{43}$$

$$1 \ : \ v + \alpha_0^2 - \lambda^2 - 2\mu = 0, \tag{44}$$

$$2 \ : \ \alpha_0\alpha_1 - 3\lambda = 0, \tag{45}$$

$$3 \ : \ \frac{1}{3}\alpha_1^2 - 2 = 0. \tag{46}$$

Solving the algebraic equations above with aid Maple,yields

$$R = 0, \quad \mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v \ \ , \alpha_0 = \frac{\sqrt{6}}{2}\lambda, \quad \alpha_1 = \sqrt{6} \tag{47}$$

$$R = 0, \quad \mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v \ \ , \alpha_0 = -\frac{\sqrt{6}}{2}\lambda, \quad \alpha_1 = -\sqrt{6} \tag{48}$$

λ, v are arbitrary constants.
By using (47), expression (40) can be written as

$$u(\xi) = \sqrt{6}(\frac{G'}{G}) + \frac{\sqrt{6}}{2}\lambda \tag{49}$$

where $\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v.Eq.(49)$ is the formula of a solution of Eq.(34).
Substituting the general solutions of Eq.(36) into Eq.(49) we have three types of traveling wave solutions of the mKdV equation as follows:
When $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{\sqrt{6}}{2}\sqrt{\lambda^2 - 4\mu}(\frac{c_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + c_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{c_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + c_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}), \tag{50}$$

$\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ and c_1, c_2 are arbitrary constants.
When $\lambda^2 - 4\mu < 0$,

$$u_2(\xi) = \frac{\sqrt{6}}{2}\sqrt{4\mu - \lambda^2}(\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{c_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}), \tag{51}$$

$\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ and c_1, c_2 are arbitrary constants.
When $\lambda^2 - 4\mu = 0$,

$$u_3(\xi) = \frac{\sqrt{6}c_2}{c_1 + c_2\xi}. \tag{52}$$

$\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ and c_1, c_2 are arbitrary constants.
By using (48), expression (40) can be written as

$$u(\xi) = -\sqrt{6}(\frac{G'}{G}) - \frac{\sqrt{6}}{2}\lambda \tag{53}$$

where $\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v.Eq.(53)$ is the formula of a solution of Eq.(34).
Substituting the general solutions of Eq.(36) into Eq.(53) we have three types of traveling wave solutions of the mKdV equation as follows:
When $\lambda^2 - 4\mu > 0$,

$$u_4(\xi) = -\frac{\sqrt{6}}{2}\sqrt{\lambda^2 - 4\mu}(\frac{c_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + c_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{c_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + c_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}), \tag{54}$$

$\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ and c_1, c_2 are arbitrary constants.
When $\lambda^2 - 4\mu < 0$,

$$u_5(\xi) = -\frac{\sqrt{6}}{2}\sqrt{4\mu - \lambda^2}(\frac{-c_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{c_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}), \tag{55}$$

$\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ and c_1, c_2 are arbitrary constants.
When $\lambda^2 - 4\mu = 0$,

$$u_6(\xi) = \frac{-\sqrt{6}c_2}{c_1 + c_2\xi}. \tag{56}$$

$\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ and c_1, c_2 are arbitrary constants.

6 Comparison of two methods of first integral and $(\frac{G'}{G})$-expansion

In this section, we compare two methods of first integral and $(\frac{G'}{G})$-expansion and we describe the advantages of these two methods.
At first, we describe the advantages of $(\frac{G'}{G})$-expansion method. We can use $(\frac{G'}{G})$-expansion method to solve the nonlinear equations with any degree of derivative but when we can use the first integral method that the nonlinear equation is a second-order nonlinear differential equation or it can become to a second-order nonlinear differential equation. In this case, there are a lot of nonlinear equations which have been solved by $(\frac{G'}{G})$-expansion method but we can not solve them by first integral method.
For example, Kupershmidt equation [23], as follows:

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_xu_{xx} + 20u^2u_x$$

which we can not convert this equation to a second-order nonlinear differential equation.
If an equation can be solved by two these methods, through the first integral method we can obtain more solution of equation because with every step we can put different m and find many solutions of the equation. But we obtain limited solutions of equation in $(\frac{G'}{G})$-expansion method.
In both methods, because the calculations are done easily with aim of Maple, so we have not problem of difficulty in the calculations or time-consuming calculations for any of the two methods.
In this paper, we have obtained the exact solutions of the mKdV equation by using these two methods. Now, we compare the solutions:
Remark 1. In expression (50), if $c_1 > 0$ and $c_1^2 > c_2^2$, then $u_1 = u_1(\xi)$ can be written as:

$$u_1(\xi) = \frac{\sqrt{6}}{2}\sqrt{\lambda^2 - 4\mu}\tanh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + \xi_0),$$

where $\xi = x - vt, \mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ and $\xi_0 = \tanh^{-1}(\frac{c_2}{c_1})$ that is equivalent to the expression (27).
Remark 2. In expression (51), if we put $c_1 = 0$, then $u_2 = u_2(\xi)$ can be written as:

$$u_2(\xi) = \frac{-\sqrt{6}}{2}\sqrt{4\mu - \lambda^2}\tan(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi),$$

where $\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ that is equivalent to the expression (28) if we put $\xi_0 = 0$.
Remark 3. In expression (54), if $c_1 > 0$ and $c_1^2 > c_2^2$, then $u_4 = u_4(\xi)$ can be written as:

$$u_4(\xi) = \frac{-\sqrt{6}}{2}\sqrt{\lambda^2 - 4\mu}\tanh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + \xi_0),$$

where $\xi = x - vt, \mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ and $\xi_0 = \tanh^{-1}(\frac{c_2}{c_1})$ that is equivalent to the expression (30).
Remark 4. In expression (55), if we put $c_1 = 0$, then $u_5 = u_5(\xi)$ can be written as:

$$u_5(\xi) = \frac{\sqrt{6}}{2}\sqrt{4\mu - \lambda^2}\tan(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi),$$

where $\xi = x - vt$ and $\mu = \frac{1}{4}\lambda^2 + \frac{1}{2}v$ that is equivalent to the expression (31) if we put $\xi_0 = 0$.

7 Conclusion

The first integral method and the $(\frac{G'}{G})$-expansion method are used to find new exact traveling wave solutions. Thus, we can say that the proposed methods can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

ACKNOWLEDGEMENTS. Authors thank from reviewers..

References

[1] M.Ablowitz, P.A. Clarkson, Soliton Nonlinear Evolution Equation and Inverse Scattering, Cambrridge University Press, New York, 1991.

[2] M.Wadati, Wave Propagation in nonlinear lattice, J. Phys. Soc. Jpn. $1975, 38, 673 - 680$.

[3] M.R. Miura, Backlund Transformation, Spring-Verlag,Berlin, 1978.

[4] V.A. Matveev, M.A. Salle, Darboux Transformation and Solitons, Spring-Verlag,Berlin, Heidelberg, 1991.

[5] C.H.Gu et al, Darboux Transformation in Solitons Theory and Geometry Applications, Shangai Science Technology Press, Shanghai, 1999.

[6] Ryogo Hirota, Exact Solution of the Korteweg-de Vries Equation for Mutiple Collisions of Solitons, Phys. Rev. Lett. $1971, 27, 192 - 1194$.

[7] Ryogo Hirota, The Direct Method in Soliton Theory, Cambridge University Press, 2004.

[8] M.L. Wang, Solitary wave solutions for vaiant Boussinesq equations, Phys. Lett. A $1995, 199, 169 - 172$.

[9] E.G. Fan, H.Q.Zhang, A note on the homogeneous balance method, Phys. Lett. A $1998, 246, 403 - 406$.

[10] Y.Cheng, Q. Wang, A new general algebraic method with symbolic computation to construct new travelling wave solution for the $(1 + 1)$-dimensional dispersive long wave equation, Appl. Math. Comput. $2005, 168, 1189 - 1204$.

[11] E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A $2000, 277, 212 - 218$.

[12] Z.Y. Yan, New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations, Phys. Lett. A $2001, 292, 100 - 106$.

[13] E.G. Fan, Travelling wave solutions in terms of special functions for nonlinear coupled evolution systems,Phys. Lett. A $2002, 300, 43 - 249$.

[14] E.G. Fan, A new algebraic method for finding the line soliton solutions and doubly periodic wave solution to a two-dimensional perturbed Kdv equation, Chaos Solitons Fract. $2003, 15, 67-574$.

[15] M.A. Abdou, An improved generalized F-expansion method and its applications, J. Comput. Appl. Math. $2008, 214, 202-208$.

[16] Emmanuel Yomba, he modified extended Fan sub-equation method and its application to the (2+1)-dimensional Broer-Kaup-Kupershmidt equation, Chaos Solitons Fract. $2006, 27, 187-196$.

[17] Z.S. Feng, The first integral method to study the Burgers-KdV equation, J. Phys. A: Math. Gen. $2002, 35, 343-349$.

[18] Z.S. Feng, X.H. Wang, The first integral method to the two-dimensional Burgers-KdV equation, Phys. Lett. A $2003, 308, 173-178$.

[19] Z.S. Feng, Traveling wave behavior for a generalized fisher equation, Chaos Solitons Fract. $2008, 38, 481-488$.

[20] K.R. Raslan, The first integral method for solving some important nonlinear partial differential equations, Nonlinear Dynam. $2008, 53, 281-286$.

[21] ML. Wang, XZ Li and JL. Zhang, The $(\frac{G'}{G})$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A $2008, 372, 417-423$.

[22] T.R. Ding and C.Z. Li, Ordinary differential equations, Peking University Press, Peking, 1996.

[23] N. Taghizadeh, F. Farahrooz, M. Mirzazadeh, Exact solutions of Kupershmidt equation by $(\frac{G'}{G})$-expansion method, J. Comput. Appl. Math. $2010, 2(4), 32-37$.

Received: April, 2012