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Some Results in Asymmetric Metric Spaces

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Abstract

In this paper, we recall some definitions and theorems in asymmetric metric spaces and then prove some results in these spaces.

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1.Introduction

Asymmetric metric spaces are defined as metric spaces, but without the requirement that the (asymmetric) metric d has to satisfy $d(x, y) = d(y, x)$.

In the realms of applied mathematics and materials science we find many recent applications metric spaces; for example, in rate-independent models for plasticity [1], shape-memory alloys[2], and models for material failure[3].

There are other applications of asymmetric metrics both in pure and applied mathematics; for example, asymmetric metric spaces have recently been studied with questions of existence and uniqueness of Hamilton-Jacobi equations[4] in mind.

The study of asymmetric metrics apparently goes back to Wilson[5].Following his terminology, asymmetric metrics are often called quasi-metrics. Author in [6], has completely discussed on asymmetric metric spaces.

In this work, we prove some theorems in asymmetric metric spaces. We start with some elementary definitions from [6].

Definition1.1. A function $d: X \times X \to \mathbb{R}$ is an asymmetric metric and (X, d) is an asymmetric metric space if:

(1) For every $x, y \in X$, $d(x,y) \geq 0$ and $d(x,y) = 0$ hold if and only if $x = y$,
(2) For every $x, y, z \in X$, we have $d(x,y) \leq d(x,z) + d(z,y)$.

Henceforth, (X,d) shall be an asymmetric metric space.

Example1.2. Let $\alpha > 0$. Then $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$ defined by

$$d(x,y) = \begin{cases} x - y & x \geq y \\ \alpha(y - x) & y > x \end{cases}$$

Is obviously an asymmetric metric.

Definition1.3. The forward topology τ_+ induced by d is the topology generated by the forward open balls

$$B_+(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\} \qquad \text{for } x \in X, \varepsilon > 0$$

Likewise, the backward topology τ_- induced by d is the topology generated by the backward open balls

$$B_-(x,\varepsilon) = \{y \in X : d(y,x) < \varepsilon\} \qquad \text{for } x \in X, \varepsilon > 0$$

Definition1.4. A sequence $\{x_k\}_{k\in\mathbb{N}}$ forward converges to $x_o \in X$, respectively backward converges to $x_o \in X$ if and only if

$$\lim_{k\to\infty} d(x_0, x_k) = 0 \text{ respectively } \lim_{k\to\infty} d(x_k, x_0) = 0$$

Then we write $x_k \xrightarrow{f} x_0$, $x_k \xrightarrow{b} x_0$ respectively.

Example1.5. Let $(\mathbb{R}, d)$ be an asymmetric space, where d is as Example1.2. It is easy to show that the sequence $\{x + \frac{1}{n}\}_{n\in\mathbb{N}}$ $(x \in X)$ is both forward and backward converges to x.

Definition1.6. Suppose that (X, d_X) and (Y, d_Y) are asymmetric metric spaces. Let $f: X \to Y$ be a function. We say that f is forward continuous at $x \in X$, respectively backward continuous, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in B_+(x,\delta)$ implies $f(y) \in B_+(f(x),\varepsilon)$, respectively, $f(y) \in B_-(f(x),\varepsilon)$.

However, note that uniform forward continuity and uniform backward continuity are the same.

Definition1.7. A set $S \subseteq X$ is forward compact if every open cover of S in the forward topology has a finite subcover. We say that S is is forward relatively compact, if $\bar{S}$ is forward compact, where $\bar{S}$ denotes the closure of S in the forward topology. We say S is forward sequentially compact if every sequence has a forward convergent subsequence with limit in S. Finally, $S \subseteq X$ is forward complete if every forward Cauchy sequence is forward convergent.

Note that there is a corresponding backward definition in each case, which is obtained by replacing “forward” with “backward” in each definition.

Lemma1.8[6]. Let $d: X \times X \to \mathbb{R}^{\geq 0}$ be an asymmetric metric. If (X, d) is forward sequentially compact and $x_n \overset{b}{\to} x_0$, then $x_n \overset{f}{\to} x_0$.

Notation1.9. We introduce some further notations. Y^X denotes the space of functions from X to Y. The uniform metric on Y^X is

$$\bar{\rho}(f, g) = \sup \{\bar{d}(f(x), g(x)) : x \in X\}$$

Where $\bar{d}(x, y) = \min \{d(x, y), 1\}$ and d is the asymmetric metric associated with Y.

2.Main Results

Throughout this section let (X, d_X) and (Y, d_Y) be asymmetric metric spaces.

Lemma2.1. Let Y be forward(backward) complete. Then Y^X is also.

Proof. Let $\{f_n\} \subset Y^X$ be an arbitrary forward Cauchy sequence. By definition, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m \geq n \geq N$, $\bar{\rho}(f_n, f_m) < \varepsilon$ holds. Fix $x \in X$. Clearly, $\{f_n(x)\}$ is a forward Cauchy sequence in Y.Since Y is forward complete, so $\{f_n(x)\}$ is convergent., say $f_n(x) \overset{f}{\to} f(x)$. Thus there is $N \in \mathbb{N}$ such that $n \geq N$ implies that

$$d_Y(f(x), f_n(x)) < \varepsilon \qquad (1)$$

Since $x \in X$ was arbitrary, by taking supermom on $x \in X$ in the both side of (1), we obtain $f_n \overset{f}{\to} f$ in the uniform metric $\bar{\rho}$. □

Theorem2.2. Let $\mathfrak{F} \subset Y^X$ be a family of forward continuous functions. Suppose further, Y is forward complete and forward convergence implies backward convergence in Y. Then $\mathfrak{F}$ is forward complete.

Proof. Let $\{f_n\} \subset \mathfrak{F}$ such that $f_n \overset{f}{\to} f$. Since Y^X is forward complete(Lemma2.1) and $\mathfrak{F} \subset Y^X$, so it is sufficient to show that $f \in \mathfrak{F}$ .Given $\varepsilon > 0$ and $x \in X$, there is $\delta > 0$ such that for each $y \in X$ which $d(x, y) < \delta$, we have

$$d_Y(f_n(x), f_n(y)) < \frac{\varepsilon}{3} \qquad (n \in \mathbb{N})$$

Also, there is $N \in \mathbb{N}$ so that

$$d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$$

For all $n \geq N$. Now, since forward convergence implies backward convergence in Y, so

$$d_Y(f_n(x), f(x)) < \frac{\varepsilon}{3}$$

Therefore

$$d_Y\big(f(x),f(y)\big) \leq d_Y\big(f(x),f_n(x)\big) + d_Y\big(f_n(x),f_n(y)\big) + d_Y\big(f_n(x),f(x)\big) < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, so the proof is completed. □

Theorem2.3. Let $\{f_n\} \subset Y^X$ be a sequence of forward continuous functions with $f_n \xrightarrow{b} f$ uniform in the uniform metric $\bar{\rho}$ corresponding to d_Y. Also, Let Y be forward sequentially compact. Then f is forward continuous.

Proof. Fix $\varepsilon > 0$ and $x \in X$. Choose $\delta > 0$ such that for all $y \in X$ which $d(x,y) < \delta$,

$$d_Y\big(f_n(x),f_n(y)\big) < \frac{\varepsilon}{4} \qquad (n \in \mathbb{N})$$

Holds. Since $f_n \xrightarrow{b} f$ in the uniform metric $\bar{\rho}$, so $f_n(x) \xrightarrow{b} f(x)$. Hence, there exists $N_1 \in \mathbb{N}$ such that

$$d_Y\big(f_n(x),f(x)\big) < \frac{\varepsilon}{4}$$

For all $n \geq N_1$. On the other hand, Y is forward sequentially compact. Thus Lemma1.8 implies that $f_n(x) \xrightarrow{f} f(x)$. So there exists $N_2 \in \mathbb{N}$ so that

$$d_Y\big(f(x),f_n(x)\big) < \frac{\varepsilon}{4}$$

For all $n \geq N_2$.Set $N \coloneqq \max\{N_1,N_2\}$. Then for each $y \in X$ which $d(x,y) < \delta$, we have($m \geq n \geq N$)

$$d_Y\big(f(x),f(y)\big) \leq d_Y\big(f(x),f_n(x)\big) + d_Y\big(f_n(x),f_m(x)\big) + d_Y\big(f_m(x),f_m(y)\big) + d_Y\big(f_m(y),f(y)\big) < \varepsilon$$

As desired. □

Finally, we prove the following result:

Theorem2.4. Let $\{f_n\} \subset Y^X$ be a sequence of uniformly forward continuous functions with $f_n \xrightarrow{b} f$ in the uniform metric $\bar{\rho}$ corresponding to d_Y.If forward convergence implies backward convergence in Y, then f is uniformly forward continuous.

Proof. Fix $\varepsilon > 0$. Then there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x,y \in X$ which $d(x,y) < \delta$, we have

$$d_Y\big(f_n(x),f_n(y)\big) < \frac{\varepsilon}{3} \qquad (n \in \mathbb{N})$$

Furthermore, there is $N \in \mathbb{N}$ such that

$$\bar{\rho}(f,f_n) < \frac{\varepsilon}{3}$$

For all $n \geq N$. It can be seen easily that

$$d_Y\big(f(x), f_n(x)\big) < \frac{\varepsilon}{3}$$

For all $n \geq N$. Now, by hypotheses, we have

$$d_Y\big(f_n(x), f(x)\big) < \frac{\varepsilon}{3}$$

For all $n \geq N$. Finally, if $d(x, y) < \delta$, then

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f_n(x)\big) + d_Y\big(f_n(x), f_n(y)\big) + d_Y\big(f_n(x), f(x)\big) < \varepsilon$$

Which means f is uniformly forward continuous. □

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