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$\mathbf{M}_1$ SURFACES OF BIHARMONIC $\mathfrak{B}$-GENERAL HELICES ACCORDING TO BISHOP FRAME IN HEISENBERG GROUP Heis^3

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Abstract

In this paper, we study $\mathbf{M}_1$ surfaces of biharmonic $\mathfrak{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

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1 Introduction

Developable surfaces have several practical applications. Many cartographic projections involve projecting the Earth to a developable surface and then "unrolling" the surface into a region on the plane. Since they may be constructed by bending a flat sheet, they are also important in manufacturing objects from sheet metal, cardboard, and plywood.

In this paper, we study $\mathbf{M}_1$ surfaces of biharmonic $\mathfrak{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. We give necessary and sufficient conditions for $\mathfrak{B}$-general helices to be biharmonic according to Bishop frame. We characterize the $\mathbf{M}_1$ surfaces of biharmonic $\mathfrak{B}$-general helices in terms of Bishop frame in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

2 The Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}) \qquad (2.1)$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial x},\quad \mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\quad \mathbf{e}_3=\frac{\partial}{\partial z}. \qquad (2.2)$$

3 Biharmonic $\mathfrak{B}$-General Helices with Bishop Frame In The Heisenberg Group Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},\end{aligned} \qquad (3.1)$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{N},\mathbf{N})=1,\ g(\mathbf{B},\mathbf{B})=1,\\ g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.\end{aligned} \qquad (3.2)$$

In the rest of the paper, we suppose everywhere $\kappa\neq 0$ and $\tau\neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = 1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ g(\mathbf{T},\mathbf{M}_1) &= g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures. where $\theta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_2^2 + k_1^2}$.

4 $\mathbf{M}_1$ Surface of Biharmonic $\mathfrak{B}$-General Helices with Bishop Frame In The Heisenberg Group Heis^3

The purpose of this section is to study $\mathbf{M}_1$ surfaces of biharmonic $\mathfrak{B}$-general helices with Bishop frame in the Heisenberg group Heis^3.

The $\mathbf{M}_1$ surface of $\gamma_{\mathfrak{B}}$ is a ruled surface

$$\mathcal{P}(s,u) = \gamma_{\mathfrak{B}}(s) + u\mathbf{M}_1(s). \tag{4.1}$$

Theorem 4.1. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-general* *helix with non-zero natural curvatures. Then the* $\mathbf{M}_1$ *surface of* $\gamma_{\mathfrak{B}}$ *is*

$$\begin{aligned}
\mathcal{P}(s,u) &= [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +u\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2]\mathbf{e}_1 \\
&\quad +[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad -u\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3]\mathbf{e}_2 \qquad (4.2) \\
&\quad +[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
&\quad [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
&\quad +(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}} \\
&\quad -\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_4]\mathbf{e}_3,
\end{aligned}$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 *are constants of integration.*

Proof. Using orthonormal basis (2.2) and (3.8), we obtain

$$\begin{aligned}
\mathbf{T} &= (\sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0],\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0], \\
&\quad \cos\theta+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin^2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \qquad (4.3) \\
&\quad +\zeta_1\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]),
\end{aligned}$$

where ζ_1 is constant of integration.

$$\begin{aligned}
\mathbf{T} &= \sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_1+\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_2 \\
&\quad +\cos\theta\mathbf{e}_3. \qquad (4.4)
\end{aligned}$$

On the other hand, using Bishop formulas (3.3) and (2.1), we have

$$\mathbf{M}_1=\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_1-\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_2. \quad (4.5)$$

Using above equation, we have (4.2), the theorem is proved.

Thus, we have following theorem.

Theorem 4.2. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix with non-zero natural curvatures. Then the normal surface of $\gamma_{\mathfrak{B}}$ are*

$$\begin{aligned}
x_{\mathcal{P}}(s,u) &= [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +u\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2], \\
y_{\mathcal{P}}(s,u) &= [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad -u\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3], \\
z_{\mathcal{P}}(s,u) &= [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +u\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
&\quad [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad -u\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
&\quad +[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
&\quad [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
&\quad +(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}} \\
&\quad -\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_4],
\end{aligned}$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 are constants of integration.

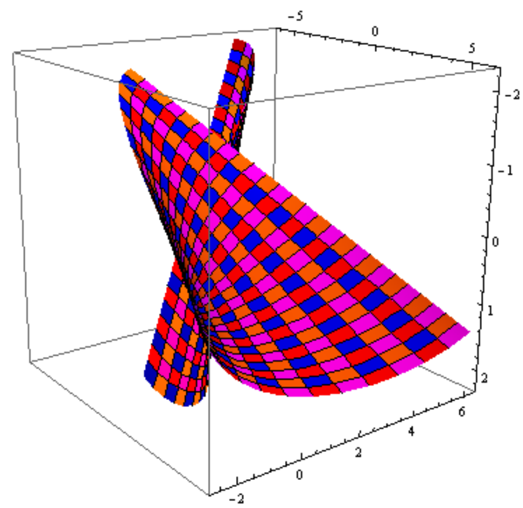

Fig.1

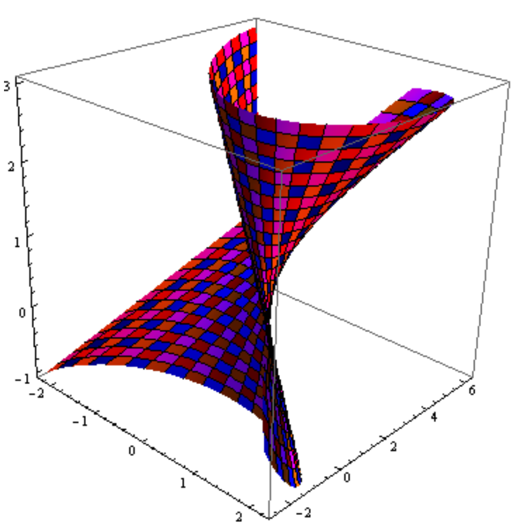

Fig.2

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