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A Conformal Approach to Bour's Theorem

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Abstract

In this paper, we give relation between Bour's theorem and conformal map in Euclidean $3-$space. We prove that a spiral surface and a helicoidal surface have a conformal relation. So, a helix on the helicoid correspond to a spiral on the spiral surface. Moreover we obtain that a spiral surface and a rotation surface have a conformal relation. So, spirals on the spiral surface correspond to parallel circles on the rotation surface. When the conformal map is an isometry we obtain the Bour's theorem ,i.e, we obtain an isometric relation between the helisoidal surface and the rotation surface, which was given by Bour in [1]. Thus this paper is a generalization of Bour's theorem.

Mathematics Subject Classification: 53A05, 53C10S

Keywords: Bour's theorem, spiral surface, rotation surface, helicoid

1 Introduction

Surface theory in Euclidean 3−space has been studied for a long time. In classical differential geometry, rotation surfaces with constant curvature and the right helicoid (resp catenoid) which is the only ruled minimal surface have been known. Also, a pair of these two surfaces has interesting properties namely, they are both members of a one parameter family of isometric minimal surfaces, and if they have the same Gauss map then they are minimal surfaces.

Moreover, Bour [1] showed that a helicoidal surface and a rotation surface are isometric in Euclidean 3−space. In this theory, orginal properties such as, minimality and preservation of the Gauss map are generaly unmaintained. In [11], Ikawa showed that a helicoidal surface and a rotation surface are isometric according to Bour's theorem in Euclidean 3−space. He determined with an additional condition that the pairs of surfaces have the same Gauss map by Bour's theorem. Also, in [10], Ikawa gave a classification of the surfaces by types of axis and the profile curves named as (axis's type, profile curve's type)-type. For example, the (S, L)-type surface means that the surface has a spacelike axis and a lightlike profile curve. After these studies, Güler considered the null (lightlike) profile curves of helicoidal and rotation surfaces in Bour's theorem and he showed that Bour's theorem is true in 3-dimensional Minkowski space.

Carmo and Dajczer [3] found that there exist a two-parameter family of helicoidal surfaces which is isometric to a given helicoidal surface by using a result of Bour's theorem. Furthermore, they characterized helicoidal surfaces which have constant mean curvature with the help of this parametrization.

Spiral curves and surfaces are most fascinating objects. Because they have important properties such as the size increases without altering the shape. Their properties are seen on different objects around us in differential geometry, in science and nature. Let us, we mention some phenomena seen curves which are similar to the spirals. For examples, the approach of a hawk to its prey, the approach of an insect to a light source (see for details [2]), the arms of a spiral galaxy, the arms of the tropical cyclones, the nerves of the cornea and several biological structures, e.g. Romanesco broccoli, Convallaria majalis, some spiral roses, sunflower heads, Nautilus shells and so on because of the fact that these curves are named also growth spirals.

In this paper, we give relation between Bour's theorem and conformal map in Euclidean 3−space. We prove that a spiral surface and a helicoidal surface have a conformal relation. So, a helix on the helicoid correspond to a spiral on the spiral surface. Moreover we obtain that a spiral surface and a rotation surface have a conformal relation. So, spirals on the spiral surface correspond to parallel circles on the rotation surface. When the conformal map is an isometry we obtain the Bour's theorem ,i.e, we obtain an isometric relation between the helicoidal surface and the rotation surface, which was given by Bour in [1]. Thus this paper is a generalization of Bour's theorem.

2 Preliminaries

First of all, we recall elementary properties in $\mathbb{E}^3$. The inner product on the Euclidean space $\mathbb{E}^3$ is given as

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3$$

where $x = (x_1, x_2, x_3)\,, y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

The norm of the vector $x \in \mathbb{E}^3$ is defined by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

and the Euclidean vector product is given by

$$x \times y = (x_2y_3 - y_2x_{3,}x_3y_1 - y_3x_{1,}x_1y_2 - y_1x_2)\,.$$

For a surface $X(u, v)$, the coefficients of the first fundamental form E, F and G in the base (X_u, X_v) are defined as

$$\begin{aligned} E &= \langle X_u, X_u\rangle \\ F &= \langle X_u, X_v\rangle \\ G &= \langle X_v, X_v\rangle\,. \end{aligned}$$

The coefficients of the second fundamental form L, M and N of $X(u, v)$ are given as

$$\begin{aligned} L &= \langle X_{uu}, e\rangle \\ M &= \langle X_{uv}, e\rangle \\ N &= \langle X_{vv}, e\rangle\,. \end{aligned}$$

The Gauss map of the surface $X(u, v)$ is given as

$$e = \frac{X_u \times X_v}{\|X_u \times X_v\|}.$$

where $\{X_u, X_v\}$ is the natural bases.

The line element of the surface $X(u, v)$ is defined as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

Let $\gamma : I \subset \mathbb{R} \to \pi$ be a curve in a plane π in $\mathbb{R}^3$ and l be a straight line in π which dosen't intersect the curve γ. A rotation surface in $\mathbb{E}^3$ is obtained by rotating a curve γ around line l (they are called the profile curve and axis, respectively).

Suppose that a profile curve γ rotates around the axis l and it simultaneously moves parallel to l so that the speed of displacement is propertional to the speed of rotation. Then the surface is called the *generalized helicoid* or *helicoidal surface* with axis l and pitch a.

Let l be the rotation axis. Thus, there is an Euclidean transformation which transformed the axis l to the axis-z in $\mathbb{R}^3$. Parametric equation of the profile curve γ is given by

$$\gamma(u) = (\psi(u), 0, \phi(u))$$

where $\psi(u) : I \subset \mathbb{R} \to \mathbb{R}$ differentiable functions for all $u \in I$.

In this paper, we consider the parametric equation of the profile curve γ for $\psi(u) = u$, that is,

$$\gamma(u) = (u, 0, \phi(u)).$$

Therefore a helicoidal surface with the rotation axis z and the profile curve γ then the parametric equation of the helicoidal surface is given by

$$H(u, v) = (u \cos v, u \sin v, \phi(u) + av) \tag{1}$$

in $\mathbb{E}^3$. Where $u \in I$, $0 \leq v \leq 2\pi$, $a \in \mathbb{R}/\{0\}$.

In the Eq. (1), if we consider that a is zero, then we obtain the parametric equation of the rotation surface [11].

Also a spiral surface with the rotation axis z and the profile curve γ then the parametric equation of the helicoidal surface is given by

$$S(u, v) = (f(v)\, u \cos v, f(v)\, u \sin v, f(v)\, \phi(u))$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ differentiable functions for all $v \in I$.

Definition 1 *A mapping of surfaces $f : X \to Y$ is conformal provided there exist a real-valued function $\lambda > 0$ on X such that*

$$\|f_* v_p\| = \lambda(p) \|v_p\|,$$

for all tangent vectors to X. The function λ is called the scale factor of f.
i) *If λ is constant function then we say f is homothetic,*
ii) *If $\lambda = 1$ then we say f is isometry [9].*

Definition 2 *Let $f : X \to Y$ be a mapping. Then f is a conformal map if and only if*

$$E = \lambda^2 \bar{E}, F = \lambda^2 \bar{F}, G = \lambda^2 \bar{G}$$

where E, F, G and $\bar{E}$, $\bar{F}$, $\bar{G}$ are coefficients of the first fundamental forms of the X and Y, respectively [12].

3 A Conformal Approach to Bour's Theorem

Theorem 1 *(Bour's theorem) A generalized helicoid is isometric to a rotation surface so that helices on the helicoid correspond to parallel circles on the rotation surface [1].*

Theorem 2 *A generalized helicoid and a spiral surface have a conformal relation so that helices on the helicoid correspond to spiral on the spiral surface.*

Proof. Let $\gamma(u) = (u, o, \phi(u))$, $l = (0,0,1)$ be the profile curve and the rotation axis, respectively. Then a helicoidal surface H can be parametrized by

$$H(u_H, v_H) = (u_H \cos v_H, u_H \sin v_H, \phi_H(u_H) + a v_H),\ a = \text{constant}.$$

If $\phi(u)$ is a constant function, then the generalized helicoid is called *the right helicoid*. Then,

$$\begin{aligned} H_{u_H}(u_H, v_H) &= (\cos v_H, \sin v_H, \phi'_H(u_H)) \\ H_{v_H}(u_H, v_H) &= (-u_H \sin v_H, u_H \cos v_H, a) \end{aligned}$$

and coefficients of first fundamental form of the helicoid is given as

$$\begin{aligned} E_H &= \langle H_{u_H}, H_{u_H} \rangle = 1 + \phi'^2_H \\ F_H &= \langle H_{u_H}, H_{v_H} \rangle = a\phi'_H \\ G_H &= \langle H_{v_H}, H_{v_H} \rangle = u_H^2 + a^2. \end{aligned}$$

Then, the line element of generalized helicoid is

$$ds_H^2 = \left(1 + \phi'^2_H\right) du_H^2 + 2a\phi'_H du_H dv_H + \left(u_H^2 + a^2\right) dv_H^2. \tag{2}$$

Now we investigate the line element of the spiral surface. The spiral surface with the parametric equation is given as

$$S(u_S, v_S) = (e^{g'(v_S)} u_S \cos v_S, e^{g'(v_S)} u_S \sin v_S, e^{g'(v_S)} \phi_S(u_S))$$

where $g''(v_S) = b = \textit{constant}$. Then we have

$$\begin{aligned} S_{u_S}(u_S, v_S) &= (e^{g'(v_S)} \cos v_S, e^{g'(v_S)} \sin v_S, e^{g'(v_S)} \phi'_S(u_S)) \\ S_{v_S}(u_S, v_S) &= (b e^{g'(v_S)} u_S \cos v_S - e^{g'(v_S)} u_s \sin v_S, b e^{g'(v_S)} u_S \cos v_S \\ &\quad + e^{g'(v_S)} u_S \sin v_S, b e^{g'(v_S)} \phi_S(u_S)) \end{aligned}$$

and the coefficients of first fundamental form of the spiral surface is given as

$$\begin{aligned} E_S &= \langle S_{u_S}, S_{u_S} \rangle = \left(e^{g'(v_S)}\right)^2 \left(1 + \phi'^2_S\right) \\ F_S &= \langle S_{u_S}, S_{v_S} \rangle = \left(e^{g'(v_S)}\right)^2 (b u_S + b\phi_S \phi'_S) \\ G_S &= \langle S_{v_S}, S_{v_S} \rangle = \left(e^{g'(v_S)}\right)^2 (b^2(u_S^2 + \phi_S^2) + u_S^2). \end{aligned}$$

Then, the line element of spiral surface is given by

$$ds_S^2 = \left(e^{g'(v_S)}\right)^2 \left[\left(1+\phi_S'^2\right) du_S^2 + 2(bu_S + b\phi_S\phi_S')du_S dv_S + (b^2(u_S^2+\phi_S^2)+u_S^2)dv_S^2\right] \tag{3}$$

Comparing the line element of helicoidal surface in Eq. (2) and spiral surface in Eq. (3) then we obtain the function ϕ_S of the spiral surface as

$$\phi_S(u) = \frac{ab\left(b^2+1\right)\phi_H'^2 \pm \sqrt{A}}{b^4 + b^2\left(b^2+1\right)\phi_H'^2}$$

where

$$A = a^2b^2\left(b^2+1\right)^2\phi_H'^4 + \left(b^4 + b^2\left(b^2+1\right)\phi_H'^2\right)\left(b^2u^2 + b^2a^2 - a^2\left(b^2+1\right)\phi_H'^2\right).$$

So, we have the following equation:

$$E_H = \left(e^{g'(v_S)}\right)^2 E_S,\ F_H = \left(e^{g'(v_S)}\right)^2 F_S,\ G_H = \left(e^{g'(v_S)}\right)^2 G_S.$$

Since the function $e^{g'(v_S)}$ is positive we have a conformal relation between $H(u_H, v_H)$ and $S(u_S, v_S)$. Therefore a generalized helicoid

$$H(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ \phi(u) + av \end{bmatrix}$$

and the spiral surface

$$S(u,v) = \begin{bmatrix} e^{g'(v)}\sqrt{\frac{a^2+u^2-b^2\phi_s^2}{b^2+1}}\cos(v) \\ e^{g'(v)}\sqrt{\frac{a^2+u^2-b^2\phi_s^2}{b^2+1}}\sin(v) \\ e^{g'(v)}\phi_s \end{bmatrix}.\ b \neq 0.$$

have a conformal relation.
If we consider that $e^{g'(v_S)} = constant$, that is, $b = 0$ then the spiral surface reduce to rotational surface then, we obtain

$$ds_S^2 = \left(e^{g'(v_S)}\right)^2 \left[\left(1+\phi_S'^2\right) du_S^2 + \left(u_S^2\right) d\bar{v}_S^2\right].$$

So, if we consider that

$$\bar{u}_S = \int \sqrt{1+\phi_S'^2}du_S,\ f_S(\bar{u}_S) = u_S,\ \bar{v}_S = v_S,$$

then the line element of the rotation surface is rewritten as

$$ds_S^2 = \left(e^{g'(v_S)}\right)^2 \left[d\bar{u}_S^2 + f_S^2(\bar{u}_S)d\bar{v}_S^2\right]. \tag{4}$$

On the other hand, a helix on the generalized helicoid has a constant u_H. Given curve is orthogonal to a helix, we have the following characterization for the helix and its orthogonal curve

$$a\phi'_H du_H + \left(u_H^2 + a^2\right) dv_H = 0.$$

If we solve the last equation, then we have

$$v_H = -\int \frac{a\phi'_H}{u_H^2 + a^2} du + c$$

where c is constant. Hence if we consider $\bar{v}_H$ is equal to following equation we obtain a curve which is orthogonal to helix.

$$\bar{v}_H = v_H + \int \frac{a\phi'_H}{u_H^2 + a^2} du.$$

If we differentiate the last equation we obtain

$$d\bar{v}_H = dv_H + \frac{a\phi'_H}{u_H^2 + a^2} du.$$

Substituting this equation into the line element in Eq. (2), we have

$$ds_H^2 = \left(1 + \frac{u_H^2 \phi_H'^2}{u_H^2 + a^2}\right) du_H^2 + \left(u_H^2 + a^2\right) d\bar{v}_H^2.$$

If we consider that

$$\bar{u}_H = \int \sqrt{1 + \frac{u_H^2 \phi_H'^2}{u_H^2 + a^2}} \text{ and } f_H(\bar{u}) = \sqrt{u_H^2 + a^2}$$

it follows that

$$ds_H^2 = d\bar{u}_H^2 + f(\bar{u})_H^2 d\bar{v}_H^2. \tag{5}$$

which is the line element of the helisoidal surface.

Comparing the line element of helicoidal surface in Eq (4) and the line element of the rotation surface in Eq. (5) and using the foloowing equation

$$\bar{u}_S = \bar{u}_S \ , \ \bar{v} = v_S \text{ and } f(\bar{u}_H) = f_S(\bar{u}_S)$$

we obtain the function ϕ_S of the rotation surface as

$$\phi_S(u) = \int \sqrt{\frac{a^2 + u_H^2 \phi_H'^2}{u_H^2 + a^2}} du.$$

So, we obtain a homothetic relation between the helicoidal surface

$$H(u,v)=\begin{bmatrix} u\cos v \\ u\sin v \\ \phi(u)+av \end{bmatrix}$$

and the rotation surface

$$S(u,v)=\begin{bmatrix} e^{g'(v)}\sqrt{a^2+u^2}\cos(v+\int \frac{a\phi'}{a^2+u^2}du) \\ e^{g'(v)}\sqrt{a^2+u^2}\sin(v+\int \frac{a\phi'}{a^2+u^2}du) \\ e^{g'(v)}\int\sqrt{\frac{a^2+u^2\phi'^2}{u^2+a^2}} \end{bmatrix}.$$

If we consider that $e^{g'(v)}=1$ then it is clearly seen that we obtain a generalized helicoid

$$H(u,v)=\begin{bmatrix} u\cos v \\ u\sin v \\ \phi(u)+av \end{bmatrix}$$

and the rotation surface

$$S(u,v)=\begin{bmatrix} \sqrt{a^2+u^2}\cos(v+\int \frac{a\phi'}{a^2+u^2}du) \\ \sqrt{a^2+u^2}\sin(v+\int \frac{a\phi'}{a^2+u^2}du) \\ \int\sqrt{\frac{a^2+u^2\phi'^2}{u^2+a^2}} \end{bmatrix}$$

have an isometric relation,that is, we obtain the Bour's theorem. ■

Example 1 *If we consider $\phi(u)=u$, $a=1$ and $b=\frac{1}{3}$ then we obtain following figures which have a conformal relation*

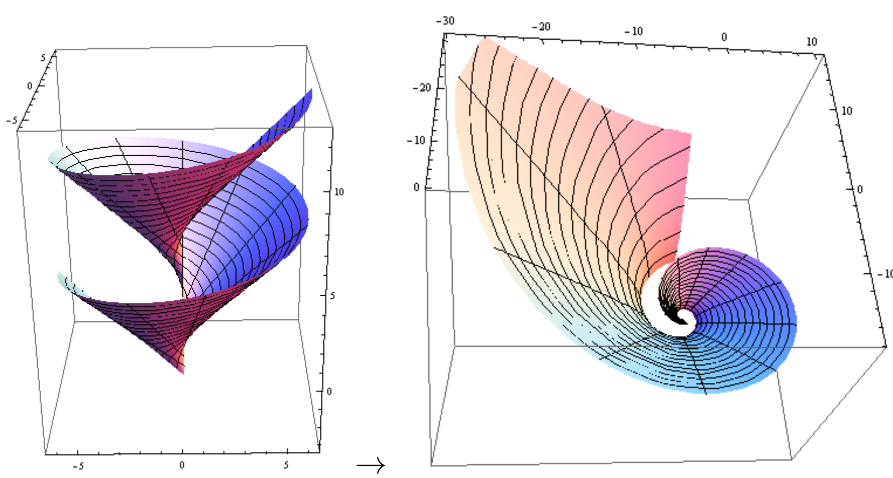

Figure1. A helicoidal surface and its images spiral surface.

Corollary 3 *A spiral surface and the rotation surface have a conformal relation. So, a spiral on the spiral surface is correspond to a circle on the rotation surface.*

Proof. Let

$$S(u_S, v_S) = (e^{g'(v_S)} u_S \cos v_S, e^{g'(v_S)} u_S \sin v_S, e^{g'(v_S)} \phi_S(u_S))$$

be a spiral surface with parametric equation for $g''(v_S) = b = constant$. Then, differentiating the last equation, we get

$$\begin{aligned} S_{u_S}(u_S, v_S) &= (e^{g'(v_S)} \cos v_S, e^{g'(v_S)} \sin v_S, e^{g'(v_S)} \phi'_S(u_S)) \\ S_{v_S}(u_S, v_S) &= (be^{g'(v_S)} u_S \cos v_S - e^{g'(v_S)} u_s \sin v_S, be^{g'(v_S)} u_S \cos v_S \\ &\quad + e^{g'(v_S)} u_S \sin v_S, be^{g'(v_S)} \phi_S(u_S)) \end{aligned}$$

and the coefficients of first fundamental form of the spiral surface is given as

$$\begin{aligned} E_S &= \langle S_{u_S}, S_{u_S} \rangle = \left(e^{g'(v_S)}\right)^2 \left(1 + \phi'^2_S\right) \\ F_S &= \langle S_{u_S}, S_{v_S} \rangle = \left(e^{g'(v_S)}\right)^2 (bu_S + b\phi_S \phi'_S) \\ G_S &= \langle S_{v_S}, S_{v_S} \rangle = \left(e^{g'(v_S)}\right)^2 (b^2(u_S^2 + \phi_S^2) + u_S^2). \end{aligned}$$

Then, the line element of spiral surface is given by

$$ds_S^2 = \left(e^{g'(v_S)}\right)^2 \left[\left(1 + \phi'^2_S\right) du_S^2 + 2(bu_S + b\phi_S \phi'_S) du_S dv_S + (b^2(u_S^2 + \phi_S^2) + u_S^2) dv_S^2\right] \tag{6}$$

a spiral on the spiral surface has a constant u_H. Given curve is orthogonal to a spiral, we have the following characterization for the spiral and its orthogonal curve

$$(bu_S + b\phi_S \phi'_S) du_S + (b^2(u_S^2 + \phi_S^2) + u_S^2) dv_S = 0.$$

from this equation, we obtain

$$v_S = -\int \frac{bu_S + b\phi_S \phi'_S}{(b^2(u_S^2 + \phi_S^2) + u_S^2)} du_s + c$$

where c is constant. Hence if we consider $\bar{v}_S$ is equal to following equation we obtain a curve which is orthogonal to helix.

$$\bar{v}_S = v_S + \int \frac{bu_S + b\phi_S \phi'_S}{(b^2(u_S^2 + \phi_S^2) + u_S^2)} du_s.$$

If we differentiate the last equation we obtain

$$d\bar{v}_S = dv_S + \frac{bu_S + b\phi_S \phi'_S}{(b^2(u_S^2 + \phi_S^2) + u_S^2)} du_S.$$

Substituting the last equation into the line element of the spiral surface in Eq. (6), we have

$$ds_S^2 = \left(e^{g'(v_S)}\right)^2 \left[\left(1+\phi_S'^2 - \frac{(bu_S + b\phi_S\phi_S')^2}{(b^2(u_S^2+\phi_S^2)+u_S^2)}\right)du_S^2 + (b^2(u_S^2+\phi_S^2)+u_S^2)d\bar{v}_S^2\right].$$

Hence if we consider that

$$\bar{u}_S = \int \sqrt{1+\phi_S'^2 - \frac{(bu_s + b\phi_S\phi_S')^2}{(b^2(u_S^2+\phi_S^2)+u_S^2)}}du_S,\ f_S(\bar{u}_S) = \sqrt{(b^2(u_S^2+\phi_S^2)+u_S^2)},$$

then it follows that

$$ds_S^2 = \left(e^{g'(v)}\right)^2 d\bar{u}^2 + \left(e^{g'(v)}\right)^2 f_S(\bar{u})^2 d\bar{v}^2 \tag{7}$$

On the other hand, the parametric equation of the rotation surface given as follows

$$R(u_R, v_R) = (u_R\cos v_R, u_R\sin v_R, \phi_R(u_R)).$$

If we differentiate the last equation we obtain

$$\begin{aligned} R_{u_R}(u_R, v_R) &= (\cos v_R, \sin v_R, \phi_R'(u_R)) \\ R_{v_R}(u_R, v_R) &= (-u_R\sin v_R, u_R\cos v_R, 0) \end{aligned}$$

and coefficients of first fundamental form of the rotation surface is given as

$$\begin{aligned} E_R &= \langle R_{u_R}, R_{u_R}\rangle = 1+\phi_R'^2 \\ F_R &= \langle R_{u_R}, R_{v_R}\rangle = a\phi_R' \\ G_R &= \langle R_{v_R}, R_{v_R}\rangle = u_R^2. \end{aligned}$$

Then, the line element of the rotation surface is given by

$$ds_R^2 = \left(1+\phi_R'^2\right)du_R^2 + 2a\phi_R' du_R dv_R + u_R^2 dv_H^2.$$

So, if we consider that

$$\bar{u}_R = \int \sqrt{1+\phi_R'^2}du_R,\ f_R(\bar{u}_R) = u_R,\ \bar{v}_R = v_R,$$

then the line element of the rotation surface is rewritten as

$$ds_R^2 = d\bar{u}_R^2 + f_R^2(\bar{u}_R)d\bar{v}_R^2. \tag{8}$$

Comparing the line element of the spiral surface in Eq. (7) and the line element of the rotation surface in Eq. (8) and if we consider that

$$\bar{u}_S = \bar{u}_R\ ,\ \bar{v}_S = v_R \text{ and } f_S(\bar{u}_S) = f_R(\bar{u}_R)$$

we get

$$\bar{u}_S = \bar{u}_R \tag{9}$$

$$\begin{aligned}
\int \sqrt{1+\phi_S'^2 - \frac{(bu_s + b\phi_S\phi_S')^2}{(b^2(u_S^2+\phi_S^2)+u_S^2)}}du_S &= \int \sqrt{1+\phi_R'^2}du_R \\
&= \int \sqrt{\left(\frac{du_R}{du_R}\right)^2 + \left(\frac{d\phi_R}{du_R}\right)^2}du_R \\
&= \int \sqrt{\left(\frac{du_R}{du_S}\frac{du_S}{du_R}\right)^2 + \left(\frac{d\phi_R}{du_S}\frac{du_S}{du_R}\right)^2}du_R \\
&= \int \sqrt{\left(\frac{du_R}{du_S}\right)^2 + \left(\frac{d\phi_R}{du_S}\right)^2}\frac{du_S}{du_R}du_R \\
&= \int \sqrt{\left(\frac{du_R}{du_S}\right)^2 + \left(\frac{d\phi_R}{du_S}\right)^2}du_S
\end{aligned}$$

and

$$f_R(\bar{u}_R) = f_S(\bar{u}_S) \tag{10}$$

$$\begin{aligned}
u_R &= \sqrt{(b^2(u_S^2+\phi_S^2)+u_S^2)} \\
\frac{du_R}{du_S} &= \frac{b^2u_S + b^2\phi_S\phi_S' + u_S}{\sqrt{(b^2(u_S^2+\phi_S^2)+u_S^2)}}.
\end{aligned}$$

Using the Eq. (9) and Eq. (10) we obtain the function ϕ_R of the rotation surface as follows

$$\phi_R(u) = \int \sqrt{\frac{b^2u_S^2\phi_S'^2+b^2\phi_S^2+u_S^2+u_S^2\phi_S'^2-2b^2\phi_S\phi_S'+(b^2+1)^2u_S^2-2(b^2+1)b^2u_S\phi_S\phi_S'-b^4\phi_S^2\phi_S'^2}{b^2(u_S^2+\phi_S^2)+u_S^2}}du_S.$$

So, we obtain a conformal relation between the spiral

$$S(u,v) = \begin{bmatrix} e^{g'(v)}u\cos v \\ e^{g'(v)}u\sin v \\ e^{g'(v)}\phi(u) \end{bmatrix}$$

and the rotation surface

$$R(u,v) = \begin{bmatrix} \sqrt{(b^2(u^2+\phi^2)+u^2)}\cos(v + \int \frac{bu+b\phi\phi'}{(b^2(u^2+\phi^2)+u^2)}du) \\ \sqrt{(b^2(u^2+\phi^2)+u^2)}\sin(v + \int \frac{bu+b\phi\phi'}{(b^2(u^2+\phi^2)+u^2)}du) \\ \phi_R \end{bmatrix}.$$

■

Example 2 *If we consider* $\phi(u) = u$, $a = 1$ *and* $b = \frac{1}{3}$ *then we obtain following figures which have a conformal relation.*

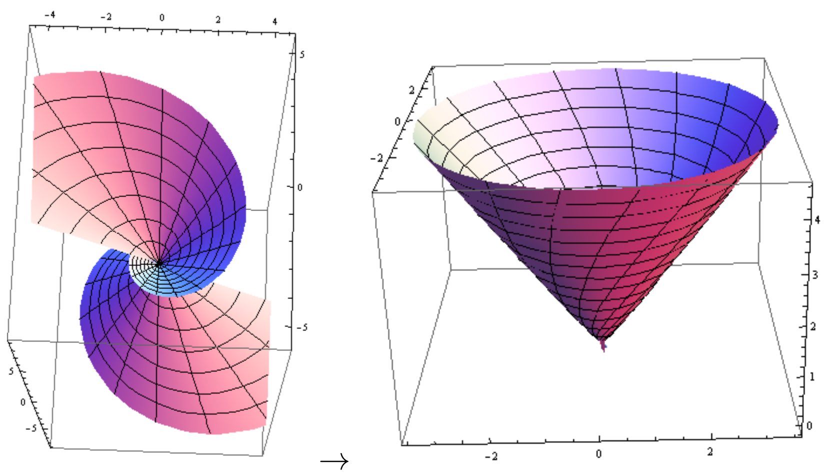

Figure 2. A spiral surface and its images rotation surface.

Corollary 4 *Consequently; helicoidal, rotation and spiral surface have following diagram which is comutative.*

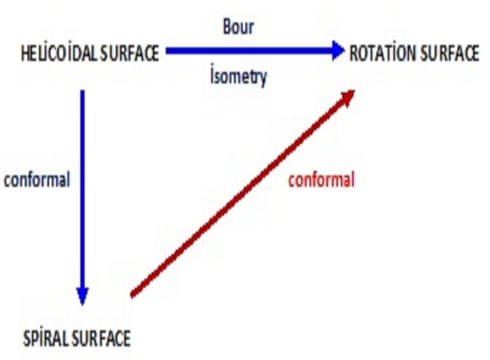

Figure 4: Relation among the helicoidal, rotational and spiral surface.

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