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Some Results on a Subclass of alpha-quazi Spirallike Mappings

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Abstract

Let $H(D)$ be the linear space of all analytic functions defined on the open unit disc $D = \{z \in C : |z| < 1\}$. A sense preserving logharmonic mapping is the solution of the non-linear elliptic partial differantial equation $\overline{f}_{\overline{z}} = w(z) f_z (\frac{\overline{f}}{f})$ where $w(z) \in H(D)$ is the second dilatation of f such that $|w(z)| < 1$ for all $z \in D$. It has been shown that if f is a non-vanishing logharmonic mapping, then f can be expressed as $f(z) = h(z).\overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in D with the normalization $h(0) \neq 0$, $g(0) = 1$. If f vanishes at $z = 0$ but it is not identically zero, then f admits the representation $f = z.|z|^{2\beta} h(z)\overline{g(z)}$, where $Re\beta > -\frac{1}{2}$ and $h(z)$, $g(z)$ are analytic in D with the normalization $h(0) \neq 0$, $g(0) = 1$. [1], [2], [3]. The class of all logharmonic mappings is denoted by S^*_{LH}.
The aim of this paper is to give an aplication of the subordination principle to the class of spirallike logharmonic mappings which was introduced by Z.Abdulhadi and W.Hengartner. [1]

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1 Introduction

Let H be the linear space of all analytic functions defined in the open unit disc $D = \{z \in C : |z| < 1\}$. A sense preserving log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = w(z) \frac{f_z}{f}, \tag{1}$$

Where $w(z)$ the second dilatation of f and $w(z) \in H(D)$, $|w(z)| < 1$ for every $z \in D$. It has been shown that if f is non vanishing logharmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)} \tag{2}$$

Where $h(z)$ and $g(z)$ are analytic in D with the normalization $h(0) \neq 0$, $g(0) = 1$. On the other hand if f vanishes at $z = 0$, but it is not identically zero, then f admits the following representation

$$f = z.\, |z|^{2\beta}\, h(z)\overline{g(z)} \tag{3}$$

where $Re\beta > -\frac{1}{2}$, $h(z)$ and $g(z)$ are analytic in the open disc D with the normalization $h(0) \neq 0$, $g(0) = 1$. Also we note that univalent logharmonic mapping have been studied extensively. [1], [2], [3] and the class of univalent logharmonic mappings is denoted by S_{LH}. Let $f = zh(z)\overline{g(z)}$ be a univalent logharmonic mapping. We say that f is a starlike logharmonic mapping if

$$\frac{\partial \arg f(re^{i\theta})}{\partial\theta} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > 0$$

for all $z \in D$, and the class of all starlike logharmonic mappings is denoted by ST^*_{LH}

Let $\varphi(z)$ be analytic in D and let α be a real number such that $|\alpha| < \frac{\pi}{2}$. If $\varphi = 0$, $\varphi'(0) \neq 0$ and if

$$Re(e^{i\alpha}z\frac{\varphi'(z)}{\varphi(z)}) > 0 \tag{4}$$

then $\varphi(z)$ is univalent [5] and is said to be spirallike. Under these conditions we have

$$e^{i\alpha}z\frac{\varphi'(z)}{\varphi(z)} = Q(z) \tag{5}$$

where $ReQ(z) > 0$ and $Q(0) = e^{i\alpha}$. Defining $P(z) = Q(z)\sec\alpha - i\tan\alpha$ we may write

$$z\frac{\varphi'(z)}{\varphi(z)} = e^{-i\alpha}[P(z)\cos\alpha + i\sin\alpha] \tag{6}$$

where $ReP(z) > 0$, $P(0) = 1$. The class of spirallike functions is denoted by S^*_α. In particular with $\alpha = 0$, S^*_0 coincides with the class of normalized starlike functions. The relationship between S^*_α and S^*_0 is indicated in the following lemma.

Lemma 1.1 *$f(z) \in S_{0,p}$ if and only if there is a $g(z) \in S_{0,p}$ such that*

$$[\frac{f(z)}{z}]^{\exp(i\alpha)} = [\frac{g(z)}{z}]^{\cos\alpha} \tag{7}$$

where the branches are chosen so that each side of the equation has the value 1, when $z = 0$.

On the other hand Z.Abdulhadi and Y.Abu Muhanna was proved the following theorem.

Theorem 1.2 *Let $f(z) = z.h(z).\overline{g(z)}$ be a logharmonic mapping in D, $0 \notin hg(D)$. Then $f \in ST^*_{LH}$ if and only if $\varphi(z) = z\frac{h(z)}{g(z)} \in ST^*$*

Finally let Ω be the family of functions $\phi(z)$ which are analytic in D and satisfying the conditions $\phi(0) = 0$ $|\phi(z)| < 1$ for every $z \in D$ and let $s_1(z) = z + a_2z^2 + a_3z^3 + ...$, $s_2(z) = z + b_2z^2 + b_3z^3 + ...$ be analytic functions in D. We say that $s_1(z)$ is subordinate to $s_2(z)$ if $s_1(z) = s_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in D$ and denote by $s_1(z) \prec s_2(z)$.

2 Main Results

Considering Lemma (1.1) and Theorem (1.2) together we obtain the following lemma.

Lemma 2.1 *$\phi(z) \in S^*_\alpha$ if and only if there is a $f(z) = zh(z)\overline{g(z)} \in ST^*_{LH}$ such that*

$$(\frac{\phi(z)}{z})^{e^{i\alpha}} = (\frac{h(z)}{g(z)})^{\cos\alpha} \tag{8}$$

where the branches are chosen so that both sides of the equation has the value 1, when $z = 0$.

Theorem 2.2 *Using Lemma 2.1 then we have the following equality,*

$$e^{i\alpha}z.\frac{\phi'(z)}{\phi(z)} = \cos\alpha[1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}] + i\sin\alpha \tag{9}$$

We have;

$$f = z.|z|^{2\beta}h(z)\overline{g(z)} \Rightarrow \{\ zf_zf = \beta + 1 + z\frac{h'(z)}{h(z)}; \frac{\overline{z}f_{\overline{z}}}{f} = \beta + \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}} \tag{10}$$

$$w(z) = \frac{\overline{f_{\overline{z}}}}{\overline{f}}\frac{f}{f_z} = \frac{\overline{\beta} + z\frac{g'(z)}{g(z)}}{1 + \beta + z\frac{h'(z)}{h(z)}} \tag{11}$$

In the equality (10) if we take $\beta = 0$ then we obtain;

$$w(z) = \frac{z\frac{g'(z)}{g(z)}}{1 + z\frac{h'(z)}{h(z)}} \tag{12}$$

Therefore we have $w(0)=0$, $|w(z)|<1$ then we can say that $w(z)$ satisfies the conditions of Schwarz Lemma, and

$$1-w(z)=\frac{1+z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}}{1+z\frac{h'(z)}{h(z)}} \tag{13}$$

$$\frac{w(z)}{1-w(z)}=\frac{z\frac{g'(z)}{g(z)}}{1+z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}} \tag{14}$$

Using the equality (12), (11) equalities (13) and (14) can be written in the following form,

$$1-w(z)=\frac{\frac{1}{\cos\alpha}[z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha]}{z\frac{f_z}{f}} \tag{15}$$

$$\frac{w(z)}{1-w(z)}=\frac{z\frac{\overline{f_{\overline{z}}}}{\overline{f}}}{\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha]} \tag{16}$$

Using the subordination principle the equalities can be written

$$\left|\frac{\frac{1}{\cos\alpha}[z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha]}{z\frac{f_z}{f}}-c_1(r)\right|\prec\rho_1(r) \tag{17}$$

$$\left|\frac{z\frac{\overline{f_{\overline{z}}}}{\overline{f}}}{\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha]}-c_2(r)\right|\prec\rho_2(r) \tag{18}$$

Because the transformations $\rho_1(r)$ and $\rho_2(r)$ map $|z|=r$ on to the discs with the centres

$$c_1(r)=[\frac{m^4(1-a)+\overline{a}(m^2-\overline{a}r^2)}{m^4-(\overline{a})^2r^2},0]$$

$$c_2(r)=[\frac{m^4a(1-a)+m^2(m^2-\overline{a})r^2}{m^4(1-a)^2-(m^2-\overline{a})^2r^2},0]$$

and the radius

$$\rho_1(r)=\frac{|m^2(m^2-\overline{a})+\overline{a}m^2(1-a)|\,r}{m^4-(\overline{a})^2r^2}$$

$$\rho_2(r)=\frac{|-m^4(1-a)-m^2a(m^2-\overline{a})|\,r}{m^4(1-a)^2-(m^2-\overline{a})^2r^2}$$

respectively using the subordination principle on the expressions (17), (18) then we get the following theorem.

Theorem 2.3 *Let $f = zh(z)\overline{g(z)}$ be a log-harmonic quazi spirallike function then*

$$F_1(r,\alpha) \le z\frac{f_z}{f} \le F_2(r,\alpha) \tag{19}$$

$$F_3(r,\alpha) \le \frac{\overline{z}\overline{f}_{\overline{z}}}{f} \le F_4(r,\alpha) \tag{20}$$

Since the transformations

$$\frac{(m^2-\overline{a})z+(m^2-m^2a)}{-\overline{a}z+m^2}$$

and

$$\frac{-m^2z+m^2a}{(m^2-\overline{a})z+m^2(1-a)}$$

map $|z| = r$ onto the discs with centres

$$c_1(r) = (\frac{m^4(1-a)+\overline{a}(m^2-\overline{a})r^2}{m^4-(\overline{a})^2r^2}, 0)$$

$$c_2(r) = (\frac{m^4a(1-a)+m^2(m^2-\overline{a})r^2}{m^4(1-a)^2-(m^2-\overline{a})^2r^2}, 0)$$

and the radius

$$\rho_1(r) = \frac{|m^2(m^2-\overline{a})+\overline{a}m^2(1-a)|\, r}{m^4-(\overline{a})^2r^2}$$

$$\rho_2(r) = \frac{|-m^4(1-a)-m^2a(m^2-\overline{a})|\, r}{m^4(1-a)^2-(m^2-\overline{a})^2r^2}$$

After simple calculations from Theorem 2.2 and using inequalities (17), (18) we get the result easily.

$$F_1(r,\alpha) = \frac{m^4-(\overline{a})^2r^2}{m^4(1-a)+\overline{a}(m^2-\overline{a})r^2+[m^2(m^2-\overline{a})+\overline{a}m^2(1-a)]r}\cdot\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)-i\sin\alpha}]$$

$$F_2(r,\alpha) = \frac{m^4-(\overline{a})^2r^2}{m^4(1-a)+\overline{a}(m^2-\overline{a})r^2-[m^2(m^2-\overline{a})+\overline{a}m^2(1-a)]r}\cdot\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)-i\sin\alpha}]$$

$$F_3(r,\alpha) = \frac{m^4a(1-a)+m^2(m^2(m^2-\overline{a})r^2-[m^4(1-a)+m^2a(m^2-\overline{a})]r}{m^4(1-a)^2-(m^2-\overline{a})^2r^2}\cdot\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)-i\sin\alpha}]$$

$$F_4(r,\alpha) = \frac{m^4a(1-a)+m^2(m^2(m^2-\overline{a})r^2+[m^4(1-a)+m^2a(m^2-\overline{a})]r}{m^4(1-a)^2-(m^2-\overline{a})^2r^2}\cdot\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)-i\sin\alpha}]$$

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