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Fuzzy relations on generalized residuated lattices

Yong Chan Kim

Department of Mathematics, Gangneung-Wonju National University,
Gangneung, Gangwondo 210-702, Korea
yck@gwnu.ac.kr

Abstract

We investigate the properties of fuzzy relations in generalized residuated lattice. In particular, we construct l-preorders (r-preorders) induced by fuzzy relations.

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1 Introduction

Wille [11] introduced the structures on lattices which are important mathematical tools for data analysis and knowledge processing. MV-algebra was introduced by Chang [2] to provide algebraic models for many valued propositional logic. Recently, it is developed many directions (BL-algebra, residuated algebra) [9,11]. On the other hand, noncommutative structures play an important role in metric spaces, algebraic structures (groups, rings, quantales, pseudo-BL-algebras)[3-8,10]. Georgescu and Iorgulescu [5] introduced pseudo MV-algebras as the generalization of the MV-algebras. Georgescu and Popescu [6] introduced generalized residuated lattice as a noncommutative structure.

In this paper, we study the properties of fuzzy relations in generalized residuated lattice. In particular, we construct l-preorders (r-preorders) induced by fuzzy relations.

2 Preliminaries

Definition 2.1 [6] A triple $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \perp, \top)$ is called a *generalized residuated lattice* iff it satisfies the following properties:

(L1) $(L, \vee, \wedge, \perp, \top)$ is a bounded lattice where $\perp$ is the bottom element and $\top$ is the top element;
(L2) $(L, \odot, \top)$ is a monoid;
(L3) adjointness properties,i.e.

$$x \leq y \rightarrow z \text{ iff } x \odot y \leq z \text{ iff } y \leq x \Rightarrow z.$$

Two maps ${}^0, {}^* : L \rightarrow L$ defined by $a^0 = a \rightarrow \perp$ and $a^* = a \Rightarrow \perp$ is called *strong negations* if $a^{0*} = a$ and $a^{*0} = a$.

In this paper, we assume that $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, {}^*, {}^0, \perp, \top)$ be a generalized residuated lattice with strong negations * and ^{0}.

Definition 2.2 Let X be a set. A function $R : X \times X \rightarrow L$ is called *l-preorder* on X if it satisfies the following conditions:
(R) (reflexive) $R(x, x) = \top$ for all $x \in X$,
(LT) (l-transitive) $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.
A function $R : X \times X \rightarrow L$ is called *r-preorder* on X if it satisfies (R) and the following condition:
(RT) (r-transitive) $R(y, z) \odot R(x, y) \leq R(x, z)$, for all $x, y, z \in X$.
The pair (X, R) is an *l-preorder (resp. r-preorder) set.*
An l-preorder (resp. r-preorder) R is called an l-order (resp. r-order) if $R(x, y) = \top$ implies $x = y$.
An l-preorder R is an $\odot$-equivalence relation if it satisfies
(S) (symmetric) $R(x, y) = R(y, x)$ for all $x \in X$.

Lemma 2.3 *For each $x, y, z, x_i, y_i \in L$, the following properties hold.*
(1) *$\odot$ is isotone in both arguments.*
(2) *$\rightarrow$ and $\Rightarrow$ are antitone in the first and isotone in the second argument.*
(3) $x \rightarrow y = \top$ *iff* $x \leq y$ *iff* $x \Rightarrow y = \top$.
(4) $x \rightarrow \top = x \Rightarrow \top = \top$ *and* $\top \rightarrow x = \top \Rightarrow x = x$.
(5) $x \odot y \leq x \wedge y$.
(6) $x \odot (\bigvee_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma}(x \odot y_i)$ *and* $(\bigvee_{i \in \Gamma} x_i) \odot y = \bigvee_{i \in \Gamma}(x_i \odot y)$.
(7) $x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma}(x \rightarrow y_i)$ *and* $(\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma}(x_i \rightarrow y)$.
(8) $x \Rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma}(x \Rightarrow y_i)$ *and* $(\bigvee_{i \in \Gamma} x_i) \Rightarrow y = \bigwedge_{i \in \Gamma}(x_i \Rightarrow y)$.
(9) $x \odot (x \Rightarrow y) \leq y$ *and* $(x \rightarrow y) \odot x \leq y$.
(10) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq (x \Rightarrow z)$ *and* $(y \rightarrow z) \odot (x \rightarrow y) \leq (x \rightarrow z)$.
(11) $x \Rightarrow y \leq (y \Rightarrow z) \rightarrow (x \Rightarrow z)$ *and* $x \rightarrow y \leq (y \rightarrow z) \Rightarrow (x \rightarrow z)$
(12) $\bigwedge_{i \in \Gamma} x_i^* = (\bigvee_{i \in \Gamma} x_i)^*$ *and* $\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*$.
(13) $\bigwedge_{i \in \Gamma} x_i^0 = (\bigvee_{i \in \Gamma} x_i)^0$ *and* $\bigvee_{i \in \Gamma} x_i^0 = (\bigwedge_{i \in \Gamma} x_i)^0$.
(14) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ *and* $(x \odot y)^0 = x \rightarrow y^0$.
(15) $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$ *and* $(x \odot y)^* = y \Rightarrow x^*$.
(16) $x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z)$ *and* $x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z)$.

Proof. (1)-(11) are proved in [6].

(12) By (8), $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$. Since $(\bigvee_{i\in\Gamma} x_i^*) \to 0 = \bigwedge (x_i^*)^0 = \bigwedge x_i$, we have $\bigvee_{i\in\Gamma} x_i^* = ((\bigvee_{i\in\Gamma} x_i^*) \to 0) \Rightarrow 0 = (\bigwedge x_i) \Rightarrow 0 = (\bigwedge x_i)^*$.

(14) Since $((x\odot y) \to z)\odot(x\odot y) \leq z$, we have $(x\odot y) \to z) \leq x \to (y \to z)$. Since $(x \to (y \to z)) \odot (x \odot y) \leq (y \to z) \odot y \leq z$, we have $x \to (y \to z) \leq ((x \odot y) \to z$.

(16) Since $\Big(y \odot (\big(x \to (y \Rightarrow z)\big)\Big) \odot x = y \odot \Big(\big(x \to (y \Rightarrow z)\big) \odot x\Big) \leq y \odot (y \Rightarrow z) \leq z$, then $x \to (y \Rightarrow z) \leq y \Rightarrow (x \to z)$.

Since $y \odot \Big(\big(y \Rightarrow (x \to z)\big) \odot x\Big) = \Big(y \odot (\big(y \Rightarrow (x \to z)\big)\Big) \odot x = (x \to z) \odot x \leq z$, then $y \Rightarrow (x \to z) \leq x \to (y \Rightarrow z)$.

Other cases are similarly proved.

3 Fuzzy relations on generalized residuated lattices

Theorem 3.1 *Let $R_1, R_2, R_3 \in L^{X\times X}$ be fuzzy relations. The compositions of R_1 and R_2 are defined as*

$$R_1 \circ R_2(x,z) = \bigvee_{y\in Y} R_1(x,y) \odot R_2(y,z)$$

$$R_1 \otimes R_2(x,z) = \bigvee_{y\in Y} R_2(y,z) \odot R_1(x,y)$$

$$(R_1 \Rightarrow R_2)(x,z) = \bigwedge_{y\in Y} (R_1(x,y) \Rightarrow R_2(y,z))$$

$$(R_1 \to R_2)(x,z) = \bigwedge_{y\in Y} (R_1(x,y) \to R_2(y,z))$$

$$(R_1 \Leftarrow R_2)(x,z) = \bigwedge_{y\in Y} (R_2(y,z) \Rightarrow R_1(x,y))$$

$$(R_1 \leftarrow R_2)(x,z) = \bigwedge_{y\in Y} (R_2(y,z) \to R_1(x,y))$$

$$R_i^s(y,x) = R_i(x,y), \forall i \in \{1,2\}.$$

Then we have the following properties.

(1) $(R_1 \circ R_2)^s = R_2^s \otimes R_1^s$ *and* $(R_1 \otimes R_2)^s = R_2^s \circ R_1^s$.

(2) $(R_1 \circ R_2)^* = R_1^* \Leftarrow R_2$ *and* $(R_1 \circ R_2)^0 = R_1 \to R_2^0$.

(3) $(R_1 \otimes R_2)^* = R_1 \Rightarrow R_2^*$ *and* $(R_1 \otimes R_2)^0 = R_1^0 \leftarrow R_2$.

(4) $(R_1 \Rightarrow R_2)^s = R_2^s \Leftarrow R_1^s$ *and* $(R_1 \to R_2)^s = R_2^s \leftarrow R_1^s$.

(5) $(R_1 \Leftarrow R_2)^s = R_2^s \Rightarrow R_1^s$ *and* $(R_1 \leftarrow R_2)^s = R_2^s \to R_1^s$.

(6) $R_1 \circ R_2 \leq R_3$ *iff* $R_1 \leq R_3 \leftarrow R_2^s$ *iff* $R_2 \leq R_1^s \Rightarrow R_3$.

(7) $R_1 \otimes R_2 \leq R_3$ *iff* $R_2 \leq R_1^s \rightarrow R_3$ *iff* $R_1 \leq R_3 \Leftarrow R_2^s$.

(8) $(R_1 \circ R_2) \rightarrow R_3 = R_1 \rightarrow (R_2 \rightarrow R_3)$ *and* $(R_1 \otimes R_2) \Rightarrow R_3 = R_1 \Rightarrow (R_2 \Rightarrow R_3)$,

(9) $(R_1 \leftarrow R_2) \leftarrow R_3 = R_1 \leftarrow (R_2 \otimes R_3)$ *and* $(R_1 \Leftarrow R_2) \Leftarrow R_3 = R_1 \Leftarrow (R_2 \circ R_3)$,

(10) $R_1 \Rightarrow (R_2 \leftarrow R_3) = ((R_1 \Rightarrow R_2) \leftarrow R_3)$ *and* $R_1 \rightarrow (R_2 \Leftarrow R_3) = ((R_1 \rightarrow R_2) \Leftarrow R_3)$.

Proof (1)

$$\begin{aligned}(R_1 \circ R_2)^s(z,x) &= (R_1 \circ R_2)(x,z)\\ &= \bigvee_{y\in V}(R_1(x,y) \odot R_2(y,z))\\ &= \bigvee_{y\in V}(R_1^s(y,x) \odot R_2^s(z,y))\\ &= (R_2^s \otimes R_1^s)(z,x).\end{aligned}$$

(2) By Lemma 2.3 (12,15), we have

$$\begin{aligned}(R_1 \circ R_2)^*(x,z) &= \Big(\bigvee_{y\in V}(R_1(x,y) \odot R_2(y,z))\Big) \Rightarrow 0\\ &= \bigwedge_{y\in V}\Big(R_2(y,z) \Rightarrow (R_1(x,y) \Rightarrow 0)\Big)\\ &= (R_1^* \Leftarrow R_2)(x,z).\end{aligned}$$

(3) By Lemma 2.3 (13,14), we have

$$\begin{aligned}(R_1 \otimes R_2)^0(x,z) &= \Big(\bigvee_{y\in V}(R_2(y,z) \odot R_1(x,y))\Big) \rightarrow 0\\ &= \bigwedge_{y\in V}\Big(R_2(y,z) \rightarrow (R_1(x,y) \rightarrow 0)\Big)\\ &= (R_1^0 \leftarrow R_2)(x,z).\end{aligned}$$

(4)

$$\begin{aligned}(R_1 \Rightarrow R_2)^s(z,x) &= (R_1 \Rightarrow R_2)(x,z)\\ &= \bigwedge_{y\in Y}(R_1(x,y) \Rightarrow R_2(y,z))\\ &= \bigwedge_{y\in Y}(R_1^s(y,x) \Rightarrow R_2^s(z,y))\\ &= (R_2^s \Leftarrow R_1^s)(z,x).\end{aligned}$$

(6) We have $R_1 \circ R_2 \leq R_3$ iff $R_1 \leq (R_3 \leftarrow R_2^s)$ iff $R_2 \leq (R_1^s \Rightarrow R_3)$ because

$$\begin{aligned}&R_1(x,y) \odot R_2(y,z) \leq R_3(x,z) \text{ iff } R_1(x,y) \leq R_2(y,z) \rightarrow R_3(x,z)\\ &R_1(x,y) \odot R_2(y,z) \leq R_3(x,z) \text{ iff } R_2(y,z) \leq R_1(x,y) \Rightarrow R_3(x,z).\end{aligned}$$

(8) By Lemma 2.3 (8,15), we have

$$\begin{aligned}((R_1 \otimes R_2) \Rightarrow R_3)(x,p) &= \bigwedge_{z\in X}((R_1 \otimes R_2)(x,z) \Rightarrow R_3(z,p))\\ &= \bigwedge_{z\in X}((\bigvee_{y\in X}(R_2(y,z) \odot R_1(x,y)) \Rightarrow R_3(z,p)))\\ &= \bigwedge_{z\in X}\bigwedge_{y\in X}(R_1(x,y) \Rightarrow (R_2(y,z) \Rightarrow R_3(z,p)))\\ &= \bigwedge_{y\in X}(R_1(x,y) \Rightarrow (R_2 \Rightarrow R_3)(y,p)))\\ &= (R_1 \Rightarrow (R_2 \Rightarrow R_3))(x,p).\end{aligned}$$

(9)By Lemma 2.3 (14), we have

$$
\begin{aligned}
((R_1 \leftarrow R_2) \leftarrow R_3)(x,p) &= \textstyle\bigwedge_{z\in X}(R_3(z,p) \rightarrow (R_1 \leftarrow R_2)(x,z)) \\
&= \textstyle\bigwedge_{z\in X}(R_3(z,p) \rightarrow \bigwedge_{y\in X}(R_2(y,z) \rightarrow R_1(x,y))) \\
&= \textstyle\bigwedge_{z\in X}\bigwedge_{y\in X}(R_3(z,p) \rightarrow (R_2(y,z) \rightarrow R_1(x,y))) \\
&= \textstyle\bigwedge_{y\in X}((\bigvee_{z\in X}(R_3(z,p) \odot R_2(y,z)) \rightarrow R_1(x,y))) \\
&= \textstyle\bigwedge_{y\in X}((R_2 \otimes R_3)(y,p) \rightarrow R_1(x,y))) \\
&= (R_1 \leftarrow (R_2 \otimes R_3))(x,p).
\end{aligned}
$$

(10) By Lemma 2.3 (16), we have

$$
\begin{aligned}
(R_1 \Rightarrow (R_2 \leftarrow R_3))(x,p) &= \textstyle\bigwedge_{y\in X}(R_1(x,y) \Rightarrow (R_2 \leftarrow R_3)(y,p)) \\
&= \textstyle\bigwedge_{y\in X}(R_1(x,y) \Rightarrow \bigwedge_{z\in W}(R_3(z,p) \rightarrow R_2(y,z))) \\
&= \textstyle\bigwedge_{y\in X}\bigwedge_{z\in X}(R_1(x,y) \Rightarrow (R_3(z,p) \rightarrow R_2(y,z))) \\
&= \textstyle\bigwedge_{z\in X}\bigwedge_{y\in X}(R_3(z,p) \rightarrow (R_1(x,y) \Rightarrow R_2(y,z))) \\
&= \textstyle\bigwedge_{z\in X}(R_3(z,p) \rightarrow \bigwedge_{y\in X}(R_1(x,y) \Rightarrow R_2(y,z))) \\
&= \textstyle\bigwedge_{z\in X}(R_3(z,p) \rightarrow (R_1 \Rightarrow R_2)(x,z)) \\
&= ((R_1 \Rightarrow R_2) \leftarrow R_3)(x,p).
\end{aligned}
$$

Other cases are similarly proved.

Theorem 3.2 *Let $R \in L^{X\times X}$ be a fuzzy relation. We have the following properties.*

(1) *If R is an $\odot$-equivalence relation, then R is an r-preorder.*

(2) *If R is an r-preorder and symmetric, then R is an $\odot$-equivalence relation.*

(3) *If R is an l-preorder (resp. r-preorder), then R^s is an r-preorder (resp. l-preorder).*

(4) *If R is reflexive, then $R \circ R, R \otimes R$ are reflexive, $R \leq (R \circ R)$, $R \leq (R \otimes R)$, $(R \rightarrow R) \leq R$, $(R \Rightarrow R) \leq R$, $(R \Leftarrow R) \leq R$ and $(R \leftarrow R) \leq R$.*

(5) *R is symmetric iff $(R \Rightarrow R)$ is reflexive iff $(R \Leftarrow R)$ is reflexive iff $(R \rightarrow R)$ is reflexive iff $(R \leftarrow R)$ is reflexive.*

(6) *If R is symmetric, then $(R\circ R)^s = R\otimes R$, $(R\otimes R)^s = R\circ R$, $(R \Leftarrow R)^s = R \Rightarrow R$, $(R \Rightarrow R)^s = R \Leftarrow R$, $(R \leftarrow R)^s = R \rightarrow R$, $(R \rightarrow R)^s = R \leftarrow R$.*

(7) *R is l-transitive iff $R \circ R \leq R$ iff $R \leq (R^s \Rightarrow R)$ iff $R \leq (R \leftarrow R^s)$.*

(8) *R is r-transitive iff $R \otimes R \leq R$ iff $R \leq (R^s \rightarrow R)$ iff $R \leq (R \Leftarrow R^s)$.*

(9) *If R is an l-preorder, then $R = (R \circ R) = (R^s \Rightarrow R) = (R \leftarrow R^s)$.*

(10) *If R is an r-preorder, then $R = (R \otimes R) = (R^s \rightarrow R) = (R \Leftarrow R^s)$.*

(11) *R is an $\odot$-equivalence relation iff $(R \Rightarrow R)$ and R are reflexive and $R \leq (R \Rightarrow R)$ iff $(R \Leftarrow R)$ and R are reflexive and $R \leq (R \Leftarrow R)$ iff $(R \rightarrow R)$ and R are reflexive and $R \leq (R \rightarrow R)$ iff $(R \leftarrow R)$ and R are reflexive and $R \leq (R \leftarrow R)$.*

(12) *If R is an $\odot$-equivalence relation, then* $R = (R\circ R) = R\otimes R = (R \Rightarrow R) = (R \Leftarrow R) = (R \rightarrow R) = (R \leftarrow R)$.

(13) *If R is symmetric, then $R \Rightarrow R$ and $R \leftarrow R$ are l-preorder, $R \rightarrow R$ and $R \Leftarrow R$ are r-preorder.*

(14) *Let R be reflexive. We define*

$$R^{\infty}(x,y) = \bigvee_{n\in N} R^n(x,y)$$

where $R^n = \overbrace{R\circ R ...\circ R}^{n}$. *Then R^{∞} is an l-preoder.*

(15) *Let R be reflexive. We define*

$$R^{[\infty]}(x,y) = \bigvee_{n\in N} R^{[n]}(x,y)$$

where $R^{[n]} = \overbrace{R\otimes R ...\otimes R}^{[n]}$. *Then $R^{[\infty]}$ is an r-preoder.*

(16) *U^{∞} is an l-preoder and $U^{[\infty]}$ is an r-preoder for* $U \in \{R \Rightarrow R^s, R \rightarrow R^s, R \Leftarrow R^s, R \leftarrow R^s, R^s \Rightarrow R, R^s \rightarrow R, R^s \Leftarrow R, R^s \leftarrow R\}$.

Proof (1) Since R is symmetric, R is r-transitive from:

$$R(y,z)\odot R(x,y) = R(z,y)\odot R(y,x) \le R(z,x) = R(x,z).$$

(2) Since R is symmetric and R is r-transitive, R is l-transitive.

(3) It follows from

$$R^s(y,z)\odot R^s(x,y) = R(z,y)\odot R(y,x) \le R(z,x) = R^s(x,z).$$

(4) Since $R\circ R(x,x) \ge R(x,x)\odot R(x,x) = \top$, $R\circ R$ is reflexive.

$$\begin{array}{ll}(R\Rightarrow R)(x,z) & = \bigwedge_{y\in X}(R(x,y)\rightarrow R(y,z)) \\ & \le (R(x,x)\rightarrow R(x,z)) = R(x,z).\end{array}$$

Other cases are similarly proved.

(5) It easily proved because

$$\begin{array}{l}(R\Rightarrow R)(x,x) = \bigwedge_{y\in X}(R(x,y)\rightarrow R(y,x)) = \top \\ \text{iff } R(x,y) \le R(y,x) \text{ (by Lemma 2.3 (3))}.\end{array}$$

Similarly, $R(x,y) \ge R(y,x)$. Hence R is symmetric. Other cases are similarly proved.

(6) $(R\circ R)^s = R^s \otimes R^s = R\otimes R$. $(R \Leftarrow R)^s = (R^s \Rightarrow R^s) = (R \Rightarrow R)$. Other cases are similarly proved.

(7) We have $R \circ R \leq R$ iff $R \leq (R^s \Rightarrow R)$ iff $R \leq (R \leftarrow R^s)$ because

$$R(x,y) \odot R(y,z) \leq R(x,z) \text{ iff } R(y,z) \leq R(x,y) \Rightarrow R(x,z)$$
$$R(x,y) \odot R(y,z) \leq R(x,z) \text{ iff } R(x,y) \leq R(y,z) \rightarrow R(x,z).$$

Other cases are similarly proved.

(8) It is similarly proved as in (7).

(9) If R is reflexive, then $R \leq (R \circ R)$ $R \leq (R^s \Rightarrow R)$ and $R \leq (R \leftarrow R^s)$. By (7), the results holds.

(10) It is similarly proved as in (8).

(11) It follows from (4),(7) and (8).

(12) Since $R^s = R$, by (9,10), we easily prove it.

(13) Since R is symmetric, by (5), $R \Rightarrow R$ is reflexive.

$$\begin{aligned}(R \Rightarrow R)(x,y) \odot (R \Rightarrow R)(y,z) &\leq (R(x,p) \Rightarrow R(p,y)) \odot (R(y,p) \Rightarrow R(p,z) \\ &\leq (R(x,p) \Rightarrow R(p,y)) \odot (R(p,y) \Rightarrow R(p,z) \\ &\leq R(x,p) \Rightarrow R(p,z)\end{aligned}$$

Hence $R \Rightarrow R$ is l-transitive. Thus $R \Rightarrow R$ is an l-preorder. Other cases are similarly proved.

(14) Since R is reflexive and $R(x,x) \leq R^2(x,x) \leq R^\infty(x,x)$, then R^∞ is reflexive. Suppose there exist $x, y, z \in X$ such that

$$R^\infty(x,y) \circ R^\infty(y,z) \not\leq R^\infty(x,z).$$

By the definition of $R^\infty(x,y)$, there exists $x_i \in X$ such that

$$R(x,x_1) \odot R(x_1,x_2) \odot ... \odot R(x_n,y) \circ R^\infty(y,z) \not\leq R^\infty(x,z).$$

By the definition of $R^\infty(y,z)$, there exists $y_j \in X$ such that

$$R(x,x_1) \odot R(x_1,x_2) \odot ... \odot R(x_n,y)$$

$$\odot R(y,y_1) \odot R(y_1,y_2) \odot ... \odot R(y_n,z) \not\leq R^\infty(x,z).$$

It is a contradiction for the definition of $R^\infty(x,z)$. Hence R^∞ is l-transitive.

(15) Since R is reflexive and $R(x,x) \leq R^{[2]}(x,x) \leq R^{[\infty]}(x,x)$, then $R^{[\infty]}$ is reflexive. Suppose there exist $x, y, z \in X$ such that

$$R^{[\infty]}(y,z) \otimes R^{[\infty]}(x,y) \not\leq R^{[\infty]}(x,z).$$

By the definition of $R^\infty(x,y)$, there exists $x_i \in X$ such that

$$R^{[\infty]}(y,z) \otimes R(x_n,y) ... \odot R(x_1,x_2) \odot R(x,x_1) \not\leq R^{[\infty]}(x,z).$$

By the definition of $R^{[\infty]}(y,z)$, there exists $y_j \in X$ such that

$$R(y_m,z) \odot ... \odot R(y_1,y_2) \odot R(y,y_1)$$

$$\odot R(x_n, y)... \odot R(x_1, x_2) \odot R(x, x_1) \not\leq R^{[\infty]}(x, z).$$

It is a contradiction for the definition of $R^{[\infty]}(x,z)$. Hence $R^{[\infty]}$ is r-transitive.

(16) For $U \in \{R \Rightarrow R^s, R \rightarrow R^s, R \Leftarrow R^s, R \leftarrow R^s, R^s \Rightarrow R, R^s \rightarrow R, R^s \Leftarrow R, R^s \leftarrow R\}$, U is reflexive. By (14,15), it is easily proved.

Theorem 3.3 *Let X be a set and $A \in L^X$. We define $R_{\Rightarrow}, R_{\rightarrow} : X \times X \rightarrow L$ as follows:*

$$R_{\Rightarrow}(x, y) = A(x) \Rightarrow A(y),\ R_{\rightarrow}(x, y) = A(x) \rightarrow A(y).$$

We have the following properties.

(1) $R_{\Rightarrow}$ is l-preorder and $R_{\rightarrow}$ is r-preorder.

(2) R_l is l-preorder and R_r is r-preorder with $R_r = R_l^s$ where $R_l(x,y) = R_{\Rightarrow}(x,y) \wedge R_{\rightarrow}(y,x)$ and $R_r(x,y) = R_{\rightarrow}(x,y) \wedge R_{\Rightarrow}(y,x)$. Moreover, if A is an injective function, then R_l is l-order and R_r is r-order.

Proof. (1) It follows from, by Lemma 2.3 (10),

$$(A(x) \Rightarrow A(y)) \odot (A(y) \Rightarrow A(z)) \leq (A(x) \Rightarrow A(z))$$

$$(A(y) \rightarrow A(z)) \odot (A(x) \rightarrow A(y)) \leq (A(x) \rightarrow A(z)).$$

(2)

$$\begin{array}{ll} R_l(x,y) \odot R_l(y,z) & \leq (A(x) \Rightarrow A(y)) \odot (A(y) \Rightarrow A(z)) \leq (A(x) \Rightarrow A(z)) \\ R_l(x,y) \odot R_l(y,z) & \leq (A(y) \rightarrow A(x)) \odot (A(z) \rightarrow A(y)) \leq (A(z) \rightarrow A(x)) \\ R_l(x,y) \odot R_l(y,z) & \leq R_l(x,z). \end{array}$$

$$\begin{array}{ll} R_r(y,z) \odot R_r(x,y) & \leq (A(y) \rightarrow A(z)) \odot (A(x) \rightarrow A(y)) \leq (A(x) \rightarrow A(z)) \\ R_r(x,y) \odot R_r(y,z) & \leq (A(z) \Rightarrow A(y)) \odot (A(y) \Rightarrow A(x)) \leq (A(z) \Rightarrow A(x)) \\ R_r(y,z) \odot R_r(x,y) & \leq R_r(x,z). \end{array}$$

If $R_l(x,y) = \top$, then $A(x) = A(y)$. Since A is injective, $x = y$.

Example 3.4 Let $K = \{(x,y) \in R^2 \mid x > 0\}$ be a set and we define an operation $\otimes : K \times K \rightarrow K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1).$$

Then $(K, \otimes)$ is a group with $e = (1,0)$, $(x,y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a,b) \in R^2 \mid a = 1, b \geq 0 \text{ , or } a > 1\}$ because $P \cap P^{-1} = \{(1,0)\}$, $P \odot P \subset P$, $(a,b)^{-1} \odot P \odot (a,b) = P$ and $P \cup P^{-1} = K$. For $(x_1,y_1), (x_2,y_2) \in K$, we define

$$\begin{array}{ll} (x_1,y_1) \leq (x_2,y_2) & \Leftrightarrow (x_1,y_1)^{-1} \odot (x_2,y_2) \in P,\ (x_2,y_2) \odot (x_1,y_1)^{-1} \in P \\ & \Leftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2. \end{array}$$

Then $(K, \leq \otimes)$ is a lattice-group. (ref. [1])

The structure $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1,0))$ is a generalized residuated lattice with strong negation where $\bot = (\frac{1}{2}, 1)$ is the least element and $\top = (1,0)$ is the greatest element from the following statements:

$$\begin{aligned}
(x_1,y_1)\odot(x_2,y_2) &= (x_1,y_1)\otimes(x_2,y_2)\vee(\tfrac{1}{2},1) = (x_1x_2, x_1y_2+y_1)\vee(\tfrac{1}{2},1),\\
(x_1,y_1)\Rightarrow(x_2,y_2) &= ((x_1,y_1)^{-1}\otimes(x_2,y_2))\wedge(1,0) = (\tfrac{x_2}{x_1}, \tfrac{y_2-y_1}{x_1})\wedge(1,0),\\
(x_1,y_1)\rightarrow(x_2,y_2) &= ((x_2,y_2)\otimes(x_1,y_1)^{-1})\wedge(1,0) = (\tfrac{x_2}{x_1}, -\tfrac{x_2y_1}{x_1}+y_2)\wedge(1,0).
\end{aligned}$$

Furthermore, we have $(x,y) = (x,y)^{*\circ} = (x,y)^{\circ *}$ from:

$$(x,y)^* = (x,y) \Rightarrow (\frac{1}{2}, 1) = (\frac{1}{2x}, \frac{1-y}{x}),$$

$$(x,y)^{*\circ} = (\frac{1}{2x}, \frac{1-y}{x}) \rightarrow (\frac{1}{2}, 1) = (x,y).$$

(1) Let $X = \{a, b, c\}$ be a set. Let $A \in L^X$ as

$$A(a) = (0.6, 2),\ A(b) = (0.8, -1),\ A(c) = (0.5, 3).$$

From Theorem 3.3, we obtain $R_{\Rightarrow}, R_{\rightarrow}, R_l, R_r \in L^{X\times X}$ as

$$R_{\Rightarrow} = \begin{pmatrix} (1,0) & (1,0) & (\frac{5}{6}, \frac{5}{3}) \\ (\frac{3}{4}, \frac{15}{4}) & (1,0) & (\frac{5}{8}, -1) \\ (1,0) & (1,0) & (1,0) \end{pmatrix}$$

$$R_{\rightarrow} = \begin{pmatrix} (1,0) & (1,0) & (\frac{5}{6}, \frac{4}{3}) \\ (\frac{3}{4}, \frac{11}{4}) & (1,0) & (\frac{5}{8}, \frac{29}{8}) \\ (1,0) & (1,0) & (1,0) \end{pmatrix}$$

$$R_l = \begin{pmatrix} (1,0) & (\frac{3}{4}, \frac{11}{4}) & (\frac{5}{6}, \frac{5}{3}) \\ (\frac{3}{4}, \frac{15}{4}) & (1,0) & (\frac{5}{8}, 5) \\ (\frac{5}{6}, \frac{4}{3}) & (\frac{5}{8}, \frac{29}{8}) & (1,0) \end{pmatrix}$$

$$R_r = \begin{pmatrix} (1,0) & (\frac{3}{4}, \frac{15}{4}) & (\frac{5}{6}, \frac{4}{3}) \\ (\frac{3}{4}, \frac{11}{4}) & (1,0) & (\frac{5}{8}, \frac{29}{8}) \\ (\frac{5}{6}, \frac{5}{3}) & (\frac{5}{8}, 5) & (1,0) \end{pmatrix}$$

(2) Let $X = \{a, b, c\}$ be a set. Define $R \in L^{X\times X}$ as

$$R = \begin{pmatrix} (1,0) & (0.6,2) & (0.7,1) \\ (0.6,2) & (1,0) & (0.9,-1) \\ (0.7,1) & (0.9,-1) & (1,0) \end{pmatrix}$$

Since $(0.63, -1) = R(b,c) \odot R(c,a) \not\leq R(b,a) = (0.6, 2)$, R is not l-transitive. Since $(0.63, -1) = R(c,b) \odot R(a,c) \not\leq R(a,b) = (0.6, 2)$, R is not r-transitive.

$$R \circ R = \begin{pmatrix} (1,0) & (0.63, 0.3) & (0.7,1) \\ (0.63, -0.1) & (1,0) & (0.9,-1) \\ (0.7,1) & (0.9,-1) & (1,0) \end{pmatrix}$$

Since $R\circ R = R^n = \bigvee_{n\in N} R^n = R^\infty$ for $n\geq 2$, by Theorem 3.2 (14), $R^2 = R^\infty$ is an l-order.

$$R\otimes R = \begin{pmatrix} (1,0) & (0.63,-0.1) & (0.7,1)\\ (0.63,0.3) & (1,0) & (0.9,-1)\\ (0.7,1) & (0.9,-1) & (1,0)\end{pmatrix}$$

Since $R\otimes R = R^{[n]} = \bigvee_{n\in N} R^{[n]} = R^{[\infty]}$ for $n\geq 2$, by Theorem 3.2 (15), $R^{[2]} = R^{[\infty]}$ is an r-order.

$$R\Rightarrow R = \begin{pmatrix} (1,0) & (0.6,2) & (0.7,1)\\ (0.6,2) & (1,0) & (0.9,-1)\\ (\frac{2}{3},\frac{10}{3}) & (\frac{6}{7},\frac{10}{7}) & (1,0)\end{pmatrix}$$

$$R\Leftarrow R = \begin{pmatrix} (1,0) & (0.6,2) & (\frac{2}{3},\frac{10}{3})\\ (0.6,2) & (1,0) & (\frac{6}{7},\frac{10}{7})\\ (0.7,1) & (0.9,-1) & (1,0)\end{pmatrix}$$

$$R\rightarrow R = \begin{pmatrix} (1,0) & (0.6,2) & (0.7,1)\\ (0.6,2) & (1,0) & (0.9,-1)\\ (\frac{2}{3},\frac{10}{3}) & (\frac{6}{7},\frac{8}{7}) & (1,0)\end{pmatrix}$$

$$R\leftarrow R = \begin{pmatrix} (1,0) & (0.6,2) & (\frac{2}{3},\frac{10}{3})\\ (0.6,2) & (1,0) & (\frac{6}{7},\frac{8}{7})\\ (0.7,1) & (0.9,-1) & (1,0)\end{pmatrix}$$

Since R is symmetric, by Theorem 3.2 (13), $R\Rightarrow R$ and $R\leftarrow R$ are l-preorder, $R\rightarrow R$ and $R\Leftarrow R$ are r-preorder.

(3) Let $X=\{a,b,c\}$ be a set. Define $R\in L^{X\times X}$ as

$$R = \begin{pmatrix} (\frac{1}{2},1) & (\frac{5}{6},-\frac{2}{3}) & (\frac{5}{7},\frac{2}{7})\\ (\frac{5}{6},-\frac{2}{3}) & (\frac{1}{2},1) & (\frac{5}{9},\frac{14}{9})\\ (\frac{5}{7},\frac{2}{7}) & (\frac{5}{9},\frac{14}{9}) & (\frac{1}{2},1)\end{pmatrix}$$

We obtain

$$R\circ R = \begin{pmatrix} (\frac{25}{36},-\frac{11}{9}) & (\frac{1}{2},1) & (\frac{1}{2},1)\\ (\frac{1}{2},1) & (\frac{25}{36},-\frac{11}{9}) & (\frac{25}{42},-\frac{4}{21})\\ (\frac{1}{2},1) & (\frac{25}{42},-\frac{4}{21}) & (\frac{25}{49},\frac{24}{49})\end{pmatrix}$$

$$R\Rightarrow R = \begin{pmatrix} (1,0) & (\frac{3}{5},2) & (\frac{2}{3},\frac{8}{3})\\ (\frac{3}{5},2) & (1,0) & (\frac{6}{7},\frac{8}{7})\\ (\frac{10}{7},-\frac{4}{3}) & (\frac{9}{10},1) & (1,0)\end{pmatrix}$$

$$R\rightarrow R = \begin{pmatrix} (1,0) & (\frac{3}{5},\frac{7}{5}) & (\frac{2}{3},2)\\ (\frac{3}{5},\frac{7}{5}) & (1,0) & (\frac{6}{7},\frac{6}{7})\\ (\frac{10}{7},\frac{4}{5}) & (\frac{9}{10},-\frac{2}{5}) & (1,0)\end{pmatrix}$$

Since R is symmetric, by Theorem 3.2 (6), $R \otimes R = (R \circ R)^s$, $R \Leftarrow R = (R \Rightarrow R)^s$ and $R \leftarrow R = (R \to R)^s$. Since $(\frac{25}{36}, -\frac{11}{9}) = R(a,b) \odot R(b,a) \not\leq R(a,a) = (0.5, 1)$, R is not l-transitive. Since R is symmetric, by Theorem 3.2 (13), $R \Rightarrow R$ and $R \leftarrow R$ are l-preorder and $R \to R$ and $R \Leftarrow R$ are r-preorder.

(4) Let $X = \{a, b, c\}$ be a set. Define $R \in L^{X \times X}$ as

$$R = \begin{pmatrix} (1,0) & (\frac{5}{8}, \frac{5}{2}) & (\frac{5}{6}, \frac{5}{3}) \\ (\frac{5}{7}, \frac{30}{7}) & (1,0) & (\frac{5}{8}, -\frac{5}{4}) \\ (1,-2) & (\frac{5}{7}, \frac{10}{3}) & (1,0) \end{pmatrix}$$

Since R is an l-preorder, $R = R \circ R = R^s \Rightarrow R = R \leftarrow R^s$. Furthermore, since R is an r-preorder, $R = R \otimes R = R^s \to R = R \Leftarrow R^s$.

(5)Let $X = \{a, b, c\}$ be a set. Define $R \in L^{X \times X}$ as

$$R = \begin{pmatrix} (\frac{1}{2}, 1) & (0.8, -1) & (0.6, 0) \\ (0.7, -2) & (\frac{1}{2}, 1) & (0.8, 2) \\ (0.5, 2) & (0.6, -1) & (\frac{1}{2}, 1) \end{pmatrix}$$

We obtain

$$R \circ R = \begin{pmatrix} (\frac{1}{2}, 1) & (\frac{1}{2}, 1) & (0.64, 0.6) \\ (\frac{1}{2}, 1) & (0.56, -2.7) & (\frac{1}{2}, 1) \\ (\frac{1}{2}, 1) & (\frac{1}{2}, 1) & (\frac{1}{2}, 1) \end{pmatrix}$$

Since $(0.64, 0.6) = R(a,b) \odot R(b,c) \not\leq R(a,c) = (0.6, 0)$, R is not l-transitive.

$$R \otimes R = \begin{pmatrix} (0.56, -2.7) & (\frac{1}{2}, 1) & (\frac{1}{2}, 1) \\ (\frac{1}{2}, 1) & (0.56, -2.7) & (\frac{1}{2}, 1) \\ (\frac{1}{2}, 1) & (\frac{1}{2}, 1) & (\frac{1}{2}, 1) \end{pmatrix}$$

Since $(0.56, -2.7) = R(b,a) \odot R(a,b) \not\leq R(a,a) = (0.5, 1)$, R is not r-transitive.

$$R \Rightarrow R^s = \begin{pmatrix} (1,0) & (\frac{5}{8}, \frac{5}{2}) & (\frac{3}{4}, 0) \\ (\frac{5}{7}, \frac{30}{7}) & (1,0) & (\frac{5}{8}, -\frac{5}{4}) \\ (1,-2) & (\frac{5}{6}, \frac{10}{3}) & (1,0) \end{pmatrix}$$

Since $(R \Rightarrow R^s)^n = R \Rightarrow R^s$ for all $n \in N$, $R^\infty = R \Rightarrow R^s$ is an l-order.

References

[1] G. Birkhoff, *Lattice Theory,* Amer. Math. Soc. Colloq. Publ., **25**, New York, 1968.

[2] C.C. Chang, Algebraic analysis of many valued logics, *Trans. of A.M.S.*,**88 (2)** (1958), 467-490.

[3] A. Dvurecenskij, Pseudo MV-algebras are intervals in l-groups, *J. Australian Math. Soc.*, **72** (2002), 427-445.

[4] A. Dvurecenskij, On pseudo MV-algebras, *Soft Computing*,**5** (2001), 347-354.

[5] G. Georgescu, A. lorgulescu, Pseudo MV-algebras, *Multiple-Valued Logics*, **6** (2001),193-215.

[6] G. Georgescu, A. Popescu, Non-commutative fuzzy Galois connections, *Soft Computing*, **7** (2003), 458-467.

[7] G. Georgescu, A. Popescu, Non-commutative fuzzy structures and pairs of weak negations, *Fuzzy Sets and Systems*, **143** (2004), 129-155.

[8] N. Galatos, C. Tsinakis, Generalized MV-alebras, *J. of Algebra*, **283** (2005), 254-291.

[9] U. Höhle, E. P. Klement, *Non-classical logic and their applications to fuzzy subsets*, Kluwer Academic Publisher, Boston, 1995.

[10] Y.C. Kim, J.W. Park, Pseudo MV-algebras induced by functions, *Int. Math. Forums*, **4 (2)** (2009), 89-99.

[11] E. Turunen, *Mathematics Behind Fuzzy Logic*, A Springer-Verlag Co., 1999.

[12] R. Wille, *Restructuring lattice theory; an approach based on hierarchies of concept, in: 1. Rival(Ed.), Ordered Sets,* Reidel, Dordrecht, Boston, 1982, 445-470.

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