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IMPROVED ANSWERS TO AN OPEN PROBLEM CONCERNING AN INTEGRAL INEQUALITY

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Abstract

In this paper, several integral inequalities are established to improve the results of paper[On an open question regarding an integral inequality.JIPAM,8(3)(2007)], and hence they given better answers to the open problem posed in paper [Notes on an integral inequality, JIPAM,7(4)(2006)].

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1 Introduction

In [1], the following Theorem 1.1(That is Theorem 2.3 in [1]) was proved.
Theorem 1.1 Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying $\int_t^1 f(x)dx \geq \int_t^1 xdx (\forall\ t \in [0,1])$, then $\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\beta f^\alpha(x)dx$ holds for every real number $\alpha > 1$ and $\beta > 0$.
This result was a solution to an open problem proposed in [2]. The present paper improved the result of Theorem 2.3 in [1] and give a better answer to the open problem in [2].The improved result is expressed in Theorem 2.1,which will be proved in section 2. Moreover, a further extended conclusion will be expressed in section 3.

2 Main Results

Lemma 2.1 Under the conditions of Theorem 1.1, the inequality

$$\int_0^1 x^\delta f(x)dx \geq \int_0^1 x^{\delta+1}dx \tag{2.1}$$

holds for every $\delta > 0$.
Proof. For $\delta > 0$, we have

$$\begin{aligned}\int_0^1 x^\delta f(x)dx &= \int_0^1 \delta t^{\delta-1}\int_t^1 f(x)dxdt \geq \int_0^1 \delta t^{\delta-1}\int_t^1 xdxdt \\ &= \int_0^1 x(\int_0^x \delta t^{\delta-1}dt)dx = \int_0^1 x^{\delta+1}dx.\end{aligned}$$

Lemma 2.2 Under the conditions of Theorem 1.1, the inequality

$$\int_0^1 f^\lambda(x)dx \geq \int_0^1 x^\lambda dx \tag{2.2}$$

holds for every $\lambda \geq 1$.
Proof. For every $\lambda > 1$, by General Cauchy inequality [3], we get

$$\frac{1}{\lambda}f^\lambda(x) + \frac{\lambda-1}{\lambda}x^\lambda \geq x^{\lambda-1}f(x).$$

Integrating both sides of the inequality, and using Lemma 2.1, we further have

$$\frac{1}{\lambda}\int_0^1 f^\lambda(x)dx + \frac{\lambda-1}{\lambda}\int_0^1 x^\lambda dx \geq \int_0^1 x^{\lambda-1}f(x)dx \geq \int_0^1 x^\lambda dx.$$

It is evident that (2.2) holds for $\lambda = 1$. Therefore (2.2) holds for $\lambda \geq 1$.
Theorem 2.1 Under the conditions of Theorem 1.1, the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx \tag{2.3}$$

holds for $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta \geq 1$.
Proof. Using General Cauchy inequality, we have

$$\frac{\alpha}{\alpha+\beta}x^{\alpha+\beta} + \frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x) \geq x^\alpha f^\beta(x).$$

Which yields

$$\frac{\alpha}{\alpha+\beta}\int_0^1 x^{\alpha+\beta}dx + \frac{\beta}{\alpha+\beta}\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx.$$

From this inequality and (2.2), the (2.3) can be obtained.
Remark: Inequality (2.3) generalized the results of Theorem 2.3 in [1] and Theorem 3.2, Theorem 3.3 in [2].

3 The further extended conclusion

In the following, we Assume $f(x) \geq 0$ is a continuous function on $[0, 1]$ satisfying the inequality $\int_t^1 f^\gamma(x)dx \geq \int_t^1 x^\gamma dx (\forall\ t \in [0,\ 1])$, here constant $\gamma > 0$.
Lemma 3.1 The inequality

$$\int_0^1 x^\delta f^\gamma(x)dx \geq \int_0^1 x^{\delta+\gamma}dx \tag{3.1}$$

holds for every $\delta > 0$.
The proof of Lemma3.1 runs in the nearly same way as Lemma 2.1.
Lemma 3.2 The inequality

$$\int_0^1 f^\lambda(x)dx \geq \int_0^1 x^\lambda dx \tag{3.2}$$

holds for every $\lambda \geq \gamma$.
Proof. For every $\lambda > \gamma$, employing General Cauchy inequality, we have

$$\frac{\gamma}{\lambda} f^\lambda(x) + \frac{\lambda-\gamma}{\lambda} x^\lambda \geq x^{\lambda-\gamma} f^\gamma(x).$$

Further, by Lemma 3.1, we get

$$\frac{\gamma}{\lambda}\int_0^1 f^\lambda(x)dx + \frac{\lambda-\gamma}{\lambda}\int_0^1 x^\lambda dx \geq \int_0^1 x^{\lambda-\gamma} f^\gamma(x)dx \geq \int_0^1 x^\lambda dx.$$

Hence (3.2) holds for $\lambda > \gamma$. It is evident that (3.2) holds for $\lambda = \gamma$. Therefore (3.2) holds for every $\lambda \geq \gamma$.

Theorem 3.1 The following inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx \tag{3.3}$$

holds for $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta \geq \gamma$.
By virtue of (3.2), the proof of Theorem 3.1 is similar to that of Theorem 2.1.
Remark: Specializing Theorem 3.1 to the case $\gamma = 1$, the Theorem 2.1 obtained.
Lastly, we propose the following open problem.
Open Problem: Assume constant $\gamma > 0$. Let $f(x) \geq 0$ be a continuous function on $[0,\ 1]$ satisfying the inequality $\int_t^1 f^\gamma(x)dx \geq \int_t^1 x^\gamma dx, \quad \forall t \in [0,\ 1]$.
Does the inequality $\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx$ hold for $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta < \gamma$?

References

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