Mathematica Aeterna, Vol. 2, 2012, no. 9, 815 - 825

Fuzzy preorders on fuzzy power structures

Yong Chan Kim and Jung Mi Ko

Department of Mathematics, Gangneung-Wonju National University,
Gangneung, Gangwondo 210-702, Korea
yck@gwnu.ac.kr; jmko@gwnu.ac.kr

Abstract

In this paper, we investigate the properties of fuzzy preorder, fuzzy closure and interior operators on fuzzy power structures. Moreover, we study the relationships between fuzzy preorder and fuzzy relations.

Mathematics Subject Classification: 03E72, 54A40,54B10

Keywords:
Residuated lattices, fuzzy preorder, isotone (antitone) maps, interior (resp. closure) operators

1 Introduction

Brink [5] introduced the notion of power structures. The properties of a structure is developed to its power structures [3-6,9]. Georgescu [6] generalized the theory of power structures to the fuzzy setting on a continuous t-norm. Zhang [9] extended it to the fuzzy setting on a complete residuated lattice.

In this paper, we investigate the properties of fuzzy preorder, fuzzy closure and interior operators on fuzzy power structures with a complete residuated lattice in a Zhang's sense. Moreover, we study the relationships between fuzzy preorder and fuzzy relations.

2 Preliminaries

Definition 2.1 [1-2, 7-11] A structure $(L, \vee, \wedge, \odot, \to, \perp, \top)$ is called a *residuated lattice* if it satisfies the following conditions:

(R1) $(L, \vee, \wedge, \top, \perp)$ is a bounded where $\top$ is the universal upper bound and $\perp$ denotes the universal lower bound;

(R2) $(L, \odot, \top)$ is a commutative monoid;

(R3) it satisfies a residuation , i.e.

$$a \odot b \le c \text{ iff } a \le b \to c.$$

We call that a residuated lattice has the law of double negation if $a = (a^*)^*$ where $a^* = a \to \bot$.

Remark 2.2 [1-2, 7-11] (1) A left-continuous t-norm $([0,1], \le, \odot)$ defined by $a \to b = \bigvee\{c \mid a \odot c \le b\}$ is a residuated lattice

(2) An MV-algebra is a residuated lattice with the law of double negation.

In this paper, we assume $(L, \wedge, \vee, \odot, \to, \bot, \top)$ is a residuated lattice with the law of double negation.

Lemma 2.3 [7-11] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \le z$, $(x \odot y) \le (x \odot z)$, $x \to y \le x \to z$ and $z \to x \le y \to x$.
(2) $x \odot y \le x \wedge y \le x \vee y$.
(3) $x \to (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma}(x \to y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \to y = \bigwedge_{i \in \Gamma}(x_i \to y)$.
(4) $x \to (\bigvee_{i \in \Gamma} y_i) \ge \bigvee_{i \in \Gamma}(x \to y_i)$.
(5) $(\bigwedge_{i \in \Gamma} x_i) \to y \ge \bigvee_{i \in \Gamma}(x_i \to y)$.
(6) $(x \odot y) \to z = x \to (y \to z)$ and $(x \odot y)^* = x \to y^*$.
(7) $x \odot (x \to y) \le y$.
(8) $(x \to y) \odot (y \to z) \le x \to z$.
(9) $(x \to z) \le (y \odot x) \to (y \odot z)$.
(10) $(x \to y) \le (y \to z) \to (x \to z)$.
(11) $x \to y = \top$ iff $x \le y$.
(12) $x \to y = y^* \to x^*$.
(13) $\bigwedge_{i \in \Gamma} x_i^* = (\bigvee_{i \in \Gamma} x_i)^*$ and $\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*$.

Definition 2.4 [1-3], [6,9] Let X be a set. A function $e_X : X \times X \to L$ is called:

(P1) (reflexive) $e_X(x, x) = 1$ for all $x \in X$,
(P2) (transitive) $e_X(x, y) \odot e_X(y, z) \le e_X(x, z)$, for all $x, y, z \in X$.
Then e_X is called a fuzzy preorder.
The pair (X, e_X) is a fuzzy preorder set.

Example 2.5 (1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then e_L is a fuzzy preorder.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as

$$e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x)).$$

Then e_{L^X} is a fuzzy preorder on L^X.

(3) Let (X, e_X) be a fuzzy preordered set. Define $e_X^{-1}(a, b) = e_X(b, a)$. Then (X, e_X^{-1}) be a fuzzy preordered set.

Definition 2.6 (1) A map $G : L^X \to L^Y$ is an isotone map if for all $A, B \in L^X$, $e_{L^X}(A, B) \leq e_{L^Y}(G(A), G(B))$.

(2) A map $G : L^X \to L^Y$ is an antitone map if for all $A, B \in L^X$, $e_{L^X}(A, B) \leq e_{L^Y}(G(B), G(A))$.

Definition 2.7 A map $C : L^X \to L^X$ is called a fuzzy closure operator if it satisfies the following conditions:

(C1) $A \leq C(A)$, for all $A \in L^X$.

(C2) $C(C(A)) = C(A)$, for all $A \in L^X$.

(C3) C is an isotone map.

A map $I : L^X \to L^X$ is called a fuzzy interior operator if it satisfies the following conditions:

(I1) $I(A) \leq A$, for all $A \in L^X$.

(I2) $I(I(A)) = I(A)$, for all $A \in L^X$.

(I3) I is an isotone map.

3 Fuzzy preorders on fuzzy power structures

Theorem 3.1 *Let $S : L^X \times L^Y \to L$ be a fuzzy relation. Define $s : L^Y \to L^{L^X}$ and $r : L^X \to L^{L^Y}$ as*

$$s(B)(A) = S(A, B), \quad r(A)(B) = S(A, B)$$

$\sigma(S) : L^Y \times L^Y \to L$ and $\rho(S) : L^X \times L^X \to L$ as

$$\sigma(S)(B_1, B_2) = e_{L^{L^Y}}(s(B_1), s(B_2)) = \bigwedge_{C \in L^X} (s(B_1)(C) \to s(B_2)(C)),$$

$$\rho(S)(A_1, A_2) = e_{L^{L^X}}(r(A_1), r(A_2)) = \bigwedge_{C \in L^X} (r(A_1)(C) \to r(A_2)(C)).$$

Then; (1) *$\sigma(S)$ is a fuzzy preorder on L^Y.*

(2) *$\rho(S)$ is a fuzzy preorder on L^X.*

(3) *S is a fuzzy preorder iff $S(A, B) = \sigma(S)(A, B)$ iff $S(A, B) = \rho(S)(B, A)$.*

Proof (1) Since $\sigma(S)(B, B) = e_{L^{L^Y}}(s(B), s(B)) = \top$ and $\sigma(S)(B_1, B_2) \odot \sigma(S)(B_2, B_3) = \bigwedge_{A \in L^X}(s(B_1)(A) \to s(B_2)(A)) \odot \bigwedge_{A \in L^X}(s(B_2)(A) \to s(B_3)(A)) \leq \bigwedge_{A \in L^X}(s(B_1)(A) \to s(B_3)(A)) = \sigma(R)(B_1, B_3)$ from Lemma 2.3(8), $\sigma(R)$ is a fuzzy preorder L^Y.

(2) It is similarly proved as in (1).

(3) Since $S(A, B) \odot S(B, C) \leq S(A, C)$ iff $S(B, C) \leq S(A, B) \to S(A, C)$ iff $S(A, B) \leq S(B, C) \to S(A, C)$, then S is transitive iff $S(B, C) \leq e_{L^{L^Y}}(s(B), s(C)) = \sigma(S)(B, C)$ iff $S(A, B) \leq e_{L^{L^Y}}(r(B), r(A)) = \rho(S)(B, A)$. Moreover, $S(A, A) = \top$ iff $S(A, B) = s(A)(A) \to s(B)(A) \geq \sigma(S)(A, B)$ iff $S(A, B) = r(B)(B) \to r(A)(B) \geq \rho(S)(B, A)$. Thus, the results hold.

Theorem 3.2 *The following statements are equivalent:*
(1) S *is a fuzzy preorder.*
(2) *There exists a family* $\{h_i \in L^{L^X} \mid i \in I\}$ *such that*

$$S(A,B) = \bigwedge_{i\in I}(h_i(A) \to h_i(B)).$$

(3) *There exists a family* $\{g_j \in L^{L^X} \mid j \in J\}$ *such that*

$$S(A,B) = \bigwedge_{j\in J}(g_j(B) \to g_j(A)).$$

Proof (1) $\Rightarrow$ (2). Since $S(A,B) \odot S(B,C) \leq S(A,C)$ iff $S(B,C) \leq S(A,B) \to S(A,C)$, then there exists a family $\{r(A) \in L^{L^X} \mid r(A)(B) = S(A,B), A \in L^X\}$ such that $S(B,C) = \bigwedge_{A\in L^X}(r(A)(B) \to r(A)(C))$.

(1) $\Rightarrow$ (3). Since $S(A,B) \odot S(B,C) \leq S(A,C)$ iff $S(A,B) \leq S(B,C) \to S(A,C)$, then there exists a family $\{s(C) \in L^X \mid s(C)(B) = S(B,C), C \in L^X\}$ such that $S(A,C) = \bigwedge_{C\in L^X}(s(C)(B) \to s(C)(A))$.

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are easily proved.

Definition 3.3 Let $R \in L^{X\times X}$ be a fuzzy relation. For each $A, B \in L^X$, we define the following maps:
(1) $R^{\odot}, (R^{-1})^{\odot} : L^X \to L^X$ as:

$$R^{\odot}(A)(y) = \bigvee_{x\in X}(R(x,y) \odot A(x)), \;\; (R^{-1})^{\odot}(B)(x) = \bigvee_{y\in Y}(R(x,y) \odot B(y));$$

(2) $S^{\odot}, (S^{-1})^{\odot} : L^X \times L^X \to L$ as:

$$S^{\odot}(A,B) = e_{L^X}(A, (R^{-1})^{\odot}(B)), \;\; (S^{-1})^{\odot}(A,B) = e_{L^X}(B, R^{\odot}(A));$$

(3) $R^{\to}, R^{\leftarrow} : L^X \to L^X$ as:

$$R^{\to}(A)(y) = \bigwedge_{x\in X}(R(x,y) \to A(x)), \;\; R^{\leftarrow}(B)(x) = \bigwedge_{x\in X}(R(x,y) \to B(y));$$

(4) $S^{\to}, S^{\leftarrow} : L^X \times L^X \to L$ as:

$$S^{\to}(A,B) = e_{L^X}(R^{\to}(A), B), \;\; S^{\leftarrow}(A,B) = e_{L^X}(R^{\leftarrow}(B), A).$$

Theorem 3.4 *Let* $R \in L^{X\times X}$ *be a fuzzy relation. For each* $A, B \in L^X$*, the following properties hold.*
(1)

$$\begin{aligned} e_{L^X}(A,B) &= e_{L^{L^X}}(e_{L^X}(-,A), e_{L^X}(-,B)) \\ &= e_{L^{L^X}}(e_{L^X}(B,-), e_{L^X}(A,-)). \end{aligned}$$

(2) $\sigma(S^{\odot})(A,B) = e_{L^X}((R^{-1})^{\odot}(A),(R^{-1})^{\odot}(B))$.
(3) $\rho((S^{-1})^{\odot})(A,B) = e_{L^X}(R^{\odot}(A),R^{\odot}(B))$.
(4) $e_{L^X}(A,B) \le e_{L^X}((R^{-1})^{\odot}(A),(R^{-1})^{\odot}(B)) = \sigma(S^{\odot})(A,B)$.
(5) $e_{L^X}(B,A) \le \sigma((S^{-1})^{\odot})(A,B)$ *and* $e_{L^X}(B,A) \le \rho(S^{\odot})(A,B)$.
(6) $e_{L^X}(A,B) \le e_{L^X}(R^{\odot}(A),R^{\odot}(B)) = \rho((S^{-1})^{\odot})(A,B)$.

Proof (1) Define $s : L^Y \to L^{L^X}$ and $r : L^X \to L^{L^Y}$ as

$$s(B)(A) = e_{L^X}(-,B)(A) = S(A,B),$$

$$r(A)(B) = S(A,B) = e_{L^X}(A,-)(B) = e_{L^X}(A,B)$$

Put $S = e_{L^X}$. Then $S = e_{L^X}$ is a fuzzy preorder on L^X. From Theorem 3.1 (3),

$$\begin{aligned} e_{L^X}(A,B) &= \sigma(e_{L^X})(A,B) = e_{L^{L^X}}(s(A),s(B)) \\ &= e_{L^{L^X}}(e_{L^X}(-,A),e_{L^X}(-,B)) \\ &= \rho(e_{L^X})(B,A) = e_{L^{L^X}}(r(B),r(A)) \\ &= e_{L^{L^X}}(e_{L^X}(B,-),e_{L^X}(A,-)). \end{aligned}$$

(2) For $s : L^X \to L^{L^X}$ with $s(B)(C) = S^{\odot}(C,B)$, we have

$$\begin{aligned} &\sigma(S^{\odot})(A,B) = e_{L^{L^X}}(s(A),s(B)) \\ &= \textstyle\bigwedge_{C\in L^X}(S^{\odot}(C,A) \to S^{\odot}(C,B)) \\ &= e_{L^{L^X}}(e_{L^X}(-,(R^{-1})^{\odot}(A)),e_{L^X}(-,(R^{-1})^{\odot}(B))) \\ &= e_{L^X}((R^{-1})^{\odot}(A),(R^{-1})^{\odot}(B)) \text{ (by (1))}. \end{aligned}$$

(3) For $r : L^X \to L^{L^X}$ with $r(A)(C) = (S^{-1})^{\odot}(A,C)$, we have

$$\begin{aligned} &\rho(S^{\odot})(A,B) = e_{L^{L^X}}(r(A),r(B)) \\ &= \textstyle\bigwedge_{C\in L^X}((S^{-1})^{\odot}(A,C) \to (S^{-1})^{\odot}(B,C)) \\ &= e_{L^{L^X}}(e_{L^X}(-,R^{\odot}(A)),e_{L^X}(-,R^{\odot}(B))) \\ &= e_{L^X}(R^{\odot}(A),R^{\odot}(B)) \text{ (by (1))}. \end{aligned}$$

(4) Since $(R^{-1})^{\odot}(A)(x) \odot e_{L^X}(A,B) \le \bigvee_{y\in X}(R(x,y) \odot A(y) \odot (A(y) \to B(y)) \le \bigvee_{y\in X}(R(x,y)\odot B(y))$, we have $e_{L^X}(A,B) \le e_{L^X}((R^{-1})^{\odot}(A),(R^{-1})^{\odot}(B)) = \sigma(S^{\odot})(A,B)$.

(5)

$$\begin{aligned} &\sigma((S^{-1})^{\odot})(A,B) = \textstyle\bigwedge_{C\in L^X}((S^{-1})^{\odot}(C,A) \to (S^{-1})^{\odot}(C,B)) \\ &= e_{L^{L^X}}(e_{L^X}(A,R^{\odot}(C)),e_{L^X}(B,R^{\odot}(C))) \\ &\ge e_{L^{L^X}}(e_{L^X}(A,D),e_{L^X}(B,D)) \\ &= e_{L^X}(B,A) \text{ (by (1))}. \end{aligned}$$

For $r : L^X \to L^{L^X}$ with $r(A)(C) = S^{\odot}(A, C)$, we have

$$\begin{aligned}
&\rho(S^{\odot})(A, B) = e_{L^{L^X}}(r(A), r(B)) \\
&= \bigwedge_{C \in L^X}(S^{\odot}(A, C) \to S^{\odot}(B, C)) \\
&= \bigwedge_{C \in L^X}(e_{L^X}(A, (R^{-1})^{\odot}(C)) \to e_{L^X}(B, (R^{-1})^{\odot}(C))) \\
&\geq \bigwedge_{D \in L^X}(e_{L^X}(A, D) \to e_{L^X}(B, D)) \\
&= e_{L^X}(B, A) \text{ (by (1))}.
\end{aligned}$$

(6) It is similarly proved as in (4).

Theorem 3.5 *Let $R \in L^{X \times X}$ be a fuzzy relation. For each $A, B \in L^X$, the following properties hold.*

(1) $\sigma(S^{\leftarrow})(A, B) = e_{L^X}(R^{\leftarrow}(B), R^{\leftarrow}(A))$.
(2) $\rho(S^{\rightarrow})(A, B) = e_{L^X}(R^{\rightarrow}(B), R^{\rightarrow}(A))$.
(3) $e_{L^X}(A, B) \leq e_{L^X}(R^{\rightarrow}(A), R^{\rightarrow}(B)) = \rho(S^{\rightarrow})(B, A)$.
(4) $e_{L^X}(A, B) \leq e_{L^X}(R^{\leftarrow}(A), R^{\leftarrow}(B)) = \sigma(S^{\leftarrow})(B, A)$.
(5) $e_{L^X}(A, B) \leq \sigma(S^{\rightarrow})(A, B)$ *and* $e_{L^X}(A, B) \leq \rho(S^{\leftarrow})(A, B)$.
(6) $(R^{\rightarrow}(A))^* = R^{\odot}(A^*)$ *and* $(R^{\leftarrow}(A))^* = (R^{-1})^{\odot}(A^*)$.
(7) $S^{\rightarrow}(A, B) = (S^{-1})^{\odot}(A^*, B^*)$ *and* $S^{\leftarrow}(A, B) = S^{\odot}(A^*, B^*)$.
(8) $\sigma(S^{\leftarrow})(A, B) = \sigma(S^{\odot})(A^*, B^*)$ *and* $\rho(S^{\rightarrow})(A, B) = \rho((S^{-1})^{\odot})(A^*, B^*)$.

Proof (1) For $s : L^X \to L^{L^X}$ with $s(B)(C) = S^{\leftarrow}(C, B)$, we have

$$\begin{aligned}
&\sigma(S^{\leftarrow})(A, B) = e_{L^{L^X}}(s(A), s(B)) \\
&= \bigwedge_{C \in L^X}(S^{\leftarrow}(C, A) \to S^{\leftarrow}(C, B)) \\
&= e_{L^{L^X}}(e_{L^X}(R^{\leftarrow}(A), -), e_{L^X}(R^{\leftarrow}(B), -)) \\
&= e_{L^X}(R^{\leftarrow}(B), R^{\leftarrow}(A)) \text{ (by Theorem 3.4(1))}.
\end{aligned}$$

(2) For $r : L^X \to L^{L^X}$ with $r(A)(C) = S^{\rightarrow}(A, C)$, we have

$$\begin{aligned}
&\rho(S^{\rightarrow})(A, B) = e_{L^{L^X}}(r(A), r(B)) \\
&= \bigwedge_{C \in L^X}(S^{\rightarrow}(A, C) \to S^{\rightarrow}(B, C)) \\
&= e_{L^{L^X}}(e_{L^X}(R^{\rightarrow}(A), -), e_{L^X}(R^{\rightarrow}(B), -)) \\
&= e_{L^X}(R^{\rightarrow}(B), R^{\rightarrow}(A)) \text{ (by Theorem 3.4(1))}.
\end{aligned}$$

(3) Since $(R(x, y) \to A(x)) \odot (A(x) \to B(x)) \leq R(x, y) \to B(x)$ iff $A(x) \to B(x) \leq (R(x, y) \to A(x)) \to (R(x, y) \to B(x))$, then $e_{L^X}(A, B) \leq e_{L^X}(R^{\rightarrow}(A), R^{\rightarrow}(B))$.

(4) It is similarly proved as in (3).

(5) For $s : L^X \to L^{L^X}$ with $s(B)(C) = S^{\rightarrow}(C, B)$, we have

$$\begin{aligned}
&\sigma(S^{\rightarrow})(A, B) = e_{L^{L^X}}(s(A), s(B)) \\
&= \bigwedge_{C \in L^X}(S^{\rightarrow}(C, A) \to S^{\rightarrow}(C, B)) \\
&= \bigwedge_{C \in L^X} e_{L^X}(R^{\rightarrow}(C), A) \to e_{L^X}(R^{\rightarrow}(C), B) \\
&\geq e_{L^{L^X}}(e_{L^X}(-, A), e_{L^X}(-, B)) \\
&= e_{L^X}(A, B) \text{ (by Theorem 3.4(1))}.
\end{aligned}$$

For $r : L^X \to L^{L^X}$ with $r(A)(C) = S^{\leftarrow}(A, C)$, we have

$$\begin{aligned}
&\rho(S^{\leftarrow})(A,B) = e_{L^{L^X}}(r(A), r(B)) \\
&= \bigwedge_{C\in L^X}(S^{\leftarrow}(A,C) \to S^{\leftarrow}(B,C)) \\
&= \bigwedge_{C\in L^X} e_{L^X}(R^{\leftarrow}(C), A) \to e_{L^X}(R^{\leftarrow}(C), B) \\
&\geq e_{L^{L^X}}(e_{L^X}(-,A), e_{L^X}(-,B)) \\
&= e_{L^X}(A,B) \text{ (by Theorem 3.4(1)).}
\end{aligned}$$

(6) From Lemma 2.3(6), we have

$$\begin{aligned}
(R^{\rightarrow}(A)(y))^* &= \Big(\bigwedge_{x\in X}(R(x,y) \to A(x))\Big)^* \\
&= \bigvee_{x\in X}(R(x,y) \odot A^*(x)) = R^{\odot}(A^*)(y).
\end{aligned}$$

Similarly, $(R^{\leftarrow}(A))^* = (R^{-1})^{\odot}(A^*)$.

(7)

$$\begin{aligned}
S^{\rightarrow}(A,B) &= e_{L^X}(R^{\rightarrow}(A), B) = e_{L^X}(B^*, (R^{\rightarrow}(A))^*) \\
&= e_{L^X}(B^*, R^{\odot}(A^*)) = (S^{-1})^{\odot}(A^*, B^*).
\end{aligned}$$

$$\begin{aligned}
S^{\leftarrow}(A,B) &= e_{L^X}(R^{\leftarrow}(B), A) = e_{L^X}(A^*, (R^{\leftarrow}(B))^*) \\
&= e_{L^X}(A^*, (R^{-1})^{\odot}(B^*)) = S^{\odot}(A^*, B^*).
\end{aligned}$$

(8)

$$\begin{aligned}
\sigma(S^{\leftarrow})(A,B) &= e_{L^X}(R^{\leftarrow}(B), R^{\leftarrow}(A)) = e_{L^X}((R^{\leftarrow}(A))^*, (R^{\leftarrow}(B))^*) \\
&= e_{L^X}((R^{-1})^{\odot}(A^*), (R^{-1})^{\odot}(B^*)) = \sigma(S^{\odot})(A^*, B^*).
\end{aligned}$$

$$\begin{aligned}
\rho(S^{\rightarrow})(A,B)(A,B) &= e_{L^X}(R^{\rightarrow}(B), R^{\rightarrow}(A)) = e_{L^X}((R^{\rightarrow}(A))^*, (R^{\rightarrow}(B))^*) \\
&= e_{L^X}(R^{\odot}(A^*), R^{\odot}(B^*)) = \rho((S^{-1})^{\odot})(A^*, B^*).
\end{aligned}$$

The following two theorems are proved in [9].

Theorem 3.6 [9] *Let $R \in L^{X\times X}$ be a fuzzy relation. Then the following conditions are equivalent:*
(1) *R is a fuzzy preorder on X.*
(2) *$R^{\odot}$ is a closure operator.*
(3) *$(R^{-1})^{\odot}$ is a closure operator.*

Theorem 3.7 [9] *Let $R \in L^{X\times X}$ be a fuzzy relation. Then the following conditions are equivalent:*
(1) *R is a fuzzy preorder on X.*
(2) *$(S^{-1})^{\odot}(A,B) = e_{L^X}(R^{\odot}(B), R^{\odot}(A)) = \rho((S^{-1})^{\odot})(B,A)$.*
(3) *$S^{\odot}(A,B) = e_{L^X}((R^{-1})^{\odot}(A), (R^{-1})^{\odot}(B)) = \sigma(R^{\odot})(A,B)$.*
(4) *$(S^{-1})^{\odot}$ is a fuzzy preorder on L^X.*
(5) *$S^{\odot}$ is a fuzzy preorder on L^X.*

Theorem 3.8 *Let $R \in L^{X\times X}$ be a fuzzy relation. Then the following conditions are equivalent:*
(1) *R is a fuzzy preorder on X.*
(2) *$R^{\to}$ is an interior operator.*
(3) *$R^{\leftarrow}$ is an interior operator.*

Proof (1) $\Rightarrow$ (2). (I1) Since $R(y,y) = \top$, $R^{\to}(A)(y) = \bigwedge_x (R(x,y) \to A(x)) \leq R(y,y) \to A(y) = A(y)$.

(I2) $R^{\to}(R^{\to}(A))(z) = \bigwedge_{y\in X}(R(y,z) \to R^{\to}(A)(y)) = \bigwedge_{y\in X}(R(y,z) \to \bigwedge_{x\in X}(R(x,y) \to A(x))) = \bigwedge_{x\in X}(\bigvee_{y\in X}(R(x,y)\odot R(y,z)) \to A(x)) = \bigwedge_{x\in X}(R(x,z) \to A(x)) = R^{\to}(A)(z)$.

(I3) Since $R^{\to}(A)(y) \odot e_{L^X}(A,B) \leq (R(x,y) \to A(x)) \odot (A(x) \to B(x)) \leq R(x,y) \to B(x)$, then $e_{L^X}(A,B) \leq e_{L^X}(R^{\to}(A), R^{\to}(B))$.

(2) $\Rightarrow$ (1). We have $R^{\to}(\chi^c_{\{x\}})(y) = R^*(x,y)$. Since $R^*(x,x) = R^{\to}(\chi^c_{\{x\}})(x) \leq \chi^c_{\{x\}}(x) = \perp$, $R(x,x) = \top$.

$R^{\to}(R^{\to}(\chi^c_{\{x\}}))(z) = \bigwedge_{y\in X}(R(y,z) \to R^{\to}(\chi^c_{\{x\}})(y)) = \bigwedge_{y\in X}(R(y,z) \to R^*(x,y)) = (\bigvee_{y\in X} R(x,y)\odot R(y,z))^* = R^{\to}(\chi^c_{\{x\}})(z) = R^*(x,z)$. Thus, $\bigvee_{y\in X} R(x,y)\odot R(y,z) = R(x,z)$.

(1) $\Rightarrow$ (3). Since $e_{L^X}(A,B) \odot R^{\leftarrow}(A)(y) \leq (A(y) \to B(y)) \odot (R(x,y) \to A(y)) \leq R(x,y) \to B(y)$, we have $e_{L^X}(A,B) \leq e_{L^X}(R^{\leftarrow}(A), R^{\leftarrow}(B))$. Other cases are similarly proved as in (1).

(3) $\Rightarrow$ (1). We have $R^{\leftarrow}(\chi^c_{\{y\}})(x) = R^*(x,y)$. Since $R^*(x,x) = R^{\leftarrow}(\chi^c_{\{x\}})(x) \leq \chi^c_{\{x\}}(x) = \perp$, then $R(x,x) = \top$. Moreover, $R^{\leftarrow}(R^{\leftarrow}(\chi^c_{\{z\}}))(x) = \bigwedge_{y\in X}(R(x,y) \to R^{\leftarrow}(\chi^c_{\{z\}})(y)) = \bigwedge_{y\in X}(R(x,y) \to R^*(y,z)) = (\bigvee_{y\in X} R(x,y) \odot R(y,z))^* = R^{\to}(\chi^c_{\{z\}})(x) = R^*(x,z)$. Thus, R is transitive.

Theorem 3.9 *Let $R \in L^{X\times X}$ be a fuzzy relation. Then the following conditions are equivalent:*
(1) *R is a fuzzy preorder on X.*
(2) *$S^{\to}(A,B) = e_{L^X}(R^{\to}(A), R^{\to}(B)) = (\rho(S^{\to}))(B,A)$.*
(3) *$S^{\leftarrow}(A,B) = e_{L^X}(R^{\leftarrow}(B), R^{\leftarrow}(A)) = (\sigma(S^{\leftarrow}))(A,B)$.*
(4) *$S^{\to}$ is a fuzzy preorder on L^X.*
(5) *$S^{\leftarrow}$ is a fuzzy preorder on L^X.*

Proof (1) $\Rightarrow$ (2). We have $S^{\to}(A,B) = e_{L^X}(R^{\to}(A), R^{\to}(B))$ from

$$\begin{aligned}
& e_{L^X}(R^{\to}(A), R^{\to}(B)) \\
&= e_{L^X}(R^{\to}(A), R^{\to}(B)) \odot e_{L^X}(R^{\to}(B), B) \\
&\leq e_{L^X}(R^{\to}(A), B) = S^{\to}(A,B) \\
&\leq e_{L^X}(R^{\to}(R^{\to}(A)), R^{\to}(B)) \\
&= e_{L^X}(R^{\to}(A), R^{\to}(B))
\end{aligned}$$

(2) $\Rightarrow$ (4).

$$\begin{aligned}&S^{\to}(A,B)\odot S^{\to}(B,C)\\&= e_{L^X}(R^{\to}(A),R^{\to}(B))\odot e_{L^X}(R^{\to}(B),R^{\to}(C))\\&\le e_{L^X}(R^{\to}(A),R^{\to}(C)) = S^{\to}(A,C).\end{aligned}$$

Hence $S^{\to}$ is a fuzzy preorder on L^X.

(4) $\Rightarrow$ (1). Since $R^{\to}(\chi^c_{\{x\}})(y) = R^*(x,y)$ and $S^{\to}(\chi^c_{\{x\}},\chi^c_{\{y\}}) = e_{L^X}(R^{\to}(\chi^c_{\{x\}}),\chi^c_{\{y\}}) = R^{**}(x,y) = R(x,y)$,

$$\begin{aligned}&R(x,y)\odot R(y,z) = S^{\to}(\chi^c_{\{x\}},\chi^c_{\{y\}})\odot S^{\to}(\chi^c_{\{x\}},\chi^c_{\{y\}})\\&\le S^{\to}(\chi^c_{\{x\}},\chi^c_{\{z\}}) = R(x,z).\end{aligned}$$

(1) $\Rightarrow$ (3).

$$\begin{aligned}&e_{L^X}(R^{\leftarrow}(B),R^{\leftarrow}(A))\\&= e_{L^X}(R^{\leftarrow}(A),A)\odot e_{L^X}(R^{\leftarrow}(B),R^{\leftarrow}(A))\\&\le e_{L^X}(R^{\leftarrow}(B),A) = S^{\leftarrow}(A,B)\\&\le e_{L^X}(R^{\leftarrow}(R^{\leftarrow}(B)),R^{\leftarrow}(A))\\&= e_{L^X}(R^{\leftarrow}(B),R^{\leftarrow}(A)).\end{aligned}$$

(3) $\Rightarrow$ (5).

$$\begin{aligned}&S^{\leftarrow}(A,B)\odot S^{\leftarrow}(B,C)\\&= e_{L^X}(R^{\leftarrow}(B),R^{\leftarrow}(A))\odot e_{L^X}(R^{\leftarrow}(C),R^{\leftarrow}(B))\\&\le e_{L^X}(R^{\leftarrow}(C),R^{\leftarrow}(A)) = S^{\leftarrow}(A,C).\end{aligned}$$

Hence $S^{\leftarrow}$ is a fuzzy preorder on L^X.

(5) $\Rightarrow$ (1).

Since $R^{\leftarrow}(\chi^c_{\{y\}})(x) = R^*(x,y)$ and $S^{\leftarrow}(\chi^c_{\{x\}},\chi^c_{\{y\}}) = e_{L^X}(R^{\leftarrow}(\chi^c_{\{y\}}),\chi^c_{\{x\}}) = R^{**}(x,y) = R(x,y)$,

$$\begin{aligned}&R(x,y)\odot R(y,z) = S^{\leftarrow}(\chi^c_{\{x\}},\chi^c_{\{y\}})\odot S^{\leftarrow}(\chi^c_{\{y\}},\chi^c_{\{z\}})\\&\le S^{\leftarrow}(\chi^c_{\{x\}},\chi^c_{\{z\}}) = R(x,z).\end{aligned}$$

Example 3.10 Define a binary operation $\odot$ (called Łukasiewicz conjection) on $L=[0,1]$ by

$$x\odot y = \max\{0, x+y-1\},\ x\to y = \min\{1-x+y, 1\}.$$

Then $([0,1],\vee,\wedge,\odot,\to)$ is a complete residuated lattice with the law of a double negation.

Let $X=\{a,b,c\}$ be a set with a fuzzy preorder R as

$$R = \begin{pmatrix} & a & b & c\\ a & 1.0 & 0.8 & 0.8\\ b & 0.6 & 1.0 & 1.0\\ c & 0.5 & 0.6 & 1.0\end{pmatrix}$$

Let $A = (A(a), A(b), A(c))^t = (0.8, 0.3, 0, 6)^t$, $B = (B(a), B(b), B(c))^t = (0.7, 0.5, 0, 9)^t$ be given.

$$R^{\odot}(A) = (08, 0.6, 0.6)^t,\ R^{\odot}(B) = (0.7, 0.5, 0.9)^t.$$

$$(R^{-1})^{\odot}(A) = (08, 0.6, 0.6)^t,\ (R^{-1})^{\odot}(B) = (0.7, 0.9, 0.9)^t.$$

$$e_{L^X}(A, B) = 0.9,\ e_{L^X}(B, A) = 0.7.$$

$$e_{L^X}(R^{\odot}(A), R^{\odot}(B)) = 0.9,\ e_{L^X}(R^{\odot}(B), R^{\odot}(A)) = 0.7.$$

$$e_{L^X}((R^{-1})^{\odot}(A), (R^{-1})^{\odot}(B)) = 0.9,\ e_{L^X}((R^{-1})^{\odot}(B), (R^{-1})^{\odot}(A)) = 0.7.$$

$$S^{\odot}(A, B) = 0.9,\ S^{\odot}(B, A) = 0.7.$$

$$(S^{-1})^{\odot}(A, B) = 0.7,\ (S^{-1})^{\odot}(B, A) = 0.9.$$

$$R^{\rightarrow}(A) = (07, 0.3, 0.3)^t,\ R^{\rightarrow}(B) = (0.7, 0.5, 0..5)^t.$$

$$R^{\leftarrow}(A) = (05, 0.3, 0.6)^t,\ R^{\leftarrow}(B) = (0.7, 0.5, 0.9)^t.$$

$$e_{L^X}(R^{\rightarrow}(A), R^{\rightarrow}(B)) = 1,\ e_{L^X}(R^{\rightarrow}(B), R^{\rightarrow}(A)) = 0.8.$$

$$e_{L^X}(R^{\leftarrow}(A), R^{\leftarrow}(B)) = 1,\ e_{L^X}(R^{\leftarrow}(B), R^{\leftarrow}(A)) = 0.7.$$

$$S^{\rightarrow}(A, B) = 1,\ S^{\rightarrow}(B, A) = 0.8.$$

$$S^{\leftarrow}(A, B) = 0.7,\ S^{\leftarrow}(B, A) = 1.$$

References

[1] R. Bělohlávek, Fuzzy relational Systems, Kluwer Publishers, New York (2002).

[2] R. Bělohlávek, *Fuzzy equational logic,* Arch. Math. Log. 41 (2002) 83-90.

[3] Bošnjak,I., Madaráz,R., *Power algebras and generalized quotient algebras,* Algebra Univers. 45,179-189 (2001) 287-295.

[4] Bošnjak,I., Madaráz,R., Vojvodić,G., *Algebras of fuzzy sets,* Fuzzy Sets and Systems, 160(2009) 2979-2988.

[5] Brink, C., *Power structures,* Algebra Univers. 30,177-216 (1993) 287-295.

[6] G. Georgescu, *Fuzzy power structures,* Arch. Math. Log. 47 (2008) 233-261.

[7] U. Höhle, E. P. Klement, *Non-classical logic and their applications to fuzzy subsets* , Kluwer Academic Publisher, Boston, 1995.

[8] U. Höhle, S. E. Rodabaugh, *Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory,* The Handbooks of Fuzzy Sets Series, Volume 3, Kluwer Academic Publishers, Dordrecht (1999).

[9] H. Lai, D. Zhang, *Good fuzzy preorders on fuzzy power structures,* Arch. Math. Logic, 49(2010)469-489.

[10] S. E. Rodabaugh, E. P. Klement, *Toplogical And Algebraic Structures In Fuzzy Sets,* The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic 20 , Kluwer Academic Publishers, (Boston/Dordrecht/London) (2003).

[11] E. Turunen, *Mathematics Behind Fuzzy Logic,* A Springer-Verlag Co., 1999.

Received: November, 2012