**Mathematica Aeterna, Vol. 2, 2012, no. 5, 409 - 422**

# THE THIRD REGULARIZED TRACE FORMULA FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH SELF-ADJOINT NUCLEAR CLASS OPERATOR COEFFICIENTS

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**Abstract**

In this work we will obtain a formula for the third regularized trace of the self adjoint differential equation of second order with nuclear class operator coefficient given in a finite interval.

**Mathematics Subject Classification:** 34L10, 30E25, 34B24

**Keywords:** Hilbert space, self-adjoint operator, eigenvalue, spectrum, resolvent, kernel operator, regularized trace

## 1 Introduction

Let H be a separable Hilbert space. In the Hilbert space $H_1 = L_2([0,\pi]; H)$, we consider the self-adjoint operators $L_0$ and $L$ generated by the differential expressions

$$l_0(y) = -y''(x),$$

$$l(y) = -y''(x) + Q(x)\,y(x) \tag{1}$$

and the boundary conditions

$$y(0) = 0, \quad y'(\pi) = 0 \tag{2}$$

Suppose that the operator function $Q(x)$ in the expression $l(y)$ satisfies the following conditions:

1. For all $x \in (0,\pi)$, $Q(x) : H \to H$ is a self-adjoint nuclear operator. Morever, $Q(x)$ hasacontinuous derivative of fourth order with respect to the norm in space $\sigma_1(H)$ in the interval $[0,\pi]$ and for $x \in (0,\pi)$, $Q^{(i)}_{(x)} : H \to H$ are self-adjoint operators $(i = 1,2,3,4)$.

2. $\sup\limits_{0\le x\le\pi} \|Q(x)\|_H < 1$.

3. There is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of the space $H$ such that

$$\sum_{n=1}^{\infty} \|Q(x)\varphi_n\|_{H_1} < \infty.$$

$$4. \int_0^{\pi} Q(x)\,dx = 0$$

$$5. Q^{(2i-1)}_{(x)} = Q^{(2i)-1}(\pi) = 0, \quad (i = 1,2)$$

Here $\sigma_1(H)$ denotes the space of the nuclear operators from $H$ to $H$.

The spectrum of the operator $L_0$ is the set $\left\{\left(m-\frac{1}{2}\right)^2\right\}_{m=1}^{\infty}$, and every point of this set is an eigenvalue of $L_0$ with infinite multiplicity. The orthonormal eigenfunctions corresponding to the eigenvalue have the form

$$\psi^0_{mn} = \sqrt{\frac{2}{\pi}}\sin\left(m-\frac{1}{2}\right)x\cdot\varphi_n, \quad n = 1,2,... \tag{3}$$

In this work, we will fund a formula for the sum of the series

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda^3_{mn} - \left(m-\frac{1}{2}\right)^6\right) - \frac{3\left(m-\frac{1}{2}\right)^2}{4\pi}\int_0^{\pi} trQ^2(x)\,dx - \frac{3}{16\pi}\times\right.$$

$$\left.\int_0^{\pi} tr\left[Q^I(x)\right]^2 dx - \frac{1}{\pi}\int_0^{\pi} g(x)\,dx + h\right] = \frac{3}{64}\left[trQ^{(IV)}(\pi) - trQ^{(IV)}(o)\right] -$$

$$\frac{3}{8\pi}\left[trQ^{II}(\pi)Q(\pi) - trQ^{II}(o)\right] + \frac{1}{4\pi}\left[g(\pi) - g(o)\right] - \frac{h}{2},$$

where $h = \frac{15}{8}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\beta_{ij}|$,

$$\beta_{ij} = \frac{1}{\pi^3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\cos ix dx\int_0^{\pi}(Q(x)\varphi_q,\varphi_S)_H$$

$$\cos(i-j)xdx \times \int_0^{\pi}(Q(x)\varphi_S,\varphi_n)\cos jxdx,$$

$$g(x) = \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H(Q(x)\varphi_q,\varphi_S)_H(Q(x)\varphi_s,\varphi_n)$$

This formula is said to be the third regularized trace formula.

The sequences $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of the operator $L$ for every $m$=*1,2,...* which belong to the interval

$$\left[\left(m-\frac{1}{2}\right)^2-\|Q\|_{H_1},\left(m-\frac{1}{2}\right)^2+\|Q\|_{H_1}\right].$$

The trace formulas for the selfadjoint Sturm-Liouville differential operators have been found by Gelfand and Levitan [10], Dikiy [7], Gasimov and Levitan [9], Sadovnichiy and Podolskiy [14] and many others.

The trace formulas of the abstract self-adjoint operators with a continuous spectrum were first analyzed by Krein [12], Faddeyev [8], Bayramoglu [5]. On the other hand, trace formulas for the differential operators with operator coefficient have been investigated by Abdukadirov [1], Maksudov et al. [13], Adiguzelov and Baksi [2], Adiguzelov and Sezer [13], Albayrak et al. [4].

## 2 Investigating the spectrum of the operator $L$ some equalities about the resolvents

Let $R_\lambda^o$ and $R_\lambda$ be the resolvents of the operators $L_0$ and $L$, respectively.

**Lemma 2.1.** If the operator function $Q(x)$ satis fies the condition 3 and $\lambda\in\rho(L_0)$, then $QR_\lambda^0 : H_1\to H_1$ is a nuclear operator: $QR_\lambda^0\in\sigma_1(H_1)$.

**Proof.** The eigenfunctions system $\{\psi_{mn}^0\}_{m,n=1}^{\infty}$ of the operator $L_0$ is an orthonormal basis of the space $H_1$. As known from [6], to show that $QR_\lambda^0$ is a nuclear operator, it is enough to prove that the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\|QR_\lambda^0\psi_{mn}^0\right\|_{H_1} \tag{4}$$

is convergent.

From (3) we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|QR_\lambda^o\psi_{mn}^o\|_{H_1}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(m-\tfrac{1}{2}\right)^2-\lambda\right|^{-1}\|Q\psi_{mn}^o\|_{H_1}=$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(m-\tfrac{1}{2}\right)^2-\lambda\right|^{-1}\left[\int_0^{\pi}\tfrac{2}{\pi}\sin^2\left(m-\tfrac{1}{2}\right)x\,\|Q(x)\varphi_n\|_H^2\,dx\right]^{1/2}\leq$$

$$\leq\sqrt{\tfrac{2}{\pi}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(m-\tfrac{1}{2}\right)^2-\lambda\right|^{-1}\|Q(x)\varphi_n\|_{H_1}=\sum_{m=1}^{\infty}\left|\left(m-\tfrac{1}{2}\right)^2-\lambda\right|^{-1}$$

$$\sum_{n=1}^{\infty}\|Q(x)\varphi_n\|_{H_1}\leq C_\lambda\sum_{m=1}^{\infty}m^{-2}\sum_{n=1}^{\infty}\|Q(x)\varphi_n\|_{H_1} \tag{5}$$

Here $C_\lambda$ is a positive constant related only to $\lambda$. By virtue of condition 3 from (5) we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|QR_\lambda^o\psi_n^o\|_{H_1}<\infty.$$

**Teorem 2.1.** If the operator function $Q(x)$ satisfies conditions 1-3, then the spectrum of the operator $L$ is a subset of the union of the disjoint intervals

$$\Omega_m=\left[\left(m-\frac{1}{2}\right)^2-\|Q\|_{H_1},\left(m-\frac{1}{2}\right)+\|Q\|_{H_1}\right],\quad m=1,2,3,...$$

and the following conditions are satisfied:

a) Each point of the spectrum of the operator $L$ which is different from $\left(m-\frac{1}{2}\right)^2$ in $\Omega_m$ is as isolated eigenvalue which has finite multiplicity.

b) $\left(m-\frac{1}{2}\right)^2$ may be an eigenvalue of the operator $L$ which has finite or infinite multiplicity.

The equality $\lim\limits_{n\to\infty}\lambda_{mn}=\left(m-\frac{1}{2}\right)^2$ holds. Here $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues, belonging to the interval $\Omega_m$ of the operator $L$, and each eigenvalue has been repeated according to multiplicity.

**Proof.** The resolvent $R_\lambda$ of the operator $L$ satisfies the equation

$$R_\lambda^o-R_\lambda QR_\lambda^o=R_\lambda \tag{6}$$

If $\lambda\in R\Big|\bigcup\limits_{m=1}^{\infty}\left[\left(m-\frac{1}{2}\right)^2-\|Q\|_{H_1},\left(m-\frac{1}{2}\right)^2+\|Q\|_{H_1}\right]$, then we have

$$\left|\lambda-\left(m-\frac{1}{2}\right)^2\right|>\|Q\|_{H_1},m=1,2,..., \tag{7}$$

For the self-adjoint operator $R_\lambda^o=(L_0-\lambda I)^{-1}$,

$$\|R_\lambda^o\|_{H_1}=\max_m\left|\lambda-\left(m-\frac{1}{2}\right)^2\right|^{-1}$$

holds . From here and (7), we obtain

$$\|R_\lambda^o\|_{H_1}<\|Q\|_{H_1}^{-1}.$$

And so, we get

$$\|QR_\lambda^o\|_{H_1}\le\|Q\|_{H_1}\|R_\lambda^o\|<1.$$

Thus, $A(B)=R_\lambda^o-BQR_\lambda^o$ is a contraction operator from $L(H_1)$ to $L(H_1)$. Here $L(H_1)$ is the linear bounded operator space from $H_1$ to $H_1$. According to this, $A(R_\lambda)R_\lambda$, that is, equation (6) has a unique solution $R_\lambda\in L(H_1)$. Thus, every point $\lambda \notin \bigcup\limits_{m=1}^{\infty} \Omega_m$ is the regular point of the self-adjoint operator $L$. So, the spectrum of operator $L$ is $\sigma(L) \subset U_{m=1}^{\infty}\Omega_m$. From formula (6) and lemma 2.1, for every $\lambda \in \rho(L) \bigcap \rho(L_0)$, $R_\lambda - R_\lambda^o$ belongs to $\sigma_1(H_1)$, that is, $R_\lambda - R_\lambda^o$ is a nuclear operator. In this case, as it was proved in Kato [11, *page*244], the continuous parts of the spectra of operators $L_0$ and $L$ coincide. According to this, and since the spectrum of operator $L_0$ is continuous, the continuous part of the spectra operator $L$ is the set $\left\{\left(m-\frac{1}{2}\right)^2\right\}_{m=1}^{\infty}$. This also means that assertions (a), (b) and (c) of Theorem 2.1 are satisfied.

In the case when conditions (b) and (c) are satisfied, it can be proved that the series

$$\sum_{n=1}^{\infty}\left[\lambda_{mn}-\left(m-\frac{1}{2}\right)^2\right], \quad (m=1,2,...)$$

are absolutely convergent. On the other hand, if we consider $R_\lambda - R_\lambda^o \in \sigma_1(H_1)$ for every $\lambda \in \rho(L)$, then we get

$$tr\left(R_\lambda - R_\lambda^o\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_{mn}-\lambda} - \frac{1}{\left(m-\frac{1}{2}\right)^2-\lambda}\right].$$

If we multiply both sides of this equality by $\frac{\lambda^3}{2\pi i}$ and integrate this equality over the circle $|\lambda| = bp = \left(p-\frac{1}{2}\right)^2 + p$, $(p \geq 1)$, then we find

$$\frac{1}{2\pi i}\int_{|\lambda|=bp}\lambda^3 tr\left(R_\lambda - R_\lambda^0\right)d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=b_P}\lambda^3\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_{mn}-\lambda}-\frac{1}{\left(m-\frac{1}{2}\right)^2-\lambda}\right]d\lambda$$

$$+\frac{1}{2\pi i}\int_{|\lambda|=bp}\lambda^3\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_{mn}-\lambda}-\frac{1}{\left(m-\frac{1}{2}\right)^2-\lambda}\right]d\lambda \tag{8}$$

For $m \leq p$ and $p \geq 1$ by condition 2, we can write

$$\left(m-\frac{1}{2}\right)^2 - \|Q\|_{H_1} \leq \lambda_{mn} \leq \left(m-\frac{1}{2}\right)^2 + \|Q\|_{H_1} < \left(p-\frac{1}{2}\right)^2 + p = b_P.$$

There fore we get

$$|\lambda_{mn}| < b_p, \quad m \leq p, \quad p \geq 1, \quad n = 1,2,... \tag{9}$$

Moreover, for $m > p$,

$$\lambda_{mn} \geq \left(m-\frac{1}{2}\right)^2 - \|Q\|_{H_1} \geq \left(p+1-\frac{1}{2}\right)^2 - \|Q\|_{H_1} > \left(p-\frac{1}{2}\right)^2 + p = b_p$$

Thus we get

$$\lambda_{mn} > b_p; \quad m > p; \quad n = 1,2,... \tag{10}$$

Using (9), (10), from (8) we have

$$\frac{1}{2\pi i}\int_{|\lambda|=b_P}\lambda^3 tr\left(R_\lambda-R^o_\lambda\right)d\lambda=$$
$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{|\lambda|=bp}\frac{\lambda^3 d\lambda}{\lambda-\left(m-\frac{1}{2}\right)^2}-\frac{1}{2\pi i}\int_{|\lambda|=b_P}\frac{\lambda^3 d\lambda}{\lambda-\lambda_{mn}}\right]+$$
$$+\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{|\lambda|=b_P}\frac{\lambda^3 d\lambda}{\lambda-\left(m-\frac{1}{2}\right)^2}-\frac{1}{2\pi i}\int_{|\lambda|=b_P}\frac{\lambda^3 d\lambda}{\lambda-\lambda_{mn}}\right]=$$
$$=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\left(m-\frac{1}{2}\right)^6-\lambda^3_{mn}\right] \tag{11}$$

Moreover from the formula $R_\lambda = R^o_\lambda \cdot R_\lambda Q R^o_\lambda$, we obtain the following equality

$$R_\lambda-R^o_\lambda=\sum_{j=1}^{N}(-1)^j R^o_\lambda\left(QR^0_\lambda\right)^j+(-1)^{N+1}R_\lambda\left(QR^o_\lambda\right)^{N+1} \tag{12}$$

where $N$ is any natural number.

From (11) and (12), we have

$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\left(m-\frac{1}{2}\right)^6-\lambda^3_{mn}\right]=\sum_{j=1}^{N}\frac{(-1)^j}{2\pi i}\int_{|\lambda|=b_P}\lambda^3 tr\left[R^o_\lambda QR^o_\lambda\right]^j d\lambda+$$
$$+\frac{(-1)^{N+1}}{2\pi i}\int_{|\lambda|=b_P}\lambda^3 tr\left[R_\lambda\left(QR^o_\lambda\right)^{N+1}\right]d\lambda \tag{13}$$

Let

$$M_{pj}=\frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_P}\lambda^3 tr\left[R^o_\lambda\left(QR^o_\lambda\right)^j\right]d\lambda \tag{14}$$

$$M_{PN}=\frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_P}\lambda^3 tr\left[R^o_\lambda\left(QR^o_\lambda\right)^{N+1}\right]d\lambda \tag{15}$$

Then from (13), (14) and (15) ve have

$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\lambda^3_{mn}-\left(m-\frac{1}{2}\right)^6\right]=\sum_{j=1}^{N}M_{Pj}+M_{PN} \tag{16}$$

Since the operator function $QR^o_\lambda$ in the domain $C\left|\left\{m-\frac{1}{2}\right\}^2_{m=1}\right.$ is analytic with respect to the norm in space $\sigma_1(H_1)$, one can show for $M_{Pj}$ the following formula is true

$$M_{Pj}=\frac{3(-1)^j}{2\pi ij}\int_{|\lambda|=b_P}\lambda^2 tr\left[QR^o_\lambda\right]d\lambda \tag{17}$$

# 3 The formula of the regularized trace of the operator $L$

In this section we will find a formula for the regularized trace of the operator $L$

According to formula (17)

$$M_{Pj} = \frac{3}{2\pi i} \int_{|\lambda|=b_P} \lambda^2 tr \left[ Q R_\lambda^o \right] d\lambda.$$

Since the eigenfunctions $\{\psi_{mn}^o\}_{m=1,n=1}^{\infty}$ on the operator $L_0$ is an orthonormal basis of the space $H_1$, then

$$M_{Pj} = \tfrac{3}{2\pi i} \textstyle\int_{|\lambda|=bp} \lambda^2 tr \left[ Q R_\lambda^o \right] d\lambda = -\tfrac{3}{2\pi i} \int_{|\lambda|=b_P} \lambda^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( Q R_\lambda^o \psi_{mn}^o, \psi_{mn}^o \right)_{H_1} d\lambda =$$

$$= 3 \textstyle\sum_{m=1}^{p} \sum_{n=1}^{\infty} \left( \frac{1}{2\pi i} \int_{|\lambda|=b_P} \frac{\lambda^2 d\lambda}{\lambda - \left(m-\frac{1}{2}\right)^2} \right) \left( Q \psi_{mn}^o, \psi_{mn}^o \right)_{H_1} =$$

$$= 3 \textstyle\sum_{m=1}^{p} \sum_{n=1}^{\infty} \left( m - \frac{1}{2} \right)^4 \left( Q \psi_{mn}^o, \psi_{mn}^o \right) =$$

$$= 3 \textstyle\sum_{m=1}^{p} \sum_{n=1}^{\infty} \left( m - \frac{1}{2} \right)^4 \frac{2}{\pi} \int_0^\pi \left( Q(x) \varphi_n, \varphi_n \right) \sin^2 \left( m - \frac{1}{2} \right) x dx =$$

$$= \tfrac{3}{\pi} \textstyle\sum_{m=1}^{p} \sum_{n=1}^{\infty} \left( m - \frac{1}{2} \right)^4 \int_0^\pi \left( Q(x) \varphi_n, \varphi_n \right)_H dx -$$

$$- \tfrac{3}{\pi} \textstyle\sum_{m=1}^{p} \sum_{n=1}^{\infty} \left( m - \frac{1}{2} \right)^4 \int_0^\pi \left( Q(x) \varphi_n, \varphi_n \right) \cos (2m-1) x dx. \tag{18}$$

If we take into account the inequality

$$\left| \textstyle\sum_{n=1}^{q} \left( Q(x) \varphi_n, \varphi_n \right)_H \right| \leq \sum_{n=1}^{\infty} \left| \left( Q(x) \varphi_n, \varphi_n \right)_H \right| \leq \| Q(x) \|_{\sigma_1(H)},$$

$$\left| \textstyle\sum_{n=1}^{q} \left( Q(x) \varphi_n, \varphi_n \right) \cos (2m-1) x \right| \leq \sum_{n=1}^{\infty} \left| \left( Q(x) \varphi_n, \varphi_n \right)_H \right| \leq \| Q(x) \|_{\sigma_1(H)}$$

and condition

$$\int_0^\pi \| Q(x) \|_{\sigma_1(H)} dx < \infty$$

then by Leveque's theorem we obtain

$$\sum_{n=1}^{\infty} \int_0^\pi \left( Q(x) \varphi_n, \varphi_n \right)_H dx = \int_0^\pi \sum_{n=1}^{\infty} \left( Q(x) \varphi_n, \varphi_n \right)_H dx = \int_0^\pi tr Q(x) dx \tag{19}$$

$$\sum_{n=1}^{\infty} \int_0^\pi \left( Q(x) \varphi_n, \varphi_n \right)_H dx \cos (2m-1) x dx = \int_0^\pi tr Q(x) \cdot \cos (2m-1) x dx \tag{20}$$

From (18), (19) and (20) we have

$$M_{P1}=-\frac{3}{\pi}\sum_{m=1}^{P}\left(m-\frac{1}{2}\right)^{4}\int_{0}^{\pi}trQ\left(x\right)\cdot\cos\left(2m-1\right)xdx+$$

$$+\frac{3}{\pi}\sum_{m=1}^{P}\left(m-\frac{1}{2}\right)^{2}\cdot\int_{0}^{\pi}trQ\left(x\right)dx \qquad (21)$$

Then after the application of integration on parts in the first integral of equality (21) for four times we achieve

$$M_{P1}=-\frac{3}{16\pi}\sum_{m=1}^{P}\int_{0}^{\pi}trQ^{(IV)}\left(x\right)\cdot\cos\left(2m-1\right)xdx+$$

$$+\frac{3}{\pi}\sum_{m=1}^{P}\left(m-\frac{1}{2}\right)^{2}\cdot\int_{0}^{\pi}trQ\left(x\right)dx \qquad (22)$$

Let us calculate $M_{P2}$. According to formula (17)

$$M_{P2}=\frac{3}{4\pi i}\int_{|\lambda|=b_P}\lambda^{2}tr\left[\left(QR_{\lambda}^{o}\right)^{2}\right]d\lambda=\frac{3}{4\pi i}\int_{|\lambda|=b_P}\lambda^{2}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\left(QR_{\lambda}^{o}\right)^{2}\psi_{mn}^{o},\psi_{mn}^{o}\right]d\lambda \qquad (23)$$

Moreover

$$QR_{\lambda}^{o}\psi_{mn}^{o}=\left[\left(m-\frac{1}{2}\right)^{2}-\lambda\right]^{-1}Q\psi_{mn}^{o}$$

$$QR_{\lambda}^{o}\psi_{mn}^{o}=\left[\left(m-\frac{1}{2}\right)^{2}-\lambda\right]^{-1}QR_{\lambda}^{o}\left\{\sum_{i=1}^{\infty}\sum_{q=1}^{\infty}\left(Q\psi_{mn}^{o},\psi_{rq}^{o}\right)\psi_{rq}\right\}=$$

$$=\left[\left(m-\frac{1}{2}\right)^{2}-\lambda\right]^{-1}\sum_{i=1}^{\infty}\sum_{q=1}^{\infty}\left[\left(r-\frac{1}{2}\right)^{2}-\lambda\right]^{-1}\left(Q\psi_{mn}^{o},\psi_{rq}^{o}\right)Q\psi_{mn}^{o} \qquad (24)$$

From (23) and (24) we get

$$M_{P2}=\frac{3}{4\pi i}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\psi_{mn}^{0},\psi_{rq}^{o}\right)\right|^{2}\int_{|\lambda|=bp}\frac{\lambda^{2}d\lambda}{\left[\lambda-\left(m-\frac{1}{2}\right)^{2}\right]\left[\lambda-\left(r-\frac{1}{2}\right)^{2}\right]}=$$

$$=\frac{3}{4\pi i}\sum_{m=1}^{P}\sum_{n=1}^{\infty}\sum_{r=1}^{P}\sum_{q=1}^{\infty}\left|\left(Q\psi_{mn}^{0},\psi_{rq}^{o}\right)\right|^{2}\int_{|\lambda|=bp}\frac{\lambda^{2}d\lambda}{\left[\lambda-\left(m-\frac{1}{2}\right)^{2}\right]\left[\lambda-\left(r-\frac{1}{2}\right)^{2}\right]}+$$

$$+\frac{3}{4\pi i}\sum_{m=1}^{P}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\psi_{mn}^{0},\psi_{rq}^{o}\right)\right|^{2}\int_{|\lambda|=bp}\frac{\lambda^{2}d\lambda}{\left[\lambda-\left(m-\frac{1}{2}\right)^{2}\right]\left[\lambda-\left(r-\frac{1}{2}\right)^{2}\right]}+$$

$$+\frac{3}{4\pi i}\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{P}\sum_{q=1}^{\infty}\left|\left(Q\psi_{mn}^{0},\psi_{rq}^{o}\right)\right|^{2}\int_{|\lambda|=bp}\frac{\lambda^{2}d\lambda}{\left[\lambda-\left(m-\frac{1}{2}\right)^{2}\right]\left[\lambda-\left(r-\frac{1}{2}\right)^{2}\right]}+$$

$$+\frac{3}{4\pi i}\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\psi_{mn}^{0},\psi_{rq}^{o}\right)\right|^{2}\int_{|\lambda|=bp}\frac{\lambda^{2}d\lambda}{\left[\lambda-\left(m-\frac{1}{2}\right)^{2}\right]\left[\lambda-\left(r-\frac{1}{2}\right)^{2}\right]}=$$

$$=\frac{3}{2}\sum_{m=1}^{P}\sum_{n=1}^{\infty}\sum_{r=1}^{P}\sum_{q=1}^{\infty}\left[\left(m-\frac{1}{2}\right)^{2}+\left(r-\frac{1}{2}\right)^{2}\right]\left|\left(Q\psi_{mn}^{0},\psi_{mn}^{o}\right)\right|^{2}+$$

$$+3\sum_{m=1}^{P}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\frac{\left(m-\frac{1}{2}\right)^{4}}{\left(m-\frac{1}{2}\right)^{2}-\left(r-\frac{1}{2}\right)^{2}}\cdot\left|\left(Q\psi_{mn}^{0},\psi_{rq}^{o}\right)\right|^{2} \tag{25}$$

Here we use that in the case $m,r<p$

$$\int_{|\lambda|=bp}\frac{\lambda^{2}d\lambda}{\left[\lambda-\left(m-\frac{1}{2}\right)^{2}\right]\left[\lambda-\left(r-\frac{1}{2}\right)^{2}\right]}=0, \tag{26}$$

In the case $m=r$

$$\int_{|\lambda|=bp}\frac{\lambda^{2}d\lambda}{\left[\lambda-\left(m-\frac{1}{2}\right)^{2}\right]^{2}}=0.$$

If in the expression

$$\sum_{m=1}^{P}\sum_{n=1}^{\infty}\sum_{r=1}^{P}\sum_{q=1}^{\infty}\left(r-\frac{1}{2}\right)^{2}\left|\left(Q\psi_{mn}^{0},\psi_{mn}^{o}\right)\right|^{2}$$

to after the index $m$ with $r$ and $n$ with $q$, then we obtain

$$\sum_{m=1}^{P}\sum_{n=1}^{\infty}\sum_{r=1}^{P}\sum_{q=1}^{\infty}\left(r-\frac{1}{2}\right)^{2}\left|\left(Q\psi_{mn}^{0},\psi_{mn}^{o}\right)\right|^{2}=$$

$$=\sum_{m=1}^{P}\sum_{n=1}^{\infty}\sum_{r=1}^{P}\sum_{q=1}^{\infty}\left(m-\frac{1}{2}\right)^{2}\left|\left(Q\psi_{rq}^{0},\psi_{mn}^{o}\right)\right|^{2} \tag{27}$$

hence

$$M_{P2}=3\sum_{m=1}\sum_{n=1}\left(m-\frac{1}{2}\right)^{2}\left\|Q\psi_{mn}^{0}\right\|^{2}-$$

$$-3\sum_{m=1}^{P}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\frac{\left(m-\frac{1}{2}\right)^{2}\left(r-\frac{1}{2}\right)^{2}}{\left(r-\frac{1}{2}\right)^{2}\left(m-\frac{1}{2}\right)^{2}}\left|\left(Q\psi_{mn}^{0},\psi_{rq}^{o}\right)\right|^{2} \tag{28}$$

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$$\alpha_P = \sum_{m=1}^{P} \sum_{r=p+1}^{\infty} \frac{\left(m-\frac{1}{2}\right)^2 \left(r-\frac{1}{2}\right)^2}{\left(r-\frac{1}{2}\right)^2 \left(m-\frac{1}{2}\right)^2} \left|\left(Q\psi_{mn}^0, \psi_{rq}^o\right)\right|^2 \tag{29}$$

We present $\alpha_P$ in the following form

$$\alpha_P = \alpha_{P1} - \alpha_{P2} + \alpha_{P3}, \tag{30}$$

where

$$\alpha_{P1} = \frac{1}{\pi^2} \sum_{m=1}^{P} \sum_{r=p+1}^{\infty} \frac{\left(m-\frac{1}{2}\right)^2 \left(r-\frac{1}{2}\right)^2}{\left(r-\frac{1}{2}\right)^2 \left(m-\frac{1}{2}\right)^2} \left|\int_0^\pi (Q(x)\varphi_n, \varphi_q)_H \cos(r-m)\,xdx\right|^2 \tag{31}$$

$$\alpha_{P2} = \frac{1}{2\pi^2} \sum_{m=1}^{P} \sum_{r=p+1}^{\infty} \frac{\left(m-\frac{1}{2}\right)^2 \left(r-\frac{1}{2}\right)^2}{\left(r-\frac{1}{2}\right)^2 \left(m-\frac{1}{2}\right)^2} \cdot Re\left[\int_0^\pi (Q(x)\varphi_n, \varphi_q)_H \cos(r-m)\,xdx \times\right.$$

$$\left. \times \int_0^\pi \overline{(Q(x)\varphi_n, \varphi_q)}_H \cos(r-m)\,xdx \right] \tag{32}$$

$$\alpha_{P3} = \frac{1}{\pi^2} \sum_{m=1}^{P} \sum_{r=p+1}^{\infty} \frac{\left(r-\frac{1}{2}\right)^2 \left(m-\frac{1}{2}\right)^2}{\left(r-\frac{1}{2}\right)^2 \left(m-\frac{1}{2}\right)^2} \left|\int_0^\pi (Q(x)\varphi_n, \varphi_q)_H \cos(r+m-1)\,xdx\right|^2 \tag{33}$$

From (28)-(33) we obtain

$$M_{P2} = 3\sum_{m=1}^{P} \sum_{n=1}^{\infty} \left(m-\frac{1}{2}\right)^2 \|Q\psi_{mn}^o\|^2 - 3\sum_{n=1}^{\infty} \sum_{q=1}^{\infty} (\alpha_{P1} - \alpha_{P2} + \alpha_{P3}) \tag{34}$$

Separately evaluating $\alpha_{P1}, \alpha_{P2}$ and $\alpha_{P3}$, we obtain

$$M_{P2} = \sum_{m=1}^{P} \left[\frac{3}{4\pi}\left(m-\frac{1}{2}\right)^2 \int_0^\pi trQ^2(x)\,dx + \frac{3}{16\pi}\int_0^\pi tr\left[Q^I(x)\right]^2 dx + \right.$$

$$\left. + \frac{3}{2\pi}\int_0^\pi tr\left[Q^{II}(x)\,Q(x) + \left[Q^I(x)\right]^2\right] \cos(2m-1)\,xdx + O\left(p^{-1}\right), \right. \tag{35}$$

where $|O(p^{-1})| < const \cdot p^{-1}$.

Researching in detail $M_{P3}$, after very complicated calculation we obtain.

$$M_{P3} = \frac{P}{\pi}\int_0^\pi g(x)\,dx - \frac{1}{\pi}\sum_{\ell=1}^{P} \int_0^\pi g(x)\cos(2\ell-1)\,xdx -$$

$$-\frac{15}{16}(2p+1)\sum_{i=1}^{P}\sum_{j=1}^{P} \beta_{ij} + o(1), \tag{36}$$

where

$$g(x)=\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\left(Q(x)\varphi_n,\varphi_q\right)_H\left(Q(x)\varphi_q,\varphi_s\right)_H\cdot\left(Q(x)\varphi_s,\varphi_n\right) \tag{37}$$

$$\beta_{ij}=\frac{1}{\pi^3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\int_0^{\pi}\left(Q(x)\varphi_n,\varphi_q\right)_H\cos ixdx\cdot\int_0^{\pi}\left(Q(x)\varphi_q,\varphi_s\right)_H\cos(i-j)xdx\times$$
$$\times\int_0^{\pi}\left(Q(x)\varphi_s,\varphi_n\right)_H\cos jxdx \tag{38}$$

Thus, as $|\lambda|=bp=p^2+\frac{1}{4}$, there is such an stable $c$, where the following inequality is true:

$$\|QR_\lambda^o\|\,\sigma_1(H_1)<C,\quad \|R_\lambda^o\|<Cp^{-1},\quad \|R_\lambda\|<Cp^{-1} \tag{39}$$

From (14), (40) and condition 2 we obtain

$$|M_{pj}|\le\tfrac{1}{j}\int_{|\lambda|=bp}\left|\lambda^2 tr\left[QR_\lambda^o\right]^j d\lambda\right|\le\tfrac{b_p^2}{j}\int_{|\lambda|=b_p}\left\|\left(QR_\lambda^o\right)^j\right\|_{\sigma_1(H_1)}|d\lambda|\le$$

$$\le\tfrac{b_p^2}{j}\int_{|\lambda|=b_p}\|(QR_\lambda^o)\|_{\sigma_1(H_1)}\cdot\left\|\left(QR_\lambda^o\right)^{j-1}\right\|\,|d\lambda|\le\tfrac{Cb_p^2}{j}\int_{|\lambda|=b_p}\|Q\|^j\cdot\|R_\lambda\|^{j-1}|d\lambda|<$$

$$<\tfrac{C^j b_p^2}{j}\int_{|\lambda|=bp}p^{1-j}|d\lambda|<C_1b_p^3\cdot p^{1-j}<C_2p^{7-j}$$

here $C_1$ and $C_2$ are stable figures.

From the last inequality we obtain

$$\lim_{p\to\infty}M_{pj}=0,\quad j\ge 8 \tag{40}$$

with $j=4,5,6,7$ we can also prove that

$$\lim_{p\to 0}M_{pj}=0. \tag{41}$$

From (40) and (41) we obtain

$$\lim_{p\to\infty}M_{pj}=0,\quad j\ge 4. \tag{42}$$

From (15) and (40) we obtain

$$|M_{PN}|\le\left|\int_{|\lambda|=bp}\lambda^3 tr\left[R_\lambda\left(QR_\lambda^o\right)^{N+1}\right]d\lambda\right|\le b_p^3\cdot\int_{|\lambda|=bp}\left\|R_\lambda\left(QR_\lambda^o\right)^{N+1}\right\|_{\sigma_1(H_1)}|d\lambda|\le$$

$$\le b_p^3\cdot\int_{|\lambda|=bp}\|R_\lambda\|\cdot\left\|\left(QR_\lambda^o\right)^{N+1}\right\|_{\sigma_1(H_1)}|d\lambda|\le$$

$$Cb_p^3\cdot p^{-1}\int_{|\lambda|=b_P}\left(\|QR_\lambda^o\|_H\right)^N\cdot\|QR_\lambda^o\|_{\sigma_1(H_1)}|d\lambda|\le$$

$$\le C_3b_p^4p^{-1-N}<C_4p^{7-N}$$

here $C_3$ and $C_4$ are stable figure.

From the last inequality we obtain

$$\lim_{p\to\infty} M_{PN} = 0, \quad N \geq 8 \tag{43}$$

The main result of this article is given by the following theorem.

**Theorem 3.1.** If the operator function $Q(x)$ satisfies conditions 1.-5., then

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}^{3}-\left(m-\frac{1}{2}\right)^{6}\right)-\frac{3\left(m-\frac{1}{2}\right)^{2}}{4\pi}\int_{0}^{\pi} trQ^{2}(x)\,dx-\right.$$

$$\left.-\frac{3}{16\pi}\int_{0}^{\pi} tr\left(Q^{I}(x)\right)^{2}dx-\frac{1}{\pi}\int_{0}^{\pi} g(x)\,dx+h\right]=\frac{3}{64}\left[trQ^{(IV)}(\pi)-trQ^{(IV)}(o)\right]-$$

$$-\frac{3}{8\pi}\left[trQ^{II}(\pi)Q(\pi)-trQ^{II}(o)\cdot Q(o)\right]+\frac{1}{4\pi}\left[g(\pi)-g(o)\right]-\frac{h}{2}, \tag{44}$$

here $h = \frac{15}{8}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta_{ij}$, $\beta_{ij}-$ figure are defined by means of equality (38), but the function $g(x)$ is defined from the equality (36).

The series on the left side of this equality is called the third regularized trace of the operator $L$.

**Proof.** From formulas (16), (22), (36), (42) and (43), we obtain.

$$\sum_{m=1}^{P}\sum_{n=1}^{\infty}\left[\lambda_{mn}^{3}-\left(m-\frac{1}{2}\right)^{6}\right]=-\frac{3}{16\pi}\sum_{m=1}^{P}\int_{0}^{\pi} trQ^{(IV)}(x)\cos(2m-1)\,xdx+$$

$$+\sum_{m=1}^{P}\left[\frac{3\left(m-\frac{1}{2}\right)^{2}}{4\pi}\int_{0}^{\pi} trQ^{2}(x)\,dx+\frac{3}{16\pi}\int_{0}^{\pi} tr\left(Q^{I}(x)\right)^{2}dx+\right.$$

$$+\frac{3}{2\pi}\int_{0}^{\pi} tr\left[Q^{II}(x)\,Q(x)+\left[Q^{I}(x)\right]^{2}\right]\cos(2m-1)\,xdx+$$

$$+\frac{P}{\pi}\int_{0}^{\pi} g(x)\,dx-\frac{1}{\pi}\sum_{\ell=1}^{P}\int_{0}^{\pi} g(x)\cos(2\ell-1)\,xdx-$$

$$-\frac{15P}{8}\sum_{i=1}^{P}\sum_{j=1}^{P}\beta_{ij}-\frac{15}{16}\sum_{i=1}^{P}\sum_{j=1}^{P}\beta_{ij}+o(1) \tag{45}$$

From the Fourier theory it is known that if the function $f(x)$ is continuous on the segment $[0,\pi]$, then the following equality is true:

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{n=1}^{\pi} f(x)\cos(2m-1)\,xdx=\frac{1}{4}\left[f(o)-f(\pi)\right]. \tag{46}$$

From (45), (46) and condition 5. we obtain

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}^{3}-\left(m-\frac{1}{2}\right)^{6}-\frac{3\left(m-\frac{1}{2}\right)}{4\pi}\int_{0}^{\pi}trQ^{2}\left(x\right)dx-\frac{3}{16\pi}\int_{0}^{\pi}tr\left[Q^{I}\left(x\right)\right]^{2}dx-\right.\right.$$

$$\left.-\frac{1}{\pi}\int_{0}^{\pi}g\left(x\right)dx+\frac{15}{8}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta_{ij}\right]=\frac{3}{64}\left[trQ^{(IV)}\left(\pi\right)-trQ^{(IV)}\left(o\right)\right]-$$

$$-\frac{3}{8\pi}\left[trQ^{II}\left(\pi\right)Q\left(\pi\right)-trQ\left(\pi\right)Q\left(o\right)\right]+\frac{1}{4\pi}\left[g\left(\pi\right)-g\left(o\right)\right]-\frac{15}{16}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta_{ij}.$$

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**Received: June, 2012**