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Relations and modal operators

Yong Chan Kim

Department of Mathematics, Gangneung-Wonju National University,
Gangneung, Gangwondo 210-702, Korea
yck@gwnu.ac.kr

Abstract

We show that reflexive, transitive, symmetric relations can be induced by modal, necessity, sufficiency and co-sufficiency operators. We give their examples.

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1 Introduction

Pawlak [8] introduced rough set theory to generalize the classical set theory. Rough approximations are defined by a partition of the universe which is corresponding to the equivalence relation about information. An information consists of (X, A) where X is a set of objects and A is a set of attributes, a map $a : X \to P(A_a)$ where A_a is the value set of the attribute a. Recently, intensional modal-like logics with the propositional operators induced by relations are important mathematical tools for data analysis and knowledge processing [1-9]. In [6], we investigated the properties of modal, necessity, sufficiency and co-sufficiency operators.

In this paper, we show that reflexive, transitive, symmetric relations can be induced by modal, necessity, sufficiency and co-sufficiency operators. We give their examples.

2 Preliminaries

Definition 2.1 [3,6] Let $P(X), P(Y)$ be the families of subsets on X and Y, respectively. Then a map $F : P(X) \to P(Y)$ is called

(1) *modal operator* if $F(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i),\ F(\emptyset) = \emptyset$.

(2) *necessity operator* if $F(\bigcap_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i),\ F(X) = Y$.
(3) *sufficiency operator* if $F(\bigcup_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i),\ F(\emptyset) = Y$.
(4) *co-sufficiency operator* if $F(\bigcap_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i),\ F(X) = \emptyset$.
(5) a dual operator F^∂ is defined by $F^\partial(A) = F(A^c)^c$. Moreover, its complementary counterpart $F^c(A) = (F(A))^c$ and $F^*(A) = F(A^c)$.

Let $R \in L^{X\times X}$ be a relation. R is called:
(1) *reflexive* if $(x,x) \in R$ for all $x \in X$.
(2) *symmetric* if $(x,y) \in R$ implies $(y,x) \in R$ for all $x,y \in X$.
(3) *transitive* if $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$ for $x,y,z \in X$.

Definition 2.2 [3,7] Let $R \subset P(X\times Y)$ be a relation. For each $A \in P(X)$, we define operations $(y,x) \in R^{-1}$ iff $(x,y) \in R$ and $[R], [[R]], \langle R\rangle, \langle\langle R\rangle\rangle, [R]^*, \langle R\rangle^* : P(X) \to P(Y)$ as follows:

$$[R](A) = \{y \in Y \mid (\forall x)((x,y) \in R \to x \in A)\},$$

$$[[R]](A) = \{y \in Y \mid (\forall x \in X)(x \in A \to (x,y) \in R)\}$$
$$\langle R\rangle(A) = \{y \in Y \mid (\exists x \in X)((x,y) \in R,\ x \in A)\}$$
$$\langle\langle R\rangle\rangle(A) = \{y \in Y \mid (\exists x \in X)((x,y) \in R^c,\ x \in A^c)\}.$$
$$[R]^*(A) = \{y \in Y \mid (\forall x \in X)((x,y) \in R \to x \in A^c)\}$$
$$\langle R\rangle^*(A) = \{y \in Y \mid (\exists x \in X)((x,y) \in R,\ x \in A^c)\}.$$

Lemma 2.3 [3,6] (1) *A map $F : P(X) \to P(Y)$ is a modal operator iff $F^\partial : P(X) \to P(Y)$ is a necessity operator.*
(2) *A map $F : P(X) \to P(Y)$ is a sufficiency operator iff $F^\partial : P(X) \to P(Y)$ is a co-sufficiency operator operator.*
(3) *A map $F : P(X) \to P(Y)$ is a modal operator iff $F^c : P(X) \to P(Y)$ is a sufficient operator.*
(4) *A map $F : P(X) \to P(Y)$ is a co-sufficiency operator iff $F^c : P(X) \to P(Y)$ is a necessity operator operator.*
(5) *A map $F : P(X) \to P(Y)$ is a sufficiency operator iff $F^* : P(X) \to P(Y)$ is a necessity operator operator.*
(6) *A map $F : P(X) \to P(Y)$ is a modal operator iff $F^* : P(X) \to P(Y)$ is a co-sufficiency operator.*

Theorem 2.4 [3,6] *Let $R \subset P(X \times Y)$ be a relation.*
(1) *$\langle R\rangle$ is a modal operator and $[R]$ is a necessity operator with $\langle R\rangle(A) = ([R](A^c))^c = [R]^\partial(A)$, for each $A \in P(X)$*
(2) *If $F : P(X) \to P(Y)$ is a modal operator on $P(X)$, there exists a unique relation $R_F \subset P(X \times Y)$ such that $\langle R_F\rangle = F$ and $[R_F] = F^\partial$ where $(x,y) \in R_F$ iff $y \in F(\{x\})$.*
(3) *$R_{\langle R\rangle} = R$.*

Theorem 2.5 [6] *Let $R \in P(X \times Y)$ be a relation.*

(1) *$[R]^*$ is a sufficiency operator and $\langle R\rangle^*$ is a co-sufficiency operator with $[R]^*(A) = (\langle R\rangle^*(A^c))^c$.*

(2) *If $F : P(X) \to P(Y)$ is a sufficiency operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $[R_F]^* = F$ and $\langle R_F\rangle^* = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\})^c$.*

(3) *$R_{[R]^*} = R$.*

Theorem 2.6 [6] *Let $R \subset P(X \times Y)$ be a relation.*

(1) *If $F : P(X) \to P(Y)$ is a necessity operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $[R_F] = F$ and $\langle R_F\rangle = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\}^c)^c$.*

(2) *$R_{[R]} = R$.*

Theorem 2.7 [6] *Let $R \in P(X \times Y)$ be a relation.*

(1) *If $F : P(X) \to P(Y)$ is a co-sufficiency operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $\langle R_F\rangle^* = F$ and $[R_F]^* = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\}^c)$.*

(2) *$R_{\langle R_F\rangle^*} = R$.*

3 Relations and modal operators

Theorem 3.1 *Let $F : P(X) \to P(X)$ be a modal operator. Define $(x, y) \in R_F$ iff $y \in F(\{x\})$. Then we have the following properties:*

(1) *R_F is reflexive iff $A \subset F(A)$ for all $A \in P(X)$ iff $\{x\} \subset F(\{x\})$ for all $x \in X$.*

(2) *R_F is transitive iff $F(F(A)) \subset F(A)$ for all $A \in P(X)$ iff $F(F(\{x\})) \subset F(\{x\})$ for all $x \in X$.*

(3) *R_F is symmetric iff $F(F^\partial(A)) \subset A$ for all $A \in P(X)$ iff $F(F^\partial(\{x\}^c)) \subset \{x\}^c$ for all $x \in X$.*

Proof. (1) First, we will show that $A \subset F(A)$ for all $A \in P(X)$ iff $\{x\} \subset F(\{x\})$ for all $x \in X$. Conversely, since $\{x\} \subset F(\{x\})$ and $A = \bigcup_{x\in A}\{x\}$, we have

$$A = \bigcup_{x\in A}\{x\} \subset \bigcup_{x\in A} F(\{x\}) = F(\bigcup_{x\in A}\{x\}) = F(A).$$

Second, R_F is reflexive iff $A \subset F(A)$ for all $A \in P(X)$ iff $\{x\} \subset F(\{x\})$ for all $x \in X$.

Let R_F be reflexive. Since $(x,x) \in R_F$, then $\{x\} \subset F(\{x\})$. Conversely, since $\{x\} \subset F(\{x\})$. Hence $(x,x) \in R_F$.

(2) First, $F(F(A)) \subset F(A)$ for all $A \in P(X)$ iff $F(F(\{x\})) \subset F(\{x\})$ for all $x \in X$ from:

$$\begin{aligned} F(F(A)) &= F(F(\bigcup_{x\in A}\{x\})) = F(\bigcup_{x\in A} F(\{x\})) = \bigcup_{x\in A} F(F(\{x\}) \\ &\subset \bigcup_{x\in A} F(\{x\}) = F(\bigcup_{x\in A}\{x\}) = F(A). \end{aligned}$$

Second, we will show that R_F is transitive iff $F(F(\{x\})) \subset F(\{x\})$ for all $x \in X$.

Let R_F be transitive. Since $(\exists y \in X)((x,y) \in R_F \ \&\ (y,z) \in R_F)$ iff $(\exists y \in X)(y \in F(\{x\}) \ \&\ z \in F(\{y\}))$ implies $(x,z) \in R_F$ iff $z \in F(\{x\})$, respectively and $F(\{x\}) = \bigcup_{y\in F(\{x\})}\{y\}$, we have:

$$\begin{aligned} z \in F(F(\{x\})) \quad & \text{iff } z \in F(\bigcup_{y\in F(\{x\})}\{y\}) = \bigcup_{y\in F(\{x\})} F(\{y\}) \\ & \text{iff } (\exists y)(y \in F(\{x\}) \ \&\ z \in F(\{y\})) \\ & \text{implies } \ z \in F(\{x\}). \end{aligned}$$

Conversely, since $F(F(\{x\})) \subset F(\{x\})$ and $F(\{x\}) = \bigcup_{y\in F(\{x\})}\{y\}$, we have

$$\begin{aligned} (\exists y)((x,y) \in R_F \ \&\ (y,z) \in R_F) \quad & \text{iff } (\exists y)(y \in F(\{x\}) \ \&\ z \in F(\{y\})) \\ & \text{iff } z \in F(F(\{x\})) \text{ implies } \ z \in F(\{x\}). \end{aligned}$$

Thus, $(x,z) \in R_F$.

(3) First, if R_F is symmetric, then $F(F^\partial(A)) \subset A$ for all $A \in P(X)$.

Let R_F be symmetric. Since $A = \bigcap_{x\in A^c}\{x\}^c$ and F^∂ is a necessity operator, then $F^\partial(A) = F^\partial(\bigcap_{x\in A^c}\{x\}^c) = \bigcap_{x\in A^c} F^\partial(\{x\}^c)$. and $x \in F(\{y\})$ iff $y \in F(\{x\})$, we have:

$$F(F^\partial(A)) = F(\bigcup_{y\in F^\partial(A)}\{y\}) = \bigcup_{y\in F^\partial(A)} F(\{y\}),$$

$$\begin{aligned} z \in F(F^\partial(A)) \quad & \text{iff } (\exists y)(y \in F^\partial(A) \ \&\ z \in F(\{y\}) \\ & \text{iff } (\exists y)\big((\forall x \in X)(x \in A^c \to y \in F^\partial(\{x\}^c)) \ \&\ y \in F(\{z\})\big) \\ & \text{implies } \ (\exists y)((z \in A^c \to y \in F^\partial(\{z\}^c) \ \&\ y \in F(\{z\}) \\ & \text{implies } \ (\exists y)((y \in F^\partial(\{z\}^c)^c \to z \in A) \ \&\ y \in F(\{z\}) \\ & \text{implies } \ (\exists y)((y \in F(\{z\}) \to z \in A) \ \&\ y \in F(\{z\}) \\ & \text{implies } \ z \in A. \end{aligned}$$

Second, if $F(F^\partial(A)) \subset A$ for all $A \in P(X)$, put $A = \{x\}^c$, then $F(F^\partial(\{x\}^c)) \subset \{x\}^c$ for all $x \in X$.

Finally, if $F(F^\partial(\{x\}^c)) \subset \{x\}^c$ for all $x \in X$, then R_F is symmetric from the following statements:

Since $F(F^\partial(\{x\}^c)) \subset \{x\}^c$ and $F^\partial(\{x\}^c) = \bigcup_{y \in F^\partial(\{x\}^c)} \{y\}$, we have

$$\begin{array}{rl} F(F^\partial(\{x\}^c)) & = \bigcup_{y\in F^\partial(\{x\}^c)} F(\{y\}) \subset \{x\}^c, \\ \{x\} & \subset \left(\bigcup_{y\in F^\partial(\{x\}^c)} F(\{y\})\right)^c = \bigcap_{y\in F^\partial(\{x\}^c)} F(\{y\})^c, \\ z\in\{x\} & \text{implies } (\forall y)(y \in F^\partial(\{x\}^c) \to z \in F(\{y\})^c), \\ z\in\{x\} & \text{implies } (\forall y)(z \in F(\{y\}) \to y \in F^\partial(\{x\}^c)^c). \end{array}$$

Thus, $(x, y) \in R_F \to (y, x) \in R_F$. Similarly, $(y, x) \in R_F \to (x, y) \in R_F$.

Example 3.2 Let R be a relation. Since $\langle R\rangle : P(X) \to P(X)$ is a modal operator, we define $(x, y) \in R_{\langle R\rangle}$ iff $y \in \langle R\rangle(\{x\})$. Since $R_{\langle R\rangle} = R$ and $\langle R\rangle^\partial = [R]$ from Theorem 2.4, we obtain:

(1) R is reflexive iff $A \subset \langle R\rangle(A)$ for all $A \in P(X)$ iff $\{x\} \subset \langle R\rangle(\{x\})$ for all $x \in X$.

(2) R is transitive iff $\langle R\rangle(\langle R\rangle(A)) \subset \langle R\rangle(A)$ for all $A \in P(X)$ $\langle R\rangle(\langle R\rangle(\{x\})) \subset \langle R\rangle(\{x\})$ for all $x \in X$.

(3) R is symmetric iff $\langle R\rangle([R](A)) \subset A$ for all $A \in P(X)$ iff $\langle R\rangle([R](\{x\}^c)) \subset \{x\}^c$ for all $x \in X$.

Theorem 3.3 *Let $F : P(X) \to P(X)$ be a necessity operator. Define $(x, y) \in R_F$ iff $y \in F(\{x\}^c)^c$. Then we have the following properties:*

(1) R_F is reflexive iff $F(A) \subset A$ for all $A \in P(X)$ iff $F(\{x\}^c) \subset \{x\}^c$ for all $x \in X$.

(2) R_F is transitive iff $F(A) \subset F(F(A))$ for all $A \in P(X)$ iff $F(\{x\}^c) \subset F(F(\{x\}^c))$ for all $x \in X$.

(3) R_F is symmetric iff $A \subset F(F^\partial(A))$ for all $A \in P(X)$ iff $\{x\} \subset F(F^\partial(\{x\}))$ for all $x \in X$.

Proof. (1) First, we show that if R_F is reflexive, then $F(A) \subset A$ for all $A \in P(X)$. Let R_F be reflexive. Then $\{x\} \subset F(\{x\}^c)^c$. Since $A = \bigcap_{x\in A^c}\{x\}^c$ and $F(\{x\}^c) \subset \{x\}^c$,

$$F(A) = F(\bigcap_{x\in A^c} \{x\}^c) = \bigcap_{x\in A^c} F(\{x\}^c) \subset \bigcap_{x\in A^c} \{x\}^c = A.$$

Second, if $F(A) \subset A$ for all $A \in P(X)$, put $A = \{x\}^c$, then $F(\{x\}^c) \subset \{x\}^c$ for all $x \in X$.

Finally, since $\{x\} \subset F(\{x\}^c)^c$, then $(x, x) \in R_F$.

(2) First, we easily show that $F(A) \subset F(F(A))$ for all $A \in P(X)$ iff $F(\{x\}^c) \subset F(F(\{x\}^c))$ for all $x \in X$ from:

$$\begin{array}{rl} F(F(A)) & = F(F(\bigcap_{x\in A^c}\{x\}^c))) = F(\bigcap_{x\in A^c} F(\{x\}^c)) = \bigcap_{x\in A^c} F(F(\{x\}^c)) \\ & \supset \bigcap_{x\in A^c} F(\{x\}^c) = F(\bigcap_{x\in A^c}\{x\}^c) = F(A). \end{array}$$

Second, we show that R_F is transitive iff $F(\{x\}^c) \subset F(F(\{x\}^c))$ for all $x \in X$. Let R_F be transitive. Since $(\exists y \in X)((x,y) \in R_F \;\&\; (y,z) \in R_F)$ iff $(\exists y \in X)(y \in F(\{x\}^c)^c \;\&\; z \in F(\{y\}^c)^c)$ implies $(x,z) \in R_F$ iff $z \in F(\{x\}^c)^c$, respectively and $F(\{x\}^c) = \bigcap_{y \in F(\{x\}^c)^c} \{y\}^c$, we have:

$$(\exists y \in X)(y \in F(\{x\}^c)^c \;\&\; z \in F(\{y\}^c)^c) \to z \in F(\{x\}^c)^c.$$

$$\begin{aligned} z \in F(\{x\}^c) \quad & \text{imlies } \Big((\exists y \in X)(y \in F(\{x\}^c)^c \;\&\; z \in F(\{y\}^c)^c)\Big)^c \\ & \text{iff } (\forall y \in X)(y \in F(\{x\}^c)^c \to z \in F(\{y\}^c)) \\ & \text{iff } z \in F\Big((\forall y \in X)(y \in F(\{x\}^c)^c \to \{y\}^c\Big) \\ & \text{iff } z \in F(F(\{x\}^c)). \end{aligned}$$

Conversely, since $F(\{x\}^c) \subset F(F(\{x\}^c))$ and $F(\{x\}^c) = \bigcap_{y \in F(\{x\}^c)^c} \{y\}^c$, we have

$$F(\{x\}^c) \subset F(F(\{x\}^c)) = F(\bigcap_{y \in F(\{x\}^c)} \{y\}^c) = \bigcap_{y \in F(\{x\}^c)^c} F(\{y\}^c).$$

Then

$$\begin{aligned} & \vdash (\forall z \in X)\Big(z \in F(\{x\}^c) \to z \in \textstyle\bigcap_{y \in F(\{x\}^c)^c} F(\{y\}^c)\Big) \\ & \text{iff } \vdash (\forall z \in X)\Big(z \in F(\{x\}^c) \to (y \in F(\{x\}^c)^c \to z \in F(\{y\}^c)\Big) \\ & \text{iff } \vdash (\forall z \in X)\Big((\exists y \in X)(y \in F(\{x\}^c)^c \;\&\; z \in F(\{y\}^c)^c) \to z \in F(\{x\}^c)^c\Big). \end{aligned}$$

Thus,

$$(\exists y \in X)((x,y) \in R_F \;\&\; (y,z) \in R_F) \to (x,z) \in R_F.$$

(3) First, we show that if R_F is symmetric, then $A \subset F(F^\partial(A))$ for all $A \in P(X)$. Let R_F be symmetric. Since $A = \bigcup_{x \in A} \{x\}$ and F^∂ is a modal operator, then $F^\partial(A) = F^\partial(\bigcup_{x \in A} \{x\}) = \bigcup_{x \in A} F^\partial(\{x\})$ and $x \in F(\{y\}^c)^c$ iff $y \in F(\{x\}^c)^c$, we have:

$$F(F^\partial(A)) = F(\bigcap_{y \in F^\partial(A)^c} \{y\}^c)) = \bigcap_{y \in F^\partial(A)^c} F(\{y\}^c).$$

$$\begin{aligned} x \in F(F^\partial(A)) \quad & \text{iff } (\exists y \in X)(y \in F^\partial(A)^c \to x \in F(\{y\}^c)), \\ x \in F(F^\partial(A)) \quad & \text{iff } (\exists y \in X)(x \in F(\{y\}^c)^c \to y \in F^\partial(A)), \\ x \in F(F^\partial(A)) \quad & \text{iff } (\exists y \in X)(x \in F(\{y\}^c)^c \to (\exists x \in X)(x \in A \;\&\; y \in F^\partial(\{x\})). \end{aligned}$$

Since $\vdash x \in A \to \Big((\exists y \in X)(x \in F(\{y\}^c)^c \to (\exists x \in X)(x \in A \;\&\; y \in F^\partial(\{x\}))\Big)$ iff $\vdash x \in A \to x \in F(F^\partial(A))$, then $A \subset F(F^\partial(A))$.

Second, if $A \subset F(F^\partial(A))$ for all $A \in P(X)$, put $A = \{x\}$, then $\{x\} \subset F(F^\partial(\{x\}))$ for all $x \in X$.

Finally, we show that if $\{x\} \subset F(F^\partial(\{x\}))$ for all $x \in X$, then R_F is symmetric from the following statements. Since $\{x\} \subset F(F^\partial(\{x\}))$ and $F^\partial(\{x\}) = \bigcap_{y \in F^\partial(\{x\})^c} \{y\}^c$, we have

$$F(F^\partial(\{x\})) = F(\bigcap_{y \in F^\partial(\{x\})^c} \{y\}^c) = \bigcap_{y \in F^\partial(\{x\})^c} F(\{y\}^c),$$

$$\begin{aligned}
&\vdash (\forall z \in X)(z \in \{x\} \to z \in F(F^\partial(\{x\}))) \\
&\text{iff } \vdash (\forall z \in X)(z \in \{x\} \to (\exists y)(y \in F^\partial(\{x\})^c \to z \in F(\{y\}^c)) \\
&\text{iff } \vdash ((\exists y)(y \in F^\partial(\{x\})^c \to x \in F(\{y\}^c)) \\
&\text{iff } \vdash y \in F(\{x\}^c) \to x \in F(\{y\}^c).
\end{aligned}$$

Hence $(x, y) \notin R_F$ implies $(y, x) \notin R_F$. Similarly, $(y, x) \notin R_F$ implies $(x, y) \notin R_F$. Thus R_F is a symmetric relation.

Example 3.4 Let R be a relation. Since $[R] : P(X) \to P(X)$ is a necessity operator, we define $(x, y) \in R_{[R]}$ iff $y \in [R](\{x\}^c)^c$. Since $R_{[R]} = R$ and $[R]^\partial = \langle R \rangle$ from Theorem 2.6, we obtain:

(1) R is reflexive iff $[R](A) \subset A$ for all $A \in P(X)$ iff $[R](\{x\}^c) \subset \{x\}^c$ for all $x \in X$.

(2) R is transitive iff $[R](A) \subset [R]([R](A))$ for all $A \in P(X)$ iff $[R](\{x\}^c) \subset [R]([R](\{x\}^c))$ for all $x \in X$.

(3) R is symmetric iff $A \subset [R](\langle R \rangle(A))$ for all $A \in P(X)$ iff $\{x\} \subset [R](\langle R \rangle(\{x\}))$ for all $x \in X$.

Theorem 3.5 *Let $F : P(X) \to P(X)$ be a sufficiency operator. Define $(x, y) \in R_F$ iff $y \in F(\{x\})^c$. Then we have the following properties:*

(1) R_F is reflexive iff $F(A) \subset A^c$ for all $A \in P(X)$ iff $F(\{x\}) \subset \{x\}^c$ for all $x \in X$.

(2) R_F is transitive iff $F(A) \subset F(F^c(A))$ for all $A \in P(X)$ iff $F(\{x\}) \subset F(F^c(\{x\}))$ for all $x \in X$.

(3) R_F is symmetric iff $A \subset F(F(A))$ for all $A \in P(X)$ iff $\{x\} \subset F(F(\{x\}))$ for all $x \in X$.

Proof. (1) We easily proved R_F is reflexive iff $F(\{x\}) \subset \{x\}^c$ for all $x \in X$.

$F(A) \subset A^c$ for all $A \in P(X)$ iff $F(\{x\}) \subset \{x\}^c$ for all $x \in X$ from the following statements: For $A = \bigcup_{x \in A} \{x\}$, we have

$$\begin{aligned}
F(A) &= F(\textstyle\bigcup_{x \in A} \{x\}) = \bigcap_{x \in A} F(\{x\}) \\
&\subset \textstyle\bigcap_{x \in A} \{x\}^c = A^c.
\end{aligned}$$

(2) Since $F(\{x\}^c) = \bigcup_{y\in F(\{x\})^c}\{y\}$ and F^c is a modal operator, we have:

$$\begin{aligned} F(F^c(A)) &= F(F^c(\textstyle\bigcup_{x\in A}\{x\})) = F(\bigcup_{x\in A} F^c(\{x\})) = \bigcap_{x\in A} F(F^c(\{x\}) \\ &\supset \textstyle\bigcap_{x\in A} F(\{x\}) = F(\bigcup_{x\in A}\{x\}) = F(A). \end{aligned}$$

Hence we easily prove that $F(A) \subset F(F^c(A))$ for all $A \in P(X)$ iff $F(\{x\}) \subset F(F^c(\{x\}))$ for all $x \in X$.

Let R_F be transitive. Since $(\exists y \in X)((x,y) \in R_F \ \&\ (y,z) \in R_F)$ iff $(\exists y \in X)(y \in F(\{x\})^c \ \&\ z \in F(\{y\})^c)$ implies $(x,z) \in R_F$ iff $z \in F(\{x\})^c$, respectively, then

$$\vdash z \in F(\{y\}) \to \Big((\exists y \in X)(y \in F(\{x\})^c \ \&\ z \in F(\{y\})^c)\Big)^c,$$

$$\vdash z \in F(\{y\}) \to (\forall y \in X)(y \in F(\{x\})^c \to z \in F(\{y\})),$$

$$F(\{x\}) \subset \bigcap_{y\in F(\{x\})^c} F(\{y\}) = F(\bigcup_{y\in F(\{x\})^c} \{y\}) = F(F(\{x\})^c).$$

Conversely, let $F(F^c(\{x\})) \supset F(\{x\})$ and $F^c(\{x\}) = \bigcup_{y\in F^c(\{x\})}\{y\}$, we have

$$\begin{aligned} &F(F^c(\{x\})) = F(\textstyle\bigcup_{y\in F^c(\{x\})}\{y\}) = \bigcap_{y\in F^c(\{x\})} F(\{y\}), \\ &z \in F(F^c(\{x\})) \text{ iff } (\forall y \in X)(y \in F^c(\{x\}) \to z \in F(\{y\})). \end{aligned}$$

Since $F(F^c(\{x\})) \supset F(\{x\})$, we have

$$\begin{aligned} &F(F^c(\{x\}))^c \subset F(\{x\})^c \\ &\text{iff } \Big(\textstyle\bigcap_{y\in X}(y \in F^c(\{x\}) \to z \in F(\{y\}))\Big)^c \text{ implies } z \in F(\{x\})^c \\ &\text{iff } (\exists y)(y \in F^c(\{x\}) \ \&\ z \in F^c(\{y\})) \text{ implies } z \in F(\{x\})^c. \end{aligned}$$

Thus, $(x,y) \in R_F \ \&\ (y,z) \in R_F$ implies $(x,z) \in R_F$.

(3) First, we show that if R_F is symmetric, then $A \subset F(F(A))$ for all $A \in P(X)$.

Let R_F be symmetric. Since $F(A) = \bigcup_{x\in F(A)}\{x\}$ and $z \in F(\{x\})^c$ iff $x \in F(\{z\})^c$, then

$$\begin{aligned} F(F(A)) &= F(\textstyle\bigcup_{x\in F(A)}\{x\}) = \bigcap_{x\in F(A)} F(\{x\}), \\ z \in F(F(A)) &\text{ iff } (\forall x)(x \in F(A) \to z \in F(\{x\})), \\ z \in F(F(A)) &\text{ iff } (\forall x)((\forall y)(y \in A \to x \in F(\{y\})) \to z \in F(\{x\})). \end{aligned}$$

Since $\vdash (\forall x)((z \in A \to x \in F(\{z\})) \to z \in F(\{x\})) \to (\forall x)((\forall y)(y \in A \to x \in F(\{y\})) \to z \in F(\{x\}))$ and $z \in F(\{x\})$ iff $(z,x) \notin R_F$ iff $(x,z) \notin R_F$ iff $x \in F(\{z\})$, then

$$\vdash (\forall x)(z \in A \ \&\ (z \in A \to x \in F(\{z\})) \to x \in F(\{z\})),$$

$$\vdash (\forall x)((z \in A \to x \in F(\{z\})) \to z \in F(\{x\})) \to z \in F(F(A)).$$

By Modus Ponens, $\vdash (\forall x)(z \in A \to z \in F(F(A)))$. Hence $A \subset F(F(A))$.

Second, if $A \subset F(F(A))$ for all $A \in P(X)$, put $A = \{x\}$, then $\{x\} \subset F(F(\{x\}))$ for all $x \in X$.

Finally, we show that if $\{x\} \subset F(F(\{x\}))$ for all $x \in X$, then R_F is symmetric from the following statements. Since $F(F(\{x\})) \supset \{x\}$ and $F(\{x\}) = \bigcup_{y\in F(\{x\})}\{y\}$, we have

$$\begin{aligned}
&F(F(\{x\})) = F(\textstyle\bigcup_{y\in F(\{x\})}\{y\}) = \bigcup_{y\in F(\{x\})} F(\{y\}),\\
&\vdash (\forall z \in X)\big(z \in \{x\} \to\ (\exists y)(y \in F(\{x\}) \to z \in F(\{y\})),\\
&\vdash (\forall z \in X)\big((z \in \{x\}\ \&\ y \in F(\{x\})) \to z \in F(\{y\})).
\end{aligned}$$

Hence $y \in F(\{x\}) \to x \in F(\{y\})$ iff $x \in F(\{y\})^c \to y \in F(\{x\})^c$ iff $(y,x) \in R_F \to (x,y) \in R_F$.

Example 3.6 Let R be a relation. Since $[R]^* : P(X) \to P(X)$ is a sufficiency operator, we define $(x,y) \in R_{[R]^*}$ iff $y \in [R]^*(\{x\})^c$. Since $R_{[R]^*} = R$ and $([R]^*)^\partial = \langle R\rangle^*$ from Theorem 2.5, we obtain:

(1) R is reflexive iff $[R]^*(A) \subset A^c$ for all $A \in P(X)$ iff $[R]^*(\{x\}) \subset \{x\}^c$ for all $x \in X$.

(2) R is transitive iff $[R]^*(A) \subset [R]^*(([R]^*)^c(A))$ for all $A \in P(X)$ iff $[R]^*(\{x\}) \subset [R]^*(([R]^*)^c(\{x\}))$ for all $x \in X$.

(3) R is symmetric iff $A \subset [R]^*([R]^*(A))$ for all $A \in P(X)$ iff $\{x\} \subset [R]^*([R]^*(\{x\}))$ for all $x \in X$.

Theorem 3.7 *Let $F : P(X) \to P(X)$ be a co-sufficiency operator. Define $(x,y) \in R_F$ iff $y \in F(\{x\}^c)$. Then we have the following properties:*

(1) *R_F is reflexive iff $A^c \subset F(A)$ for all $A \in P(X)$ iff $\{x\} \subset F(\{x\}^c)$ for all $x \in X$.*

(2) *R_F is transitive iff $F(F^c(A)) \subset F(A)$ for all $A \in P(X)$ iff $F(F^c(\{x\}^c)) \subset F(\{x\}^c)$ for all $x \in X$.*

(3) *R_F is symmetric iff $F(F(A)) \subset A$ for all $A \in P(X)$ iff $F(F(\{x\}^c)) \subset \{x\}^c$ for all $x \in X$.*

Proof. (1) Let R_F be reflexive. Since $A = \bigcap_{x\in A^c}\{x\}^c$ and $\{x\} \subset F(\{x\}^c)$, $F(A) = F(\bigcap_{x\in A^c}\{x\}^c) = \bigcup_{x\in A^c} F(\{x\}^c) \supset \bigcup_{x\in A^c}\{x\} = A^c$.

Put $A = \{x\}^c$. Then $\{x\} \subset F(\{x\}^c)$. Let $\{x\} \subset F(\{x\}^c)$. Then $(x,x) \in R_F$.

(2) First, we show that $F(F^c(A)) \subset F(A)$ for all $A \in P(X)$ iff $F(F^c(\{x\}^c)) \subset F(\{x\}^c)$ for all $x \in X$. Since F^c is a necessity operator, we have:

$$\begin{aligned} F(F^c(A)) &= F(F^c(\bigcap_{x\in A^c}\{x\}^c))) = F(\bigcap_{x\in A^c} F^c(\{x\}^c)) = \bigcup_{x\in A^c} F(F^c(\{x\}^c)) \\ &\subset \bigcup_{x\in A^c} F(\{x\}^c) = F(\bigcap_{x\in A^c}\{x\}^c) = F(A). \end{aligned}$$

Conversely, put $A = \{x\}^c$. It is trivial.

Second, R_F is transitive iff $F(F^c(\{x\}^c)) \subset F(\{x\}^c)$ for all $x \in X$. Let R_F be transitive. Since $(\exists y \in X)((x,y) \in R_F \ \&\ (y,z) \in R_F)$ iff $(\exists y \in X)(y \in F(\{x\}^c) \ \&\ z \in F(\{y\}^c))$ implies $(x,z) \in R_F$ iff $z \in F(\{x\}^c)$ and $F^c(\{x\}^c) = \bigcap_{y\in F(\{x\}^c)}\{y\}^c$, we have:

$$\begin{aligned} F((F^c(\{x\}^c)) &= F(\bigcap_{y\in F(\{x\}^c)}\{y\}^c) = \bigcup_{y\in F(\{x\}^c)} F(\{y\}^c) \\ z \in F((F^c(\{x\}^c)) &\text{ iff } (\exists y)(y \in F(\{x\}^c) \ \&\ z \in F(\{y\}^c) \\ &\text{ implies } z \in F(\{x\}^c). \end{aligned}$$

Hence $F((F^c(\{x\}^c)) \subset F(\{x\}^c)$.

Conversely, since $F(F^c(\{x\}^c)) \subset F(\{x\}^c)$ and $F^c(\{x\}^c) = \bigcap_{y\in F(\{x\}^c)}\{y\}^c$, we have

$$\begin{aligned} F(\{x\}^c) &\supset F(F^c(\{x\}^c)) = F(\bigcap_{y\in F(\{x\}^c)}\{y\}^c) \\ &= \bigcup_{y\in F(\{x\}^c)} F(\{y\}^c). \end{aligned}$$

Thus $z \in \bigcup_{y\in F(\{x\}^c)} F(\{y\}^c)$ implies $z \in F(\{x\}^c)$. Hence $(x,y) \in R_F \ \&\ (y,z) \in R_F \to (x,z) \in R_F$.

(3) First, we will show that if R_F is symmetric, then $F(F(A)) \subset A$ for all $A \in P(X)$. Let R_F be symmetric. Since $A = \bigcap_{x\in A^c}\{x\}^c$ and F^c is a necessity operator, then $F^c(A) = \bigcap_{x\in A^c} F^c(\{x\}^c)$, $F(A) = \bigcap_{x\in F(A)^c}\{x\}^c$ and $x \in F(\{y\}^c)$ iff $y \in F(\{x\}^c)$, we have:

$$F(F(A)) = F(\bigcap_{y\in F^c(A)}\{y\}^c) = \bigcup_{y\in F^c(A)} F(\{y\}^c)$$

$$\begin{aligned} x \in F(F(A)) &\text{ iff } x \in \bigcup_{y\in F^c(A)} F(\{y\}^c) \\ &\text{ iff } (\exists y)(y \in F^c(A) \ \&\ x \in F(\{y\}^c) \\ &\text{ iff } (\exists y)((\forall x \in X)(x \in A^c \to y \in F^c(\{x\}^c))) \ \&\ y \in F(\{x\}^c)) \\ &\text{ iff } (\exists y)(((\forall x \in X)(y \in F(\{x\}^c) \to x \in A)) \ \&\ y \in F(\{x\}^c)) \\ &\text{ implies } x \in A. \end{aligned}$$

Second, if $F(F(A)) \subset A$ for all $A \in P(X)$, put $A = \{x\}^c$, then $F(F(\{x\}^c)) \subset \{x\}^c$ for all $x \in X$.

Finally, we will show if $F(F(\{x\}^c)) \subset \{x\}^c$ for all $x \in X$, then R_F is symmetric. Since $F(F(\{x\}^c)) \subset \{x\}^c$ and $F(\{x\}^c) = \bigcap_{y\in F^c(\{x\})^c}\{y\}^c$, we have

$$F(F(\{x\}^c)) = F(\bigcap_{y\in F^c(\{x\})^c}\{y\}^c) = \bigcup_{y\in F^c(\{x\})^c} F(\{y\}^c) \subset \{x\}^c.$$

Thus $z \in \{x\} \to (\forall y \in X)(y \in F^c(\{x\}^c) \to z \in F^c(\{y\})^c)$. Put $x = z$, then $\vdash x \in F(\{y\}^c) \to y \in F(\{x\}^c)$. Similarly, $\vdash y \in F(\{x\}^c) \to x \in F(\{y\}^c)$.

Example 3.8 Let R be a relation. Since $\langle R\rangle^* : P(X) \to P(X)$ is a sufficiency operator, we define $(x, y) \in R_{\langle R\rangle^*}$ iff $y \in \langle R\rangle^*(\{x\})^c$. Since $R_{\langle R\rangle^*} = R$ and $(\langle R\rangle^*)^\partial = [R]^*$ from Theorem 2.7, we obtain:

(1) R is reflexive iff $A^c \subset \langle R\rangle^*(A)$ for all $A \in P(X)$ iff $\{x\} \subset \langle R\rangle^*(\{x\}^c)$ for all $x \in X$.

(2) R is transitive iff $\langle R\rangle^*(\langle R\rangle(A)) \subset \langle R\rangle^*(A)$ for all $A \in P(X)$ iff $\langle R\rangle^*(\langle R\rangle(\{x\}^c)) \subset \langle R\rangle^*(\{x\}^c)$ for all $x \in X$.

(3) R is symmetric iff $\langle R\rangle^*(\langle R\rangle^*(A)) \subset A$ for all $A \in P(X)$ iff $\langle R\rangle^*(\langle R\rangle^*(\{x\}^c)) \subset \{x\}^c$ for all $x \in X$.

Example 3.9 Let $X = \{a, b, c, d\}$ be a set. Define $F, G : P(X) \to P(X)$ as

$$F(\{a\}) = \{a, b\}, F(\{b\}) = \{b\}, F(\{c\}) = \{a, c\}, F(\{d\}) = \{a, d\}$$

$$G(\{a\}) = \{c, d\}, G(\{b\}) = \{c, d\}, G(\{c\}) = \{a, b\}, G(\{d\}) = \{a, b\}$$

$$H(\{b, c, d\}) = \{b, c\}, H(\{a, c, d\}) = \{c, d\}, H(\{a, b, d\}) = \{a, d\}, H(\{a, b, c\}) = \{a, b\}$$

(1) If F is a modal operator, then, by Theorem 3.1,

$$R_F = \{(a, a), (a, b), (b, b), (c, a), (c, c), (d, a), (d, d)\}$$

Since R_F is reflexive, then $A \subset F(A)$. Since $(c, a) \in R_F$ and $(a, b) \in R_F$ but $(c, b) \notin R_F$, then R_F is not transitive. Thus, $\{a, b, c\} = F(F(\{c\})) \not\subset F(\{c\}) = \{a, c\}$. Since R_F is not symmetric,

$$\{a, b, c\} = F(F^\partial(\{d\}^c)) \not\subset \{d\}^c = \{b, c\}.$$

(2) If G is a sufficiency operator, then, by Theorem 3.5,

$$R_G = \{(a, a), (a, b), (b, a), (b, b), (c, b), (c, c), (d, c), (d, d)\}.$$

Since R_G is reflexive, transitive and symmetric, then $G(A) \subset A^c$, $G(A) \subset G(G^c(A))$ and $A \subset G(G(A))$.

(3) If H is a necessity operator, then, by Theorem 3.3,

$$R_H = \{(a, a), (a, d), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}.$$

Since R_H is reflexive, then $H(A) \subset A$. Since $(b, a) \in R_H$ and $(a, d) \in R_H$ but $(b, d) \notin R_H$, then R_H is not transitive. Thus, $\{a, d\} = H(\{a, b, d\}) \not\subset H(H(\{a, b, d\})) = \{d\}$. Since R_H is not symmetric,

$$\{c\} \not\subset H(H^\partial(\{c\})) = H(\{b, c\}) = H(\{a, b, c\}) \cap H(\{b, c, d\}) = \{b\}.$$

(4) If H is a co-sufficiency operator, then by Theorem 3.7,

$$R_H = \{(a,b),(a,c),(b,c),(b,d),(c,a),(c,b),(d,a),(d,b)\}.$$

Since R_H is not reflexive, we have $\{a\}^c \not\subset H(\{a\}^c)$. Since R_H is not transitive,

$$\{a,c,d\} = H(H^c(\{a\}^c)) \not\subset H(\{a\}^c) = \{b,c\}.$$

Since R_H is not symmetric, $H(H(\{a\}^c)) = H(H(\{b,c,d\})) = H(\{b,c\}) = H(\{a,b,c\} \cap \{b,cd\} = H(\{a,b,c\}) \cup H(\{b,cd\}) = \{a,b,c\}$. Thus

$$\{a,b,c\} = H(H(\{a\}^c)) \not\subset \{a\}^c = \{b,c,d\}.$$

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