Mathematica Aeterna, Vol. 2, 2012, no. 8, 675 -687

Fuzzy consequence operators

Yong Chan Kim

Department of Mathematics, Gangneung-Wonju National University,
Gangneung, Gangwondo 210-702, Korea
yck@gwnu.ac.kr

Abstract

We investigate the properties of fuzzy consequence operators in generalized residuated lattice. In particular, we investigate the relations between right (resp. left) $\odot$-preorders and fuzzy consequence operators.

Mathematics Subject Classification: 03E72, 54A40,54B10

Keywords: Generalized residuated lattice, right(left) $\odot$-preorder, right (left) $\odot$-transitive, fuzzy consequence operators

1 Introduction

Pavelka [8] introduced the concept of fuzzy consequence operator. Recently, it is developed in the approximate reasoning context with different fuzzy logics on residuated lattices [4,5]. On the other hand, Wille [10] introduced the structures on lattices which are important mathematical tools for data analysis and knowledge processing. MV-algebra was introduced by Chang [2] to provide algebraic models for many valued propositional logic. Recently, it is developed many directions (BL-algebra, residuated algebra) [5,9,10]. In particular, noncommutative structures play an important role in metric spaces, algebraic structures (groups, rings, quantales, pseudo-BL-algebras)[3,6,7,9,10]. Georgescu and Popescu [6] introduced generalized residuated lattice as a noncommutative structure.

In this paper, we investigate the properties of fuzzy consequence operators in generalized residuated lattice. In particular, we investigate the relations between right (resp. left) $\odot$-preorders and fuzzy consequence operators.

2 Preliminaries

Definition 2.1 [6] A structure $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \perp, \top)$ is called a *generalized residuated lattice* iff it satisfies the following properties:

(L1) $(L, \vee, \wedge, \bot, \top)$ is a bounded lattice where $\bot$ is the bottom element and $\top$ is the top element;

(L2) $(L, \odot, \top)$ is a monoid;

(L3) adjointness properties,i.e.

$$x \le y \to z \text{ iff } x \odot y \le z \text{ iff } y \le x \Rightarrow z.$$

Two maps ${}^0, {}^* : L \to L$ defined by $a^0 = a \to \bot$ and $a^* = a \Rightarrow \bot$ is called *strong negations* if $a^{0*} = a$ and $a^{*0} = a$. We define

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise.} \end{cases} \quad \top_x^*(y) = \top_x^0(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, \Rightarrow, {}^*, {}^0, \bot, \top)$ be a generalized residuated lattice with strong negations * and ^{0}.

Definition 2.2 Let X be a set. A function $R : X \times X \to L$ is called a *right $\odot$-preorder* on X if it satisfies the following conditions:

(R) (reflexive) $R(x, x) = \top$ for all $x \in X$,

(LT) (right transitive) $R(x, y) \odot R(y, z) \le R(x, z)$, for all $x, y, z \in X$.

A function $R : X \times X \to L$ is called a *left $\odot$-preorder* on X if it satisfies (R) and the following condition:

(RT) (left transitive) $R(y, z) \odot R(x, y) \le R(x, z)$, for all $x, y, z \in X$.

Definition 2.3 [5] An operator $C : L^X \to L^X$ is called a *fuzzy consequence operator* iff it satisfies the following conditions:

(C1) $A \le C(A)$ for $A \in L^X$.

(C2) If $A \le B$, then $C(A) \le C(B)$ $A \in L^X$.

(C3) $C(C(A)) = C(A)$ for $A \in L^X$.

Lemma 2.4 *For each $x, y, z, x_i, y_i \in L$, the following properties hold.*

(1) *$\odot$ is isotone in both arguments.*

(2) *$\to$ and $\Rightarrow$ are antitone in the first and isotone in the second argument.*

(3) *$x \to y = \top$ iff $x \le y$ iff $x \Rightarrow y = \top$.*

(4) *$x \to \top = x \Rightarrow \top = \top$ and $\top \to x = \top \Rightarrow x = x$.*

(5) *$x \odot y \le x \wedge y$.*

(6) *$x \odot (\bigvee_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma}(x \odot y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \odot y = \bigvee_{i \in \Gamma}(x_i \odot y)$.*

(7) *$x \to (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma}(x \to y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \to y = \bigwedge_{i \in \Gamma}(x_i \to y)$.*

(8) *$x \Rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma}(x \Rightarrow y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \Rightarrow y = \bigwedge_{i \in \Gamma}(x_i \Rightarrow y)$.*

(9) *$x \odot (x \Rightarrow y) \le y$ and $(x \to y) \odot x \le y$.*

(10) *$(x \Rightarrow y) \odot (y \Rightarrow z) \le (x \Rightarrow z)$ and $(y \to z) \odot (x \to y) \le (x \to z)$.*

(11) *$x \Rightarrow y \le (y \Rightarrow z) \to (x \Rightarrow z)$ and $x \to y \le (y \to z) \Rightarrow (x \to z)$*

(12) *$\bigwedge_{i \in \Gamma} x_i^* = (\bigvee_{i \in \Gamma} x_i)^*$ and $\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*$.*

(13) *$\bigwedge_{i \in \Gamma} x_i^0 = (\bigvee_{i \in \Gamma} x_i)^0$ and $\bigvee_{i \in \Gamma} x_i^0 = (\bigwedge_{i \in \Gamma} x_i)^0$.*

(14) $(x \odot y) \to z = x \to (y \to z)$ *and* $(x \odot y)^0 = x \to y^0$.
(15) $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$ *and* $(x \odot y)^* = y \Rightarrow x^*$.
(16) $x \to (y \Rightarrow z) = y \Rightarrow (x \to z)$ *and* $x \Rightarrow (y \to z) = y \to (x \Rightarrow z)$.

Proof. (1)-(13) are proved in [6,9].

(14) Since $((x \odot y) \to z) \odot (x \odot y) \le z$, we have $(x \odot y) \to z \le x \to (y \to z)$. Since $(x \to (y \to z)) \odot (x \odot y) \le (y \to z) \odot y \le z$, we have $x \to (y \to z) \le (x \odot y) \to z$.

(16) Since $\Big(y \odot \Big(x \to (y \Rightarrow z)\Big)\Big) \odot x = y \odot \Big(\Big(x \to (y \Rightarrow z)\Big) \odot x\Big) \le y \odot (y \Rightarrow z) \le z$, then $x \to (y \Rightarrow z) \le y \Rightarrow (x \to z)$.

Since $y \odot \Big(\Big(y \Rightarrow (x \to z)\Big) \odot x\Big) = \Big(y \odot \Big(y \Rightarrow (x \to z)\Big)\Big) \odot x = (x \to z) \odot x \le z$, then $y \Rightarrow (x \to z) \le x \to (y \Rightarrow z)$.

(15) and other cases are similarly proved.

3 Fuzzy consequence operators

Definition 3.1 Let $R \in L^{X \times X}$ be a fuzzy relation. Define mappings $I^R, I_R, C^R, C_R : L^X \to L^X$ as follows:

$$I_R(A)(x) = \bigwedge_y (R(x,y) \Rightarrow A(y)) \quad I^R(A)(x) = \bigwedge_y (R(x,y) \to A(y)).$$

$$C_R(A)(x) = \bigvee_y (A(y) \odot R(y,x)) \quad C^R(x) = \bigvee_y (R(y,x) \odot A(y)).$$

Definition 3.2 (1) An operator $C : L^X \to L^X$ is called right $\odot$-coherent if

$$A(y) \odot C(\top_y)(x) \le C(A)(x).$$

(2) An operator $C : L^X \to L^X$ is called left $\odot$-coherent if

$$C(\top_y)(x) \odot A(y) \le C(A)(x).$$

Lemma 3.3 *Let* $R \in L^{X \times X}$ *be a fuzzy relation. Define*

$$R \circ R(x,z) = \bigvee_y (R(x,y) \odot R(y,z)), \quad R^{-1}(x,y) = R(y,x).$$

(1) *If* R *be a right* $\odot$*-preorder, then* R^{-1} *be a left* $\odot$*-preorder.*
(2) R *is a right* $\odot$*-preorder on* X *iff* $R \circ R = R$ *and* $R(x,x) = \top$.
(3) R *is a left* $\odot$*-preorder on* X *iff* $R^{-1} \circ R^{-1} = R^{-1}$ *and* $R(x,x) = \top$.

Proof (1), (2) and (3) are easily proved from:

$$\begin{aligned} R(x,z) &= R(x,x)\odot R(x,z)\le R\circ R(x,z)=\bigvee_y(R(x,y)\odot R(y,z))\\ &\le R(x,z)\\ R^{-1}(x,z) &= R^{-1}(x,x)\odot R^{-1}(x,z)\le R^{-1}\circ R^{-1}(x,z)\\ &=\bigvee_y(R^{-1}(x,y)\odot R^{-1}(y,z))=\bigvee_y(R(y,x)\odot R(z,y))\\ &\le R(z,x)=R^{-1}(x,z). \end{aligned}$$

Theorem 3.4 *$I_R(A^*)=(C_{R^{-1}}(A))^*$ and $I^R(A^0)=(C^{R^{-1}}(A))^0$.*

Proof (1)

$$\begin{aligned} &I_R(A^*)(x)=\bigwedge_y(R(x,y)\Rightarrow A^*(y))\\ &=(\bigvee_y(A(y)\odot R(x,y))^*=(C_{R^{-1}}(A))^*. \end{aligned}$$

(2)

$$\begin{aligned} &I^R(A^0)(x)=\bigwedge_y(R(x,y)\to A^0(y))\\ &=(\bigvee_y(R(x,y)\odot A(y))^0=(C^{R^{-1}}(A))^0. \end{aligned}$$

Theorem 3.5 (1) *Let $C : L^X \to L^X$ be a right $\odot$-coherent fuzzy consequence operator and R_C defined by*

$$R_C(x,y)=C(\top_x)(y).$$

Then R_C is a right $\odot$-preorder on X and $C_{R_C}(A)\le C(A)$ for all $A\in L^X$ with $C_{R_C}(\top_x)(y)=R_C(x,y)=C(\top_x)(y)$.

(2) *Let $C : L^X \to L^X$ be a left $\odot$-coherent fuzzy consequence operator and R_C defined by*

$$R_C(x,y)=C(\top_x)(y).$$

Then R_C is a left $\odot$-preorder on X and $C_{R_C}(A)\le C(A)$ for all $A\in L^X$ with $C_{R_C}(\top_x)(y)=R_C(x,y)=C(\top_x)(y)$.

Proof (1) Since $C : L^X \to L^X$ is right $\odot$-coherent, $C(\top_x)(y)\odot C(\top_y)(z)\le C(C(\top_x))(z)$. Thus, R_C is a right $\odot$-preorder on X from: $R_C(x,x)=C(\top_x)(x)\ge \top_x(x)=\top$ and

$$\begin{aligned} &R_C(x,y)\odot R_C(y,z)=C(\top_x)(y)\odot C(\top_y)(z)\\ &\le C(C(\top_x))(z)=C(\top_x)(z)=R_C(x,z). \end{aligned}$$

$$\begin{aligned} &C_{R_C}(A)(x)=\bigvee_y(A(y)\odot R_C(y,x))\\ &=\bigvee_y(A(y)\odot C(\top_y)(x))\le C(A)(x). \end{aligned}$$

Moreover, $C_{R_C}(\top_x)(y) = R_C(x,y) = C(\top_x)(y)$.

(2) Since $C : L^X \to L^X$ is left $\odot$-coherent, $C(\top_y)(z) \odot C(\top_x)(y) \leq C(C(\top_x))(z)$. Thus, R_C is a left $\odot$-preorder on X from:

$$\begin{array}{l} R_C(y,z) \odot R_C(x,y) = C(\top_y)(z) \odot C(\top_x)(y) \\ \leq C(C(\top_x))(z) = C(\top_x)(z) = R_C(x,z). \end{array}$$

Other cases are proved as a similar method in (1).

Theorem 3.6 *Let $R \in L^{X\times X}$ be a fuzzy relation.*

(1) *C_R is a right $\odot$-coherent operator. Moreover, R is a right $\odot$-preorder iff C_R is a fuzzy consequence operator with $R_{C_R} = R$.*

(2) *C^R is a left $\odot$-coherent operator. Moreover, R is a left $\odot$-preorder iff C^R is a fuzzy consequence operator with $R_{C^R} = R$.*

Proof (1) Since $C_R(\top_x)(y) = \bigvee_z(\top_x(z) \odot R(z,y)) = R(x,y)$, we have

$$\begin{array}{ll} C_R(A)(x) & = \bigvee_y(A(y) \odot R(y,x)) = \bigvee_y(A(y) \odot C_R(\top_y)(x)) \\ & \geq A(y) \odot C_R(\top_y)(x). \end{array}$$

Thus C_R is a right $\odot$-coherent operator. Let R be a right $\odot$-preorder. Then C_R is a fuzzy consequence operator from:

$$\begin{array}{ll} C_R(A)(x) & = \bigvee_y(A(y) \odot R(y,x)) \geq A(x) \odot R(x,x) = A(x). \\ C_R(C_R(A))(x) & = \bigvee_y(C_R(A)(y) \odot R(y,x)) \\ & = \bigvee_y(\bigvee_w(A(w) \odot R(w,y)) \odot R(y,x)) \\ & \leq \bigvee_w(A(w) \odot R(w,x)) = C_R(A)(x). \end{array}$$

Moreover, $R_{C_R}(x,y) = C_R(\top_x)(y) = \bigvee_z(\top_x(z) \odot R(z,y)) = R(x,y)$.

Conversely, since C_R is a right $\odot$-coherent fuzzy consequence operator operator, by Theorem 3.5(1), $R_{C_R} = R$ is a right $\odot$-preorder.

(2) Since $C^R(\top_x)(y) = \bigvee_z(R(z,y) \odot \top_x(z)) = R(x,y)$, we have

$$C^R(A)(x) = \bigvee_y (R(y,x) \odot A(y)) = \bigvee_y (C_R(\top_y)(x) \odot A(y)) \geq C_R(\top_y)(x) \odot A(y).$$

Hence C^R is a left $\odot$-coherent operator. Other cases are proved as a similar method in (1).

Theorem 3.7 (1) *If $C : L^X \to L^X$ is an operator with $C(A) \leq C(B)$ for $A \leq B$ and $\alpha \odot C(A) \leq C(\alpha \odot A)$ for $\alpha \in L$, then C is a right $\odot$-coherent operator.*

(2) *If $C : L^X \to L^X$ is an operator with $C(A) \leq C(B)$ for $A \leq B$ and $C(A) \odot \alpha \leq C(A \odot \alpha)$ for $\alpha \in L$, then C is a left $\odot$-coherent operator.*

Proof (1) Since $A = \bigvee_x (A(y) \odot \top_y)$, we have

$$\begin{aligned} C(A)(x) &= C(\bigvee_y (A(y) \odot \top_y))(x) \geq \bigvee_y C(A(y) \odot \top_y)(x) \\ &\geq \bigvee_y (A(y) \odot C(\top_y)(x)). \end{aligned}$$

Thus C is a right $\odot$-coherent operator.

(2) Since $A = \bigvee_x (\top_y \odot A(y))$, we have

$$\begin{aligned} C(A)(x) &= C(\bigvee_y (\top_y \odot A(y)))(x) \geq \bigvee_y C(\top_y \odot A(y))(x) \\ &\geq \bigvee_y (C(\top_y)(x) \odot A(y)). \end{aligned}$$

Thus C is a left $\odot$-coherent operator.

Theorem 3.8 *Let $R \in L^{X\times X}$ be a fuzzy relation. Define $\phi_R : L^X \to L^X$ as*

$$\phi_R(A)(x) = I_R(C^R(A))(x) = \bigwedge_w \Big(R(x,w) \Rightarrow \bigvee_y (R(y,w) \odot A(y)) \Big).$$

Then the following properties:

(1) *ϕ_R is a left $\odot$-coherent operator.*

(2) *If R is a left $\odot$-preorder, then ϕ_R is a fuzzy consequence operator with a left $\odot$-preorder as follows*

$$R_{\phi_R}(y,x) = \phi_R(\top_y)(x) = \bigwedge_w (R(x,w) \Rightarrow R(y,w)).$$

(3) *R is a reflexive relation iff $R_{\phi_R} \leq R$ or $\phi_R \leq C^R$.*

(4) *R is a left $\odot$-preorder iff $R_{\phi_R} = R$ or $\phi_R = C^R$.*

Proof (1) ϕ_R is a left $\odot$-coherent operator from:

$$\begin{aligned} \phi_R(\top_y)(x) \odot A(y) &= \bigwedge_w \Big(R(x,w) \Rightarrow \bigvee_y (R(y,w) \odot \top_y(y)) \Big) \odot A(y) \\ &= \bigwedge_w \Big(R(x,w) \Rightarrow R(y,w)) \Big) \odot A(y) \\ &\leq \bigwedge_w \Big(R(x,w) \Rightarrow R(y,w) \odot A(y) \Big) \\ &\leq \bigwedge_w \Big(R(x,w) \Rightarrow C^R(A)(w) \Big) \\ &= \phi_R(A)(x) \end{aligned}$$

(2)

$$\begin{aligned} \phi_R(A)(x) &= I_R(C^R(A))(x) \\ &= \bigwedge_w \Big(R(x,w) \Rightarrow \bigvee_y (R(y,w) \odot A(y)) \Big) \\ &= \bigwedge_w \Big(R(x,w) \Rightarrow (R(x,w) \odot A(x)) \Big) \geq A(x). \end{aligned}$$

Thus, $\phi_R(\phi_R(A)) \geq \phi_R(A)$, for all $A \in L^X$. Since R is left $\odot$- preorder, $I_R(A) \leq A \leq C_R(A)$ and $I_R(C^R(A)) \leq C^R(A)$ implies $C^R(I_R(C^R(A))) \leq C^R(C^R(A)) = C^R(A)$. Thus $\phi_R(\phi_R(A)) = \phi_R(A)$. Moreover,

$$R_{\phi_R}(y,x) = \phi_R(\top_y)(x) = \bigwedge_w (R(x,w) \Rightarrow R(y,w)).$$

(3) Since R is reflexive, $R_{\phi_R} \leq R$ and $\phi_R \leq C^R$ from

$$\begin{aligned} R_{\phi_R}(y,x) &= \phi_R(\top_y)(x) = \bigwedge_w (R(x,w) \Rightarrow R(y,w)) \\ &\leq R(x,x) \Rightarrow R(y,x) = R(y,x) \\ \phi_R(A)(x) &= I_R(C^R(A))(x) = \bigwedge_w \left(R(x,w) \Rightarrow \bigvee_y (R(y,w) \odot A(y))\right) \\ &\leq R(x,x) \Rightarrow \bigvee_y (R(y,x) \odot A(y)) = C^R(A)(x). \end{aligned}$$

Conversely,

$$\begin{aligned} R_{\phi_R}(x,x) &= \phi_R(\top_x)(x) = \bigwedge_w (R(x,w) \Rightarrow R(x,w)) = \top \\ &\leq C^R(\top_x)(x) = R(x,x). \end{aligned}$$

(4) Since $R(x,w) \odot R(y,x) \leq R(y,w)$ iff $R(y,x) \leq R(x,w) \Rightarrow R(y,w)$, we have $R \leq R_{\phi_R}$. Hence $R = R_{\phi_R}$.

Since R is left $\odot$-transitive, we have

$$R(w,y) \odot R(x,w) \odot A(x) \leq R(x,y) \odot A(x)$$

$$R(x,w) \odot A(x) \leq R(w,y) \Rightarrow R(x,y) \odot A(x)$$

Thus, $C^R(A) \leq \phi_R(A)$.

Conversely,

$$R_{\phi_R}(y,x) = \phi_R(\top_y)(x) = \bigwedge_w (R(x,w) \Rightarrow R(y,w)) \geq R(y,x) = C^R(\top_y)(x).$$

Thus $R(x,w) \odot R(y,x) \leq R(y,w)$ for all $x,y,w \in X$;i.e. R is left $\odot$-transitive.

Theorem 3.9 *Let $R \in L^{X \times X}$ be a fuzzy relation. Define $\phi^R : L^X \to L^X$ as*

$$\phi^R(A)(x) = I^R(C_R(A))(x) = \bigwedge_w \left(R(x,w) \to \bigvee_y (A(y) \odot R(y,w))\right).$$

Then the following properties:

(1) *ϕ^R is a right $\odot$-coherent operator.*

(2) *If R is a right $\odot$-preorder, then ϕ^R is a fuzzy consequence operator with a right $\odot$-preorder R_{ϕ^R} as follows*

$$R_{\phi^R}(y,x) = \phi^R(\top_y)(x) = \bigwedge_w (R(x,w) \to R(y,w)).$$

(3) *R is a reflexive relation iff $R_{\phi^R} \leq R$ or $\phi^R \leq C_R$.*

(4) *R is a right $\odot$-preorder iff $R_{\phi^R} = R$ or $\phi^R = C_R$.*

Proof (1) Since $(b\odot(a\to c))\odot a = b\odot((a\to c)\odot a)\le b\odot c$, we have $b\odot(a\to c)\le a\to b\odot c$.

$$\phi^R(A)(x) = I^R(C_R(A))(x) = \bigwedge_w \Big(R(x,w)\to \bigvee_y (A(y)\odot R(y,w))\Big).$$

$$\begin{array}{ll}\phi^R(\alpha\odot A)(x) & = I^R(C_R(\alpha\odot A))(x)\\ & = \bigwedge_w \Big(R(x,w)\to \bigvee_y(\alpha\odot A(y)\odot R(y,w))\Big)\\ & = \bigwedge_w\bigvee_y \Big(R(x,w)\to(\alpha\odot A(y)\odot R(y,w))\Big)\\ & \ge \bigwedge_w\bigvee_y \Big(\alpha\odot\Big(R(x,w)\to(A(y)\odot R(y,w))\Big)\Big)\\ & \ge \bigwedge_w \Big(\alpha\odot\bigvee_y\Big(R(x,w)\to(A(y)\odot R(y,w))\Big)\Big)\\ & \ge \alpha\odot\bigwedge_w\Big(R(x,w)\to\bigvee_y(A(y)\odot R(y,w))\Big)\\ & = \alpha\odot\phi^R(A)(x).\end{array}$$

(2)

$$\begin{array}{ll}\phi^R(A)(x) & = I^R(C_R(A))(x) = \bigwedge_w \Big(R(x,w)\to\bigvee_y(A(y)\odot R(y,w))\Big)\\ & \ge \bigwedge_w\Big(R(x,w)\to(A(x)\odot R(x,w))\Big)\ge A(x).\end{array}$$

Thus, $\phi_R(\phi_R(A))\ge\phi_R(A)$, for all $A\in L^X$. Since R is a right $\odot$- preorder, $I^R(A)\le A\le C_R(A)$ and $I^R(C_R(A))\le C_R(A)$ implies $C_R(I^R(C_R(A)))\le C_R(C_R(A))=C_R(A)$. Thus $\phi^R(\phi^R(A))=\phi^R(A)$. Moreover,

$$R_{\phi^R}(y,x)=\phi^R(\top_y)(x)=\bigwedge_w(R(x,w)\to R(y,w)).$$

(3) Since R is reflexive, $R_{\phi^R}\le R$ and $\phi^R\le C_R$ from

$$\begin{array}{ll}R_{\phi^R}(y,x) & = \phi^R(\top_y)(x)=\bigwedge_w(R(x,w)\to R(y,w))\\ & \le R(x,x)\to R(y,x)=R(y,x)\\ \phi^R(A)(x) & = I^R(C_R(A))(x)=\bigwedge_w\Big(R(x,w)\to\bigvee_y(A(y)\odot R(y,w))\Big)\\ & \le R(x,x)\to\bigvee_y(A(y)\odot R(y,x))=C_R(A)(x).\end{array}$$

Conversely,

$$\begin{array}{ll}R_{\phi^R}(x,x) & = \phi^R(\top_x)(x)=\bigwedge_w(R(x,w)\to R(x,w))=\top\\ & \le C_R(\top_x)(x)=R(x,x).\end{array}$$

(4) Since $R(x,y)\odot R(y,z)\le R(x,z)$ iff $R(x,y)\le R(y,z)\to R(x,z)$, we have $R\le R_{\phi^R}$. Hence $R=R_{\phi^R}$.

Since R is right $\odot$-transitive, we have

$$A(x)\odot R(x,y)\odot R(y,z)\le A(x)\odot R(x,z)$$

$$A(x) \odot R(x,y) \leq R(y,z) \to A(x) \odot R(x,z)$$

Thus, $C_R(A) \leq \phi^R(A)$.

Conversely,

$$R_{\phi^R}(y,x) = \phi^R(\top_y)(x) = \bigwedge_w (R(x,w) \to R(y,w)) \geq R(y,x) = C_R(\top_y)(x).$$

Thus $R(y,x) \odot R(x,w) \leq R(y,w)$ for all $x,y,w \in X$;i.e. R is right $\odot$-transitive.

Definition 3.10 Let $R \in L^{X\times Y}$ be a fuzzy relation. Define mappings $R_\uparrow, R_\Uparrow : L^X \to L^Y$ and $R^\uparrow, R^\Uparrow : L^Y \to L^X$ as follows:

$$R_\uparrow(A)(x) = \bigwedge_y (A(x) \to R(x,y)) \quad R_\Uparrow(A)(x) = \bigwedge_y (A(x) \Rightarrow R(x,y)),$$

$$R^\downarrow(B)(y) = \bigwedge_x (B(y) \to R(x,y)) \quad R^\Downarrow(B)(y) = \bigwedge_y (B(y) \Rightarrow R(x,y)).$$

Theorem 3.11 *Let $R \in L^{X\times Y}$ be a fuzzy relation. Define $\eta_R : L^X \to L^X$ as*

$$\eta_R(A)(x) = R^\downarrow(R_\Uparrow(A))(x) = \bigwedge_y \Big((\bigwedge_w A(w) \Rightarrow R(w,y)) \to R(x,y) \Big).$$

Then the following properties:

(1) η_R *is a left $\odot$-coherent operator.*

(2) η_R *is a fuzzy consequence operator with a left $\odot$-preorder as follows*

$$R_{\eta_R}(z,x) = \eta_R(\top_z)(x) = \bigwedge_w (R(z,y) \to R(x,y)).$$

(3) *If $R \in L^{X\times X}$, then R is a reflexive relation iff $R^{-1}_{\eta_R} \leq R$.*

(4) *If $R \in L^{X\times X}$, then R is a right $\odot$-preorder iff $R^{-1}_{\eta_R} = R$.*

Proof (1) Since $\Big((b \to a) \odot c\Big) \odot (c \Rightarrow b) = (b \to a) \odot (c \odot (c \Rightarrow b)) \leq (b \to a) \odot b \leq a$, we have $(b \to a) \odot c \leq (c \Rightarrow b) \to a$. It follows

$$\begin{aligned}
\eta_R(\top_z)(x) \odot A(z) &= \textstyle\bigwedge_y \Big((\bigwedge_w \top_z(w) \Rightarrow R(w,y)) \to R(x,y) \Big) \odot A(z) \\
&= \textstyle\bigwedge_y ((R(z,y) \to R(x,y)) \odot A(z) \\
&\leq \textstyle\bigwedge_y \Big((R(z,y) \to R(x,y)) \odot A(z) \Big) \text{ (by Lemma 2.4(2))} \\
&\leq \textstyle\bigwedge_y \Big((A(z) \Rightarrow R(z,y)) \to R(x,y) \Big) \text{ (by above equality)} \\
&\leq \textstyle\bigwedge_y \Big(\bigwedge_z (A(z) \Rightarrow R(z,y)) \to R(x,y) \Big) \\
&= \eta_R(A)(x).
\end{aligned}$$

Hence η_R is a left $\odot$-coherent operator.

(2)

$$\begin{aligned}\eta_R(A)(x) &= \bigwedge_y \Big((\bigwedge_w A(w) \Rightarrow R(w,y)) \to R(x,y) \Big) \\ &\geq \bigwedge_y \Big((A(x) \Rightarrow R(x,y)) \to R(x,y) \Big) \geq A(x).\end{aligned}$$

Thus, $R^{\downarrow}(R_{\Uparrow}(A)) \geq A$ implies $R_{\Uparrow}(R^{\downarrow}(R_{\Uparrow}(A))) \leq R_{\Uparrow}(A)$. Similarly,

$$\begin{aligned}R_{\Uparrow}(R^{\downarrow}(B))(y) &= \bigwedge_y \Big((\bigwedge_w B(w) \to R(x,w)) \Rightarrow R(x,y) \Big) \\ &= \bigwedge_y \Big((B(y) \to R(x,y)) \Rightarrow R(x,y) \Big) \geq B(y).\end{aligned}$$

Hence, $R_{\Uparrow}(R^{\downarrow}(R_{\Uparrow}(A))) \geq R_{\Uparrow}(A)$. Thus, $\eta_R(\eta_R(A)) = \eta_R(A)$, for all $A \in L^X$. Moreover,

$$R_{\eta_R}(z,x) = \eta_R(\top_z)(x) = \bigwedge_w (R(z,y) \to R(x,y)).$$

(3) Since R is reflexive, $R^{-1}_{\eta_R} \leq R$ from

$$\begin{aligned}R_{\eta_R}(z,x) &= \eta_R(\top_z)(x) = \bigwedge_w (R(z,w) \to R(x,w)) \\ &\leq R(z,z) \to R(x,z) = R(x,z).\end{aligned}$$

Conversely,

$$\begin{aligned}R_{\eta_R}(x,x) &= \eta_R(\top_x)(x) = \bigwedge_w (R(x,w) \to R(x,w)) = \top \\ &\leq R(x,x).\end{aligned}$$

(4) Since $R(z,x) \odot R(x,y) \leq R(z,y)$ iff $R(z,x) \leq R(x,y) \to R(z,y)$, we have $R(z,x) \leq R^{-1}_{\eta_R}(z,x)$. Hence $R = R^{-1}_{\eta_R}$.

Conversely,

$$R_{\eta_R}(z,x) = \eta_R(\top_z)(x) = \bigwedge_w (R(z,w) \to R(x,w)) \geq R(x,z).$$

Thus $R(x,z) \odot R(z,w) \leq R(x,w)$ for all $x, y, w \in X$;i.e. R is right $\odot$-transitive.

Theorem 3.12 *Let $R \in L^{X \times Y}$ be a fuzzy relation. Define $\eta^R : L^X \to L^X$ as*

$$\eta^R(A)(x) = R^{\Downarrow}(R_{\uparrow}(A))(x) = \bigwedge_y \Big((\bigwedge_w A(w) \to R(w,y)) \Rightarrow R(x,y) \Big).$$

Then the following properties:

(1) *η^R is a right $\odot$-coherent operator.*

(2) *η^R is a fuzzy consequence operator with a right $\odot$-preorder as follows*

$$R_{\eta^R}(z,x) = \eta^R(\top_z)(x) = \bigwedge_w (R(z,y) \Rightarrow R(x,y)).$$

(3) *If $R \in L^{X \times X}$, then R is a reflexive relation iff $R^{-1}_{\eta^R} \leq R$.*

(4) *If $R \in L^{X \times X}$, then R is a left $\odot$-preorder iff $R^{-1}_{\eta^R} = R$.*

Proof (1) Since $(a \to b) \odot (a \odot (b \Rightarrow c)) = ((a \to b) \odot a) \odot (b \Rightarrow c) \leq b \odot (b \Rightarrow c) \leq c$, we have $a \odot (b \Rightarrow c) \leq (c \Rightarrow b) \to a$. It follows

$$\begin{array}{ll} A(z) \odot \eta^R(\top_z)(x) & = A(z) \odot \bigwedge_y \Big((\bigwedge_w \top_z(w) \to R(w,y)) \Rightarrow R(x,y) \Big) \\ & = A(z) \odot \bigwedge_y (R(z,y) \Rightarrow R(x,y)) \\ & \leq \bigwedge_y \Big(A(z) \odot (R(z,y) \Rightarrow R(x,y)) \Big) \text{ (by Lemma 2.4(2))} \\ & \leq \bigwedge_y \Big((A(z) \to R(z,y)) \Rightarrow R(x,y) \Big) \text{ (by above equality)} \\ & \leq \bigwedge_y \Big(\bigwedge_z (A(z) \to R(z,y)) \Rightarrow R(x,y) \Big) \\ & = \eta^R(A)(x). \end{array}$$

Hence η^R is a right $\odot$-coherent operator.

(2)

$$\begin{array}{ll} \eta^R(A)(x) & = \bigwedge_y \Big((\bigwedge_w A(w) \to R(w,y)) \Rightarrow R(x,y) \Big) \\ & = \bigwedge_y \Big((A(x) \to R(x,y)) \Rightarrow R(x,y) \Big) \geq A(x). \end{array}$$

Thus, $R_\uparrow(R^\Downarrow(R_\uparrow(A))) \leq R_\uparrow(A)$. Similarly,

$$\begin{array}{ll} R_\uparrow(R^\Downarrow(B))(y) & = \bigwedge_y \Big((\bigwedge_w B(w) \Rightarrow R(x,w)) \to R(x,y) \Big) \\ & = \bigwedge_y \Big((B(y) \Rightarrow R(x,y)) \to R(x,y) \Big) \geq B(y). \end{array}$$

Hence, $R_\uparrow(R^\Downarrow(R_\uparrow(A))) \geq R_\uparrow(A)$. Thus, $\eta^R(\eta^R(A)) = \eta^R(A)$, for all $A \in L^X$. Moreover,

$$R_{\eta^R}(z,x) = \eta^R(\top_z)(x) = \bigwedge_w (R(z,y) \Rightarrow R(x,y)).$$

(3) Since R is reflexive, $R^{-1}_{\eta^R} \leq R$ from

$$\begin{array}{ll} R_{\eta^R}(z,x) & = \eta_R(\top_z)(x) = \bigwedge_w (R(z,w) \Rightarrow R(x,w)) \\ & \leq R(z,z) \Rightarrow R(x,z) = R(x,z). \end{array}$$

Conversely,

$$\begin{array}{ll} R_{\eta^R}(x,x) & = \eta^R(\top_x)(x) = \bigwedge_w (R(x,w) \Rightarrow R(x,w)) = \top \\ & \leq R(x,x). \end{array}$$

(4) Since $R(x,y) \odot R(z,x) \leq R(z,y)$ iff $R(z,x) \leq R(x,y) \Rightarrow R(z,y)$, we have $R(z,x) \leq R^{-1}_{\eta^R}(z,x)$. Hence $R = R^{-1}_{\eta^R}$.

Conversely,

$$R_{\eta^R}(z,x) = \eta^R(\top_z)(x) = \bigwedge_w (R(z,w) \Rightarrow R(x,w)) \geq R(x,z).$$

Thus $R(z,w) \odot R(x,z) \leq R(x,w)$ for all $x, y, w \in X$;i.e. R is left $\odot$-transitive.

Example 3.13 Let $K = \{(x,y) \in R^2 \mid x > 0\}$ be a set and we define an operation $\otimes : K \times K \to K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1x_2, x_1y_2 + y_1).$$

Then $(K, \otimes)$ is a group with $e = (1,0)$, $(x,y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a,b) \in R^2 \mid a = 1, b \geq 0 \text{ , or } a > 1\}$ because $P \cap P^{-1} = \{(1,0)\}$, $P \odot P \subset P$, $(a,b)^{-1} \odot P \odot (a,b) = P$ and $P \cup P^{-1} = K$. For $(x_1, y_1), (x_2, y_2) \in K$, we define

$$\begin{aligned}(x_1, y_1) \leq (x_2, y_2) \ &\Leftrightarrow (x_1, y_1)^{-1} \odot (x_2, y_2) \in P, \ (x_2, y_2) \odot (x_1, y_1)^{-1} \in P \\ &\Leftrightarrow x_1 < x_2 \ \text{ or } \ x_1 = x_2, y_1 \leq y_2.\end{aligned}$$

Then $(K, \leq \otimes)$ is a lattice-group. (ref. [1])

For $L \subset K$, the structure $(L, \odot, \Rightarrow, \to, (\frac{1}{2}, 1), (1,0))$ is a generalized residuated lattice with strong negation where $\bot = (\frac{1}{2}, 1)$ is the least element and $\top = (1,0)$ is the greatest element from the following statements:

$$\begin{aligned}(x_1, y_1) \odot (x_2, y_2) \ &= (x_1, y_1) \otimes (x_2, y_2) \vee (\tfrac{1}{2}, 1) = (x_1x_2, x_1y_2 + y_1) \vee (\tfrac{1}{2}, 1), \\ (x_1, y_1) \Rightarrow (x_2, y_2) \ &= ((x_1, y_1)^{-1} \otimes (x_2, y_2)) \wedge (1,0) = (\tfrac{x_2}{x_1}, \tfrac{y_2 - y_1}{x_1}) \wedge (1,0), \\ (x_1, y_1) \to (x_2, y_2) \ &= ((x_2, y_2) \otimes (x_1, y_1)^{-1}) \wedge (1,0) = (\tfrac{x_2}{x_1}, -\tfrac{x_2y_1}{x_1} + y_2) \wedge (1,0).\end{aligned}$$

Furthermore, we have $(x,y) = (x,y)^{*\circ} = (x,y)^{\circ *}$ from:

$$(x,y)^* = (x,y) \Rightarrow (\frac{1}{2}, 1) = (\frac{1}{2x}, \frac{1-y}{x}),$$

$$(x,y)^{*\circ} = (\frac{1}{2x}, \frac{1-y}{x}) \to (\frac{1}{2}, 1) = (x,y).$$

Let $X = \{a, b, c\}$ be a set. Define $R \in L^{X \times X}$ as

$$R = \begin{pmatrix} (1,0) & (\frac{5}{8}, \frac{5}{2}) & (\frac{5}{6}, \frac{5}{3}) \\ (\frac{5}{7}, \frac{30}{7}) & (1,0) & (\frac{5}{8}, -\frac{5}{4}) \\ (1,-2) & (\frac{5}{7}, \frac{10}{3}) & (1,0) \end{pmatrix}$$

(1) Since $R \circ R = R$, $R^{-1} \circ R^{-1} = R^{-1}$ and $R(x,x) = R^{-1}(x,x) = \top$, by Lemma 3.3, R is a right $\odot$-preorder and R is a left $\odot$-preorder.

(2) From Theorem 3.6, since R is a right $\odot$-preorder, then $R_{C_R} = R$. Since R is a left $\odot$-preorder, then $R_{C^R} = R$.

(3) From Theorem 3.9, since R is a left $\odot$-preorder, then $R_{\phi_R} = R$ or $\phi_R = C^R$ where $R_{\phi_R}(x,y) = \bigwedge_w (R(y,w) \Rightarrow R(x,w))$.

(4) From Theorem 3.10, since R is a right $\odot$-preorder, then $R_{\phi^R} = R$ or $\phi^R = C_R$ where $R_{\phi^R}(x,y) = \bigwedge_w (R(y,w) \to R(x,w))$.

(5) From Theorem 3.11, since R is a right $\odot$-preorder, then $R^{-1}_{\eta_R} = R$ where $R^{-1}_{\eta_R}(x,y) = \bigwedge_w (R(y,w) \to R(x,w))$.

(6) From Theorem 3.12, since R is a left $\odot$-preorder, then $R^{-1}_{\eta^R} = R$ where $R^{-1}_{\eta^R}(x,y) = \bigwedge_w (R(y,w) \Rightarrow R(x,w))$.

References

[1] G. Birkhoff,*Lattice Theory,* Amer. Math. Soc. Colloq. Publ., **25**, New York, 1968.

[2] C.C. Chang, Algebraic analysis of many valued logics, *Trans. of A.M.S.*,**88(2)**(1958), 467-490.

[3] A. Dvurecenskij, On pseudo MV-algebras, *Soft Computing*,**5**(2001), 347-354.

[4] J. Elorza, P. Burillo, Connecting fuzzy preorders, fuzzy consequence operators and fuzzy closure and co-closure systems, *Fuzzy Sets and Systems* **139**(3)(2003), 601-613.

[5] J. Elorza, R.F. Gonzalez, J. Bragard, P. Burillo, On the relation between fuzzy closing morphological operators, fuzzy consequence operators induced by fuzzy preorders and fuzzy closure and co-closure systems, (Article in press) *Fuzzy Sets and Systems.*

[6] G. Georgescu, A. Popescu, Non-commutative fuzzy Galois connections,*Soft Computing*, **7** (2003), 458-467.

[7] G. Georgescu, A. Popescu, Non-commutative fuzzy structures and pairs of weak negations,*Fuzzy Sets and Systems*, **143**(2004), 129-155.

[8] J. Pavelka, On fuzzy logic, Zeitschr.f. Math. Logik und Grundlagen d. Math.Bd. 25(1979), 45-52.

[9] E. Turunen, *Mathematics Behind Fuzzy Logic*, A Springer-Verlag Co., 1999.

[10] R. Wille, *Restructuring lattice theory; an approach based on hierarchies of concept, in: 1. Rival(Ed.), Ordered Sets,* Reidel, Dordrecht, Boston, 1982, 445-470.

Received: September, 2012