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New application of power increasing sequences

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Abstract

In the present paper, a main theorem dealing with $\mid \bar{N}, p_n \mid_k$ summability factors has been generalized to the $\mid \bar{N}, p_n, \theta_n \mid_k$ summability factors using a new general class of power increasing sequences. Some new results have also been obtained.

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1 Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). We write $\mathcal{BV}_\mathcal{O} = \mathcal{BV} \cap \mathcal{C}_\mathcal{O}$, where $\mathcal{C}_\mathcal{O} = \{ \ x = (x_k) \in \Omega \ : \ \lim_k |x_k| = 0 \ \}$, $\mathcal{BV}$=$\{ \ x = (x_k) \in \Omega \ : \ \sum_k |x_k - x_{k+1}| < \infty \ \}$ and Ω being the space of all real or complex-valued sequences. A positive sequence $X = (X_n)$ is said to be a quasi-δ-power increasing sequence if there exists a constant $K = K(\delta, X) \geq 1$ such that $Kn^\delta X_n \geq m^\delta X_m$ holds for all $n \geq m \geq 1$ (see [9]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n the nth (C,1) mean of the sequence (na_n), that is , $t_n = \frac{1}{n}\sum_{v=1}^n va_v$. A series $\sum a_n$ is said to be summable $\mid C, 1 \mid_k$, $k \geq 1$, if (see [6], [8])

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n \mid^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \rightarrow \infty \quad as \quad n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta \sigma_{n-1} \mid^k < \infty, \tag{4}$$

where

$$\Delta \sigma_{n-1} = \sigma_n - \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{5}$$

In the special case $p_n = 1$ for all values of n, $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ summability. Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid \Delta \sigma_{n-1} \mid^k < \infty. \tag{6}$$

If we take $\theta_n = \frac{P_n}{p_n}$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid \bar{N}, p_n \mid_k$ summability. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $\mid C, 1 \mid_k$ summability.
Furthermore, if we take $\theta_n = n$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid R, p_n \mid_k$ (see [4]) summability.

2. Known Result. In [5], we have proved the following main theorem dealing with $\mid \bar{N}, p_n \mid_k$ summability factors.

Theorem A. Let $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$ and (X_n) be a quasi-δ-power increasing sequence for some δ $(0 < \delta < 1)$. Suppose also that there exist sequences (δ_n) and (λ_n) such that

$$\mid \Delta \lambda_n \mid \leq \beta_n, \tag{7}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} n \mid \Delta \beta_n \mid X_n < \infty, \tag{9}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{10}$$

If

$$\sum_{v=1}^{n} \frac{\mid s_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{11}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{12}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{13}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$,$k \geq 1$. If we take (X_n) as an almost increasing sequence in Theorem A, then we get a result which was published in [3]. In this case the condition $(\lambda_n) \in \mathcal{BV_O}$ is not needed .

Remark. In Theorem A , we can take $(\lambda_n) \in \mathcal{BV}$ instead of $(\lambda_n) \in \mathcal{BV_O}$, because it s sufficient to prove the theorem .

2 Main Results

The aim of this paper is to generalize Theorem A for $\mid \bar{N}, p_n, \theta_n \mid_k$ summability. Now , we shall prove the following general theorem.

Theorem. Let $(\lambda_n) \in \mathcal{BV}$ and (X_n) be a quasi-δ-power increasing sequence for some δ $(0 < \delta < 1)$. If the conditions (7)-(10), (12)-(13) and

$$\sum_{v=1}^{n} \theta_v^{k-1} v^{-k} \mid s_v \mid^k = O(X_n) \quad as \quad n \to \infty, \tag{14}$$

are satisfied and $\left(\frac{\theta_n p_n}{P_n}\right)$ is a non-increasing sequence, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n, \theta_n \mid_k$, $k \geq 1$. If we take $\theta_n = \frac{P_n}{p_n}$, then we obtain Theorem A. In this case, condition (14) reduces to the condition (12) and the condition " $\left(\frac{\theta_n p_n}{P_n}\right)$ is a non-increasing sequence " is automatically satisfied.

We require the following lemmas for the proof of the theorem.

Lemma 1([9]). Except for the condition $(\lambda_n) \in \mathcal{BV}$, under the conditions on (X_n), (β_n) and (λ_n) as expressed in the statement of the theorem, we have the following :

$$nX_n \beta_n = O(1), \tag{15}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{16}$$

Lemma 2 ([10]). If the conditions (12) and (13) are satisfied, then we have that

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{17}$$

4. Proof of the theorem. Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{v p_v}. \tag{18}$$

Then , for $n \geq 1$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v}, \; n \geq 1. \tag{19}$$

Using Abel's transformation, we get

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \Delta \left(\frac{P_{v-1} P_v \lambda_v}{v p_v} \right) + \frac{\lambda_n s_n}{n} \\
&= \frac{s_n \lambda_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \frac{P_{v+1} P_v \Delta \lambda_v}{(v+1) p_{v+1}} \\
&+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta \left(\frac{P_v}{v p_v} \right) - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \lambda_v \frac{1}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{20}$$

Firstly , by using Abel's transformation, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,1} \mid^k &= \sum_{n=1}^{m} \theta_n^{k-1} n^{-k} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid\mid s_n \mid^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \theta_n^{k-1} n^{-k} \mid s_n \mid^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \theta_v^{k-1} v^{-k} \mid s_v \mid^k \\
&+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \theta_n^{k-1} n^{-k} \mid s_n \mid^k \\
&= O(1) \sum_{n=1}^{m-1} \mid \Delta \lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 1.
Now, using the fact that $P_{v+1} = O((v+1)p_{v+1})$ by (12), and applying Hölder's inequality we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{\sum_{v=1}^{n-1} \frac{P_v}{p_v} \mid s_v \mid p_v \mid \Delta\lambda_v \mid\right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v \ (\beta_v)^k \\
&\times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v \ (\beta_v)^k \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v \ (\beta_v)^k \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k \ (\beta_v)^k \left(\frac{p_v}{P_v}\right) \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} (v\beta_v)^{k-1} v\beta_v \frac{1}{v^k} \theta_v^{k-1} \mid s_v \mid^k \\
&= O(1) \sum_{v=1}^{m} v\beta_v \theta_v^{k-1} v^{-k} \mid s_v \mid^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \theta_r^{k-1} r^{-k} \mid s_r \mid^k + O(1) m\beta_m \sum_{v=1}^{m} \theta_v^{k-1} v^{-k} \mid s_v \mid^k \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m = O(1)
\end{aligned}$$

as $m \to \infty$, by virtue of the hypotheses of the theorem and Lemma 1. Again, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,3} \mid^k &= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{\sum_{v=1}^{n-1} P_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v}\right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k v^{-k} p_v \mid s_v \mid^k \mid \lambda_v \mid^k
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k} v^{-k} \mid s_v \mid^k p_v \mid \lambda_v \mid^k \sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} v^{-k}\theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1} \mid \lambda_v \mid^{k-1}\mid \lambda_v \mid\mid s_v \mid^k\\
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \theta_v^{k-1} v^{-k} \mid s_v \mid^k\\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

in view of the hypotheses of the theorem , Lemma 1 and Lemma 2.
Finally, using Hölder's inequality, as in $T_{n,3}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1} \mid T_{n,4} \mid^k &= \sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v\frac{P_v}{v}\lambda_v \mid^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v\frac{P_v}{vp_v}p_v\lambda \mid^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} \mid s_v \mid^k \left(\frac{P_v}{p_v}\right)^k v^{-k}p_v \mid \lambda_v \mid^k\\
&\times \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k} v^{-k} \mid s_v \mid^k p_v \mid \lambda_v \mid^k \frac{1}{P_v}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} v^{-k}\left(\frac{p_v}{P_v}\right)^{k-1}\theta_v^{k-1} \mid \lambda_v \mid^{k-1}\mid \lambda_v \mid\mid s_v \mid^k\\
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \theta_v^{k-1} v^{-k} \mid s_v \mid^k\\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m\to\infty.
\end{aligned}
$$

This completes the proof of the theorem. If we take $p_n = 1$ for all values of n, then we have a new result for $\mid C,1,\theta_n \mid_k$ summability. Furthermore, if we take $\theta_n = n$, then we have another new result for $\mid R, p_n \mid_k$ summability. Finally, if we take $p_n = 1$ for all values of n and $\theta_n = n$, then we get a new result dealing with $\mid C, 1 \mid_k$ summability factors of infinite series.

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