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CONFORMAL PARA - SASAKIAN MANIFOLDS

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Abstract

The aim of this paper is to study the Conformal Para - Sasakian manifolds. Section 1 is devoted to the conformal C - Killing vector field. Section 2 deals to D - Conformal vector field in a Para-Sasakian manifolds.

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1. INTRODUCTION:

Definition 1.1:

If a vector field u^{α} in a Para - Sasakian manifolds satisfies the relation

(1.1) $$L_u (g_{\alpha\beta} - \eta_{\alpha}\eta_{\beta}) = 2a(g_{\alpha\beta} - \eta_{\alpha}\eta_{\beta})$$

then the vector field u^{α} is called conformal C - Killing vector field. Wherein Lu is the Lie derivative with respect to u^{α} and a is a scalar field in a Para - Sasakian manifolds.

Definition 1.2:

If the scalar field holds the relation (1.1) then the scalar field in a Para-Sasakian manifolds is said to be an associated scalar field with regard to the conformal C - Killing vector field u^{α}.

Definition 1.3:

Let u^{α} is a conformal C - Killing vector field in a Para - Sasakian manifolds and the vector field v^{α} is defined in such a way

(1.2) $$v_{\alpha} = u_{\alpha} - u^{*}\eta_{\alpha}$$

then v^{α} is termed special conformal C - Killing vector field.

In a Para - Sasakian manifolds, we have [8]:

(1.3) $\xi^{\alpha} = g^{\alpha\beta}\eta_{\beta}$

(1.4) $\xi^{\alpha} = g^{\alpha\beta}\eta_{\beta}$

(1.5) $\phi^{\alpha}{}_{\beta} = \nabla_{\beta}\, \xi^{\alpha}$

(1.6) $\phi_{\alpha\beta} = \nabla_{\alpha}\, \eta_{\beta}$

(1.7) $\phi_{\alpha\beta} = g_{\lambda\alpha}\, \phi^{\lambda}{}_{\beta}$

(1.8) $\phi_{\alpha\beta} = \phi_{\beta\alpha}$

(1.9) $\xi^{\alpha}\, \eta_{\alpha} = 1$

(1.10) $\phi^{\alpha}{}_{\beta}\, \xi^{\beta} = 0$

(1.11) $\phi^{\alpha}{}_{\beta}\, \eta_{\alpha} = 0$

(1.12) $\text{Lu}\,\eta_\alpha f^\alpha = 0$

(1.13) $f^\alpha = \nabla_\alpha f$

(1.14) $f = \eta_\alpha u^\alpha$

(1.15) $\text{Lu}\,\xi^\alpha = -(f_\beta \xi^\beta)\xi^\alpha$

And

(1.16) $R_{\alpha\beta}\,\xi^\beta = -(n-1)\eta_\alpha$.

In a special conformal C - Killing vector field, we have [3]:

(1.17) $\nabla_\alpha u_\beta + \nabla_\beta u_\alpha = 2\{u^\lambda(\phi_{\lambda\alpha}\eta_\beta + \phi_{\lambda\beta}\eta_\alpha) + a(g_{\alpha\beta} - \eta_\alpha\eta_\beta)\}$

and

(1.18) $\nabla^\beta\nabla_\beta u_\alpha + R_{\alpha\beta}u^\alpha + 2Du\eta_\alpha$

$+ 4u_\alpha + 2(n-1)a_\alpha = 4(n+1)u^*\eta_\alpha$.

In this regard, we have the following theorems:

Theorem 1.1:

In a Para - Sasakian manifolds, a special conformal C - Killing vector field u^{α} holds the relation

$$\eta_{\alpha} \, Lu \, g^{\alpha\beta} = - \xi^{\beta}(f_{\nu}\xi^{\nu} - f_{\mu}\xi^{\mu}).$$

Proof:

By virtue of equation (1.1), we have

(1.19) $Lu \, (g^{\alpha\beta} - \xi^{\alpha}\xi^{\beta}) = - 2a(g^{\alpha\beta} - \xi^{\alpha}\xi^{\beta})$

In view of equations (1.15) and (1.19), we obtain

(1.20) $Lu \, g_{\alpha\beta} = - \{\xi^{\alpha}\xi^{\beta}(f_{\nu}\xi^{\nu} - f_{\mu}\xi^{\mu}) + 2a(g^{\alpha\beta} - \xi^{\alpha}\xi^{\beta})\}$

Contracting equation (1.20) by η_{α} and using equations (1.3) and (1.9), we get

(1.21) $\eta_{\alpha} \, Lu \, g^{\alpha\beta} = - \xi^{\beta}(f_{\nu}\xi^{\nu} - f_{\mu}\xi^{\mu})$

Theorem 1.2:

In a Para - Sasakian manifolds, a special conformal C - Killing vector field u^{α} satisfies the relation

$$u^{\lambda}\phi_{\lambda\beta} = (1/2)\xi^{\alpha}(\nabla_{\alpha}u_{\beta} + \nabla_{\beta}u_{\alpha}).$$

Proof:

Contracting equation (1.17) with ξ^{α} and using equation (1.10), we obtain

$$(1.22)\quad \xi^{\alpha}(\nabla_{\alpha}u_{\beta}+\nabla_{\beta}u_{\alpha})=2\{\xi^{\alpha}u^{\lambda}\phi_{\lambda\beta}\eta_{\alpha}+a(\xi^{\alpha}g_{\alpha\beta}-\xi^{\alpha}\eta_{\alpha}\eta_{\beta})\}$$

From equations (1.4), (1.9) and (1.22), we get

$$(1.23)\quad u^{\lambda}\phi_{\lambda\beta} = (1/2)\xi^{\alpha}(\nabla_{\alpha}u_{\beta} + \nabla_{\beta}u_{\alpha})$$

Theorem 1.3:

In a Para - Sasakian manifolds, a special conformal C - Killing vector field u^{α} holds the relation

$$\eta_{\alpha}u^{\alpha} = \{1/(n-1)\}\{\xi^{\alpha}(\nabla^{\beta}\nabla_{\beta}u_{\alpha}) + 2Du + 4\xi^{\alpha}u_{\alpha}$$
$$+ 2(n-1)\xi^{\alpha}a_{\alpha} - 4(n+1)u^{*}\}.$$

Proof:

Transvecting equation (1.18) by ξ^{α} and using equation (1.9), we obtain

$$(1.24)\quad \xi^{\alpha}(\nabla^{\beta}\nabla_{\beta}u_{\alpha}) + \xi^{\alpha}R_{\alpha\beta}u^{\alpha} + 2Du + 4\xi^{\alpha}u_{\alpha}$$

$+ 2(n-1)\xi^{\alpha}a_{\alpha} = 4(n+1)u^*$

In view of equations (1.16) and (1.24), we get

(1.25) $\eta_{\alpha}u^{\alpha} = \{1/(n-1)\}\{\xi^{\alpha}(\nabla^{\beta}\nabla_{\beta}u_{\alpha}) + 2Du + 4\xi^{\alpha}u_{\alpha}$

$+ 2(n-1)\xi^{\alpha}a_{\alpha} - 4(n+1)u^*\}.$

2. D - CONFORMAL VECTOR FIELD IN A PARA - SASAKIAN MANIFOLDS :

Definition 2.1:

If a vector field u^{α} satisfies the following relations

(2.1) $Lu\,(g_{\alpha\beta} - \eta_{\alpha}\eta_{\beta}) = 2b(g_{\alpha\beta} - \eta_{\alpha}\eta_{\beta})$

and

(2.2) $Lu\,\eta_{\alpha} = c\eta_{\alpha}$

then the vector field u^{α} in Para - Sasakian manifold is said to be D-conformal vector field with an associated function b.

Definition 2.2:

If α is constant in the definition (2.1) then D - conformal vector field is said to be D - homothetic vector field.

By the definition of D - conformal vector field, it is easy to verify

(2.3) $$\mathrm{Lu}\,(g^{\alpha\beta} - \xi^{\alpha}\xi^{\beta}) = 2b(g^{\alpha\beta} - \xi^{\alpha}\xi^{\beta})$$

and

(2.4) $$\mathrm{Lu}\,\xi^{\alpha} = - c\xi^{\alpha}.$$

In a Para - Sasakian manifolds, we have [5]:

(2.5) $$\eta_{\lambda}\,\mathrm{Lu}\{{}_{\alpha}{}^{\lambda}{}_{\beta}\} = \nabla_{\alpha}(\mathrm{Lu}\,\eta_{\beta}) - \mathrm{Lu}\,(\nabla_{\alpha}\eta_{\beta}),$$

and

(2.6) $$\mathrm{Lu}\{{}_{\alpha}{}^{\lambda}{}_{\beta}\} = (1/2)g^{\lambda\mu}\{\nabla_{\alpha}(\mathrm{Lu}\,g_{\beta\mu}) + \nabla_{\beta}(\mathrm{Lu}\,g_{\alpha\mu}) - \nabla_{\mu}(\mathrm{Lu}\,g_{\alpha\beta})\}.$$

By virtue of equations (1.6), (1.2), (2.5) and using the relation $c_{\alpha} = \nabla_{\alpha} c$, we get

(2.7) $$\mathrm{Lu}\,\phi_{\alpha\beta} + \eta_{\lambda}\,\mathrm{Lu}\{{}_{\alpha}{}^{\lambda}{}_{\beta}\} = c\phi_{\alpha\beta} + c_{\alpha}\eta_{\beta}$$

In view of equations (1.3), (1.7), (2.1), (2.2), (2.6) and using the relations $b_{\alpha} = \nabla_{\alpha} b$, $c_{\alpha} = (c_{\beta}\xi^{\beta})\eta_{\alpha}$, we obtain

(2.8) $$\mathrm{Lu}\{{}_{\alpha}{}^{\lambda}{}_{\beta}\} = (1/2)\{b_{\alpha}\{\delta^{\gamma}{}_{\beta} - \eta_{\beta}\xi^{\gamma}) - b_{\beta}\{\delta^{\gamma}{}_{\alpha} - \eta_{\alpha}\xi^{\gamma}) - b^{\gamma}\{g_{\alpha\beta} - \eta_{\alpha}\eta_{\beta})\} + 2c_{\alpha}\eta_{\beta}\xi^{\gamma} + c^{\gamma}\eta_{\alpha}\eta_{\beta} - \phi_{\alpha\beta}\xi^{\gamma} + 2c\phi_{\alpha\beta}\xi^{\gamma}$$

Contracting equation (2.8) by ξ^{α} and using equations (1.4), (1.9) and (1.10), we get

$$(2.9) \qquad Lu\{{}^{\ \lambda}_{\alpha\ \beta}\} = (1/2)\{b_{\alpha}\{\delta^{\gamma}{}_{\beta} - \eta_{\beta}\xi^{\gamma}) - b_{\beta}\{\delta^{\gamma}{}_{\alpha} - \eta_{\alpha}\xi^{\gamma}) - b^{\gamma}\{g_{\alpha\beta} - \eta_{\alpha}\eta_{\beta})\} + 2c_{\alpha}\eta_{\beta}\xi^{\gamma} + c^{\gamma}\eta_{\alpha}\eta_{\beta} - \phi_{\alpha\beta}\xi^{\gamma} + 2c\phi_{\alpha\beta}\xi^{\gamma}.$$

Theorem 2.1:

D - conformal vector field u^{α} in a Para - Sasakian manifolds holds the relation

$$Lu\ \phi_{\alpha\beta} - Lu\ \phi_{\beta\alpha}\ .$$

Proof:

By virtue of equation (2.7), we have

$$(2.10) \quad Lu\ \phi_{\beta\alpha} = c\phi_{\beta\alpha} + c_{\beta}\eta_{\alpha} - \eta_{\lambda}\ Lu\{{}^{\ \lambda}_{\beta\ \alpha}\},$$

On subtracting equation (2.10) from the equation (2.7) and using equation (1.8), we get

$$(2.11) \quad Lu\ \phi_{\alpha\beta} - Lu\ \phi_{\beta\alpha} = c_{\alpha}\eta_{\beta} - c_{\beta}\eta_{\alpha} - \eta_{\lambda}Lu\{{}^{\ \lambda}_{\alpha\ \beta}\} + \eta_{\lambda}Lu\{{}^{\ \lambda}_{\beta\ \alpha}\}$$

Since $\{{}^{\lambda}_{\alpha\ \beta}\} = \{{}^{\lambda}_{\beta\ \alpha}\}$, then we have

(2.12) $Lu\ \phi_{\alpha\beta} - Lu\ \phi_{\beta\alpha} = c_{\alpha}\eta_{\beta} - c_{\beta}\eta_{\alpha}$

In view of equation (2.12) and using the fact that $c_{\alpha} = (c_{\beta}\xi^{\beta})\eta_{\alpha}$, we obtain

(2.13) $Lu\ \phi_{\alpha\beta} - Lu\ \phi_{\beta\alpha} = 0$

This established the theorem.

Theorem 2.2:

If u^{α} is vector field of D - conformal in a Para - Sasakian manifolds then the relation

$$Lu\{{}^{\lambda}_{\alpha\ \beta}\} - Lu\{{}^{\lambda}_{\beta\ \alpha}\} = 0$$

holds good.

Proof:

On interchanging α and β in equation (2.8), then

(2.14) $Lu\{{}^{\lambda}_{\beta\ \alpha}\} = (1/2)\{b_{\beta}\{\delta^{\gamma}_{\alpha} - \eta_{\alpha}\xi^{\gamma}) - b_{\alpha}\{\delta^{\gamma}_{\beta} - \eta_{\beta}\xi^{\gamma}) - b^{\gamma}\{g_{\beta\alpha}$
$- \eta_{\alpha}\eta_{\beta})\} + 2c_{\beta}\eta_{\alpha}\xi^{\gamma} + c^{\gamma}\eta_{\alpha}\eta_{\beta} - \phi_{\beta\alpha}\xi^{\gamma} + 2c\phi_{\beta\alpha}\xi^{\gamma}$

On subtracting equation (2.14) from the equation (2.8) and using equations (1.8) and the relation $c_\alpha = (c_\beta \xi^\beta)\eta_\alpha$, we get

(2.15) $$Lu\{{}^{\lambda}_{\alpha\ \beta}\} - Lu\{{}^{\lambda}_{\beta\ \alpha}\} = 0$$

Theorem 2.3:

D - conformal vector field u^α in a Para - Sasakian manifolds holds the relation $c = f_\beta \xi^\beta$.

Proof:

Taking Lie-derivative in equation (1.9) on both sides, we get

(2.16) $$\eta_\alpha \, Lu\, \xi^\alpha + \xi^\alpha \, Lu\, \eta_\alpha = 0$$

In view of equations (2.2) and (2.16) and using equation (1.9), we obtain

(2.17) $$c = -\eta_\alpha \, Lu\, \xi^\alpha$$

From equations (1.9), (1.15) and (2.17), we get

(2.18) $c = f_\beta \xi^\beta$

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