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Timelike Curves on Timelike Parallel Surfaces in Minkowski 3-space E_1^3

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Abstract

In this paper,we study timelike curves on timelike parallel surfaces in Minkowski 3-spaces E_1^3.Using the definition of parallel surface we find images of timelike curve which lie on timelike surface in Minkowski 3-spaces.Subsequently we obtain relationships between the geodesic curvature,the normal curvature, the geodesic torsion of curve and its image curve.Besides,we give some characterization for its image curve.

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1 Introduction

In differential geometry of surfaces, a Darboux frame is a natural moving frame constructed on a surface ıt is the analog of the Frenet-serret frame as applied to surface geometry. A Darboux frame exists at any non-umbilic point of a surface embedded in Euclidean space. It is named after French mathematician Jean Gaston Darboux.In Euclidean space E^3, a solid perpendicular trihedrons Darboux instantaneous rotation vectors of a curve lying on the surface are well know.Let θ be the angle between principal normal N and surface normal Z on a point p of the curve. For the radii of geodesic torsion t_r, normal curvature k_n and geodesic curvature k_g, some relations are given in[9]. Curves lying on surface have an important role in theory of curves. Hereby, from the past to today, a lot of mathematicians have studied an curves lying on surface in different areas[12, 13, 19, 20]. Uğurlu,H.H, Kocayiğit,H[12] have studied Frenet and Darboux rotation vectors of curves on Time-like surfaces in Minkowski 3-space E_1^3.Also,Uğurlu,H.H and H.Topal[13] studied Relation between Darboux instantaneous rotain vectors of curves on a timelike surfaces.They have given the Darboux frame of the curves according to the Lorentzian characters of surfaces and the curves.Recently,S.Özkaldı and Y.Yaylı[20] studied Constant angle surface and curves in Euclidean 3-space.Moreover, analogue to the associated curves, similar relationships can be constructed between regular surfaces. For example, a surface and another surface which have constant distance with the reference surface along its surface normal have a relationship between their parametric representations. Such surfaces are called parallel surface[21]. By this definition, it is convenient to carry the points of a surface to the points of another surface. Since the curves are set of points, then the curves lying fully on a reference surface can be carry to another surface.By using this definition S.Kiziltuğ,Ö.Tarakcı and Y.Yaylı[18] investigated curves on Parallel surfaces in E^3.S.Kiziltuğ,M.Önder and Ö.Tarakcı[19] study Bertrand and Manheim curves on parallel surfaces in E^3 and obtained interesting conclusion.In this study, we consider the image timelike curves on timelike parallel surface in Minkowski 3-space. First we obtain the image curves of these curves on parallel surface. Then we investigate the relationships between reference curve and its image curve

2 Preliminaries

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3)\ \ x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3- dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The lorentz scalar product of x and y is defined by

$$\langle x, y\rangle_L = x_1y_1 + x_2y_2 - x_3y_3,$$

$E_1^3 = \left(R^3, \langle x, y\rangle_L\right)$ is called 3- dimensional Lorentzian space, Minkowski 3-space or 3- dimensional semi -euclidean space. The vector x in E_1^3 is a called a spacelike vector, null vector or a timelike vector if $\langle x, x\rangle_L > 0$ or $x = 0$, $\langle x, x\rangle_L = 0$ or $\langle x, x\rangle_L < 0$, respectively. For $x \in E_1^3$, the norm of the vecctor x defined by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$, and x is called a unit vector if $\|x\|_L = 1$. For any $x, y \in E_1^3$, Lorentzian vectoral product of x and y is defined by

$$x \times y = \det \begin{bmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2)$$

where

$$\delta_{ij} = \{ \begin{matrix} 1 & i = j \\ 0 & i \neq j \end{matrix} ,\ e_i = (\delta_{i1},\delta_{i2}, \delta_{i3}) \quad \text{and} \quad e_1{\times}e_2 = -e_3, e_2{\times}e_3 = e_1, e_3{\times}e_1 = -e_2.$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the unit speed curve $\alpha(s)$ in the Minkowski space E_1^3 , the following Frenet formulae are given,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\epsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where

$$\langle T, T\rangle_L = 1, \langle N, N\rangle_L = \epsilon = \pm 1, \langle B, B\rangle_L = -\epsilon, \langle T, N\rangle_L = \langle T, B\rangle_L = \langle N, B\rangle_L = 0$$

and k_1 and k_2 are curvature and torsion of the spacelike curve $\alpha(s)$ respectively. Here, ϵ determines the kind of spacelike curve. If $\epsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike first principal normal N and timelike binormal B.If $\epsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike first principal normal N and spacelike binormal B.Furthermore, for a timelike curve $\alpha(s)$ in the Minkowski space E_1^3, the following Frenet formulae are given in as follows,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where

$$\langle T,T\rangle_L = -1, \langle N,N\rangle_L = \epsilon = 1, \langle B,B\rangle_L = 1, \langle T,N\rangle_L = \langle T,B\rangle_L = \langle N,B\rangle_L = 0$$

and k_1 and k_2 are curvature and torsion of the timelike curve $\alpha(s)$ respectively**??**

Definition 1 *Let x and y be future pointing (or past pointing) timelike vectors in E_1^3.Then there is a unique real number $\theta \geqslant 0$ such that $\langle x,y\rangle_L = -|x|\,|y|\cosh\theta$.This number is called the hyperbolic angle between the vectors x and y .[7]*

Definition 2 *Let x and y be spacelike vector in E_1^3.Then span a timelike vector subspace.Then there is a unique real number $\theta \geqslant 0$ such that $\langle x,y\rangle_L = |x|\,|y|\cosh\theta$.This number is called the central angle between the vectors x and y.[7]*

Definition 3 *Let x and y be spacelike vector in E_1^3.Then span a spacelike vector subspace.Then there is a unique real number $\theta \geqslant 0$ such that $\langle x,y\rangle_L = |x|\,|y|\cos\theta$.This number is called the spacelike angle between the vectors x and y .[7]*

Definition 4 *Let x be a spacelike vector and y be a timelike vector in E_1^3.Then there is a unique real number $\theta \geqslant 0$ such that $\langle x,y\rangle_L = |x|\,|y|\sinh\theta$.This number is called the Lorentzian timelike angle between the vectors x and y.[7]*

Definition 5 *A surface in the Minkowski space E_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric , the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector.[7]*

3 Darboux Frame of a Curve Lying on a Surface in Minkowski 3-space E_1^3

Let M be a oriented surface Minkowski space E_1^3 and let consider a non-null curve $\alpha(s)$ lying on M fully. Since the curve $\alpha(s)$ is also in space, there exists

Frenet frame $\{T, N, B\}$ at each points of the curve where T is unit tangent vector, N is principal normal vector and B is binormal vector, respectively.Since the curve $\alpha(s)$ lying on the surface M there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{T, Y, Z\}$.In this frame T is the unit tangent of the curve, Z is the unit normal of the surface M and Y is a unit vector given by $Y = \pm Z \times T$. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors N, B, Y and Z lie on the same plane. Then, if surface M is an oriented timelike surface, the relations between these frames can be given as follows

ıf the curve $\alpha(s)$ is timelike

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

ıf the curve $\alpha(s)$ is spacelike

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

If the surface M is an oriented spacelike surface, then the curve $\alpha(s)$ lying on M is a spacelike curve. So, the relations between the frames can be given as follows

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

In all cases, θ is the angle between the vectors Y and N. According to the Lorentzian causal characters of the surface M and the curve $\alpha(s)$ lying on M, the derivative formulae of the Darboux frame can be changed as follows:

i) If the surface M is a timelike surface, then the curve $\alpha(s)$ lying on surface M can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

$$\begin{aligned} \begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} &= \begin{bmatrix} 0 & k_g & -\epsilon k_n \\ k_g & 0 & \epsilon t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix} \\ \langle T, T \rangle_L &= \epsilon = \pm 1, \langle Y, Y \rangle_L = -\epsilon, \langle Z, Z \rangle_L = 1 \\ \langle T, Y \rangle_L &= \langle T, Z \rangle_L = \langle Y, Z \rangle_L = 0 \end{aligned} \tag{1}$$

ii) If the surface M is a spacelike surface, then the curve $\alpha(s)$ lying on surface M is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}$$
$$\begin{aligned} \langle T, T \rangle_L &= \langle Y, Y \rangle_L = 1, \langle Z, Z \rangle_L = -1 \\ \langle T, Y \rangle_L &= \langle T, Z \rangle_L = \langle Y, Z \rangle_L = 0 \end{aligned} \tag{2}$$

In these formulae k_g, k_n and t_r are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively[12, 13]. The relations between geodesic curvature, normal curvature,geodesic torsion and κ, τ are given as follows

if both M and $\alpha(s)$ are timelike or spacelike

$$k_g = \kappa \cos\theta, k_n = \kappa \sin\theta, t_r = \tau + \theta', \tag{3}$$

if M timelike and $\alpha(s)$ is spacelike

$$k_g = \kappa \cosh\theta, k_n = \kappa \sinh\theta, t_r = \tau + \theta'. \tag{4}$$

Where $\theta = (\widehat{N, B})$ the angle function is between the unit normal and binormal to $\alpha(s)$.Furthermore, the geodesic curvature k_g, normal curvature k_n and geodesic torsion t_r of the curve $\alpha(s)$ can be calculated as follows

$$\begin{aligned} k_g(s) &= \left\langle \alpha''(s), Y(s) \right\rangle, \\ k_n(s) &= \left\langle \alpha''(s), Z_{\alpha(s)} \right\rangle, \\ t_r(s) &= -\left\langle Z', Y(s) \right\rangle. \end{aligned}$$

In the diffrential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M the following are well-known

i) $\alpha(s)$ is a geodesic curve iff $k_g = 0$,

ii) $\alpha(s)$ is an asymptotic curve iff $k_n = 0$,

iii) $\alpha(s)$ is a principal line iff $t_r = 0$[12]

Definition 6 *Let M be a surface in Minkowski space E_1^3 with unit normal Z. For any constant r in $\mathbb{R}$, let $M_r = (f(p) = p + rZ_p : p \in M)$.Thus if p is on M, then $f(p) = p + rZ_p$ defines a new surface M_r. The map f is called the natural map on M into M_r, and if f is univalent, then M_r is a parallel surface of M with unit normal Z, $Z_{f(p)} = Z_p$ for all p on M.[8]*

4 Timelike Curves on Timelike Parallel Surfaces in Minkowski 3-space E_1^3

In this section, we deal with the notions of curve by considering parallel surfaces. we obtain the representation of point on M_r using the representation of point on M.Let M and M_r be oriented timelike surfaces in Minkowski space E_1^3 and let consider the arc-length parameter the timelike curves $\alpha(s)$ and $\beta(s_\beta)$ lying fully on M and M_r,respectively.Then, by the definition of parallel surface, the parametrization for curve $\beta(s_\beta)$ is given by

$$\beta(s_\beta) = \alpha(s) + rZ. \tag{5}$$

Denote the Darboux frames of curve $\beta(s_\beta)$ by $[T^*, Y^*, Z^*]$. By taking derivative of (5) with respect to s and by using Darboux formulas (1) we have

$$\beta' = T^* \frac{ds_\beta}{ds} = (1 + rk_n) T + rt_r Y. \tag{6}$$

If taking norms of both sides of (6) we have

$$\frac{ds_\beta}{ds} = \sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}. \tag{7}$$

From (6) and (7) we have

$$T^* = \frac{1}{\sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}} \left((1 + rk_n) T + rt_r Y\right). \tag{8}$$

Since $Y^* = Z \times T^*$, from (8) it is obtained that

$$Y^* = \frac{1}{\sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}} \left(rt_r T + (1 + rk_n) Y\right). \tag{9}$$

This new trihedron $[T^*, Y^*, Z^*]$ be an orthonormal base of tangent space $T_{\beta(s_\beta)}E_1^3$ because of vectoral product.

Then we have the following theorem.

Theorem 7 *Let the pair (M, M_r) be a parallel surface pair and the curve $\beta(s_\beta)$ lying on M_r be the image of the curve $\alpha(s)$ lying on timelike surface M.Then, the relationships between the Darboux frames of $\alpha(s)$ and $\beta(s_\beta)$ are given as follows*

$$\begin{bmatrix} T^* \\ Y^* \\ Z^* \end{bmatrix} = \begin{bmatrix} \frac{(1+rk_n)}{\sqrt{\left|(rt_r)^2-(1+rk_n)^2\right|}} & \frac{rt_r}{\sqrt{\left|(rt_r)^2-(1+rk_n)^2\right|}} & 0 \\ \frac{rt_r}{\sqrt{\left|(rt_r)^2-(1+rk_n)^2\right|}} & \frac{(1+rk_n)}{\sqrt{\left|(rt_r)^2-(1+rk_n)^2\right|}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}$$

Theorem 8 *Let the pair (M, M_r) be a parallel surface pair and the curve $\beta(s_\beta)$ lying on M_r be the image of the curve $\alpha(s)$ lying on timelike surface M. Then, the relation between geodesic curvature k_g, normal curvature k_n, geodesic torsion*

t_r *of curve* $\alpha(s)$ *and the geodesic curvature* k_g^**, the normal curvature* k_n^**, the geodesic torsion* t_r^* *of curve* $\beta(s_\beta)$ *are given as follows*

$$i)\quad k_g^* = \frac{-rt_r\left(rk_n^{\prime} + rt_r k_g\right) + (1 + rk_n)\left(k_g(1 + rk_n) + rt_r^{\prime}\right)}{\sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}}$$

$$ii)\quad k_n^* = k_n(1 + rk_n) - rt_r^2$$

$$iii)\quad t_r^* = \frac{-t_r}{\sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}}$$

Proof. The geodesic curvature k_g^*, the normal curvature k_n^*, and the geodesic torsion t_r^* of the curve $\beta(s_\beta)$ lying on parallel surface M_r can be defined as follows

$$k_g^* = \left\langle \beta^{\prime\prime}(s_\beta), Y^* \right\rangle \tag{10}$$

$$k_n^* = \left\langle \beta^{\prime\prime}(s_\beta), Z^* \right\rangle \tag{11}$$

$$t_r^* = -\left\langle (Z^*)^{\prime}, Y^* \right\rangle \tag{12}$$

By taking derivative of (5) with respect to s twice,we get

$$\beta^{\prime\prime} = \left(rk_n^{\prime} + rt_r k_g\right)T + \left(k_g(1 + rk_n) + rt_r^{\prime}\right)Y + \left(k_n(1 + rk_n) - rt_r^2\right)Z. \tag{13}$$

Taking the inner product of (13) with (9) and by using features of inner product, the geodesic curvature of curve $\beta(s_\beta)$ as follows

$$k_g^* = \frac{-rt_r\left(rk_n^{\prime} + rt_r k_g\right) + (1 + rk_n)\left(k_g(1 + rk_n) + rt_r^{\prime}\right)}{\sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}}.$$

T

Taking the inner product of (13) with Z and and by using features of inner product, the normal curvature of curve $\beta(s_\beta)$ as follows

$$k_n^* = k_n(1 + rk_n) - rt_r^2.$$

And the geodesic torsion t_r^* of curve $\beta(s_\beta)$ is

$$t_r^* = -\left\langle (Z^*)^{\prime}, Y^* \right\rangle = -\left\langle k_n T + t_r Y\ ,\ \frac{(rt_r T + (1 + rk_n)Y)}{\sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}} \right\rangle = \frac{-t_r}{\sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}}. \tag{14}$$

This completes proof. ■

Theorem 9 *Let the timelike curve $\alpha(s)$ lying on timelike surface M in E_1^3 is a asymptotic curve. Then, the geodesic curvature k_g^*, the normal curvature k_n^* and the geodesic torsion t_r^* of the curve $\beta(s_\beta)$ are given as follows*

$$i) k_g^* = \frac{k_g\left(r^2 t_r^2 - 1\right) + r t_r^{\prime}}{\sqrt{\left|-1 + r^2 t_r^2\right|}}$$

$$ii) k_n^* = -r t_r^2$$

$$iii) t_r^* = \frac{t_r}{\sqrt{\left|-1 + r^2 t_r^2\right|}}$$

Proof. Since the timelike curve $\alpha(s)$ lying on timelike surface M is asymptotic curve, $k_n = 0$. If we replace $k_n = 0$ in Theorem (8), we obtain k_g^*, k_n^* and t_r^* as follows

$$k_g^* = \frac{k_g\left(r^2 t_r^2 - 1\right) + r t_r^{\prime}}{\sqrt{\left|-1 + r^2 t_r^2\right|}}, \quad k_n^* = -r t_r^2, \quad t_r^* = \frac{t_r}{\sqrt{\left|-1 + (r t_r)^2\right|}}.$$

■

Theorem 10 *Let the timelike curve $\alpha(s)$ lying on timelike surface M in E_1^3 is a principal line. Then, the geodesic curvature k_g^*, the normal curvature k_n^* and the geodesic torsion t_r^* of the curve $\beta(s_\beta)$ are given as follows*

$$i) k_g^* = \pm(1 + r k_n) k_g$$

$$ii) k_n^* = k_n (1 + r k_n)$$

$$iii) t_r^* = 0$$

Proof. Since the curve $\alpha(s)$ lying on M is a principal line, $t_r = 0$. If we replace $t_r = 0$ in Theorem (8), we obtain k_g^*, k_n^* and t_r^* as follows

$$k_g^* = \pm(1 + r k_n) k_g, \; k_n^* = k_n (1 + r k_n), \quad t_r^* = 0.$$

■

Thus, from Theorem(10) we have the following corollaries.

Corollary 11 *Let the curve $\alpha(s)$ lying on surface M in E_1^3 is a principal line. Then, the curve $\beta(s_\beta)$ lying on parallel surface M_r is a asymptotic curve if and only if the curve $\alpha(s)$ lying on surface M is asymptotic curve.*

Corollary 12 *Let the curve $\alpha(s)$ lying on surface M is a principal line.Then, the curve $\beta(s_\beta)$ lying on parallel surface M_r is a geodesic curve if and only if the curve $\alpha(s)$ lying on surface M is geodesic curve.*

Theorem 13 *Let the asymptotic curvature of curve $\alpha(s)$ is $k_n = -\frac{1}{r}$. Then, the geodesic curvature k_g^* ,the normal curvature k_n^* and the geodesic torsion t_r^* of the curve $\beta(s_\beta)$ are given as follows*

$$\begin{aligned} i)k_g^* &= \pm r t_r k_g \\ ii)k_n^* &= -r t_r^2 \\ iii)t_r^* &= \pm\frac{1}{r} \end{aligned}$$

Proof. If we take as $k_n = -\frac{1}{r}$ and replace $k_n = -\frac{1}{r}$ in Theorem (8), we obtain k_g^*, k_n^* and t_r^* as follows

$$k_g^* = \pm r t_r k_g, \quad k_n^* = -r t_r^2, \quad t_r^* = \pm\frac{1}{r}.$$

■

Thus, from Theorem (13) we can give the following corollaries

Corollary 14 *Let the asymptotic curvature of curve $\alpha(s)$ is $k_n = -\frac{1}{r}$.Then, the curve $\beta(s_\beta)$ lying on parallel surface M_r is a geodesic curve if and only if the curve $\alpha(s)$ lying on surface M is geodesic curve.*

Corollary 15 *Let the asymptotic curvature of curve $\alpha(s)$ is $k_n = -\frac{1}{r}$.Then, the curve $\beta(s_\beta)$ lying on parallel surface M_r is a asymptotic curve if and only if the curve $\alpha(s)$ lying on surface M is asymptotic curve.*

Theorem 16 *Let M be a constant angle timelike surface in E_1^3 and M_r be parallel surface of M. If the timelike curve $\alpha(s)$ lying on timelike surface M is both geodesic curve and principal line.Then,the curve $\alpha(s)$ lying on surface M is a slant helix if and only if the curve $\beta(s_\beta)$ lying on parallel surface M_r is a slant helix.*

Proof. Since $\alpha(s)$ is a geodesic curve lying on M, the normal of the surface coincides with the principal normal of the curve.In the case of constant angle surfaces it follows that the principal normal of the cure makes a constant angle with the fixed direction, namely

$$(\widehat{N,k}) = (\widehat{Z,k}) = \theta \tag{15}$$

where N is a principal normal of curve $\alpha(s)$.

It follows that $\alpha(s)$ is a slant helix(See[10, 20]).

From corollary (12) $\beta(s_\beta)$ is a geodesic curve and since M and M_r is a parallel surface,we have $Z = Z^*$. So, parallel surface M_r is constant angle surface.Thus, ın the case of constant angle surfaces it follows that the principal normal of the cure makes a constant angle with the fixed direction, namely

$$(\widehat{N^*,k}) = (\widehat{Z^*,k}) = (\widehat{Z,k}) = \theta$$

where N^* is a principal normal of curve $\beta(s_\beta)$. Thus, $\beta(s_\beta)$ is a slant helix. ■

Theorem 17 *Let M be a constant angle timelike surface in E_1^3 and M_r be parallel surface of M. If the timelike curve $\alpha(s)$ lying on timelike surface M is both asymptotic curve and principal line. Then, the curve $\alpha(s)$ lying on surface M is a general helix if and only if the curve $\beta(s_\beta)$ lying on parallel surface M_r is a general helix.*

Proof. Since $\alpha(s)$ is a asymptotic curve on M, we have

$$k_n = 0 \tag{16}$$

From Eq (3) one obtains $\theta = 0$ where θ, the angle function is between the unit normal and binormal to $\alpha(s)$. This actually means that the normal of the surface coincides with the binormal of the curve. In the case of constant angle surfaces it follows that the principal normal of the curve makes a constant angle with the fixed direction k, namely

$$(\widehat{B,k}) = (\widehat{Z,k}) = \theta \tag{17}$$

ıt follows that $\alpha(s)$ is a general helix(See[10, 20]).

From corollary (11) $\beta(s_\beta)$ is a asymptotic curve.Besides, M and M_r is a parallel surface,we have

$Z = Z^*$

If we replace this in (18) we get

$$(\widehat{B^*,k}) = (\widehat{Z^*,k}) = (\widehat{Z,k}) = \theta$$

where B^* is a binormal of curve $\beta(s_\beta)$. Thus, $\beta(s_\beta)$ is a general helix. ■

Theorem 18 *Let M be a timelike surface in E_1^3 and M_r be a parallel surface of M. If the timelike curve $\alpha(s)$ lying on timelike surface M is a principal line. Then, $\alpha(s)$ is a general helix if and only if the curve $\beta(s_\beta)$ lying on parallel surface M_r is a general helix.*

Proof. Since $\alpha(s)$ is a general helix, it follows that

$$\langle T,k\rangle = constant. \tag{18}$$

Let $\alpha(s)$ is a principal line on M. Then, we have $t_r = 0$ and from (8) we have $T^* = \pm T$.

If we replace this in (19) we get

$$\langle T^*,k\rangle = constant. \tag{19}$$

where T^* is tangent vector of curve $\beta(s_\beta)$ and k is the fixed direction. Thus, $\beta(s_\beta)$ is a general helix. ■

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