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The traveling and non-traveling wave solutions of (1+1)-dimensional Boussinesq equations with variable coefficients

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Abstract

Based on (G'/G)-expansion method, new traveling and non-traveling exact solutions of (1+1)-dimensional Boussinesq equations with variable coefficients are established. To obtain the traveling wave solution, we expand $\xi(x,t) = x - Vt$ to a more general form $\xi(x,t) = f(\eta)$, $\eta = x - Vt$. We also suppose the non-traveling wave solution $\xi(x,t)$ with variable separation forms, such as $\xi(x,t) = f(x) + g(t)$ or $\xi(x,t) = f(x)g(t)$. Finally, a series of important novel solutions of the equations are obtained.

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1 Introduction

The investigation of the exact solutions of nonlinear evolution equations (NLEEs) plays a vital role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, elastic media, optical fibers, etc. In recent decades, many mathematicians devote to find the exact solutions of nonlinear PDEs. Meanwhile, the development of mathematical softwares such as Mathematica, Maple and Matlab, provide more effective tools to find the exact solutions of nonlinear PDEs. Tools such as Mathematica can be used to deal with complex calculation.

Recently, a brilliant achievement is that many effective methods have been established to obtain the exact solutions of NLEEs, such as homogenous balance method [1], the F-expansion method [2], the exp-function expansion method [3], the auxiliary equation method [4], the hyperbolic function method [5], the modified extended Fan sub-equation method [6], the tanh method [7], the ansatz method [8], the Jacobi elliptic function expansion [9] etc. However, one needs to be extremely careful to apply these methods since there is a possibility that blindly applying these techniques could lead to misleading results as pointed out by Kudryashov [10-12] and Popovych indicated two more common errors concerning the similarity and linearizability of differential equations [13]. More recently, the (G'/G)-expansion method has been proposed to obtain traveling wave solutions, which is based on the homogenous balance principle and the linear ordinary differential equation (LODE) theory. By using this simple and effective method, M.L. Wang, G. Ebadi and others construct the traveling wave solution of K(m,n) equation, high-order Schödinger equation, BBM equation etc. [14-23].

Inspired by that, we use (G'/G)-expansion method to construct a new kind of exact solutions of the (1+1)-dimensional Boussinesq equations with variable coefficients (CBEVC):

$$\begin{cases} u_t + B(t)(uu_x + v_x) = 0 \\ v_t + B(t)\{[(1+v)u]_x + \frac{1}{3}u_{xxx}\} = 0 \end{cases}, \tag{1.1}$$

where $B(t)$ is an arbitrary function of time t, and $B(t) \neq 0, 1$. If $B(t) = 1$, (1.1) can be reduced to the famous (1+1)-dimensional classical Boussinesq equations, which was originally introduced to describe the propagation of long waves in the shallow water, see [24] and references therein. Eq.(1.1) appears in many areas such as waves in the deep water, fluid dynamics etc., and it seems that the variable coefficients of the nonlinear evolution equations could make those models realistic, see [25,26] and references therein. Actually, in Refs.[27], Lax pairs and Darboux transformation have already been introduced and applied to the variant-coefficient variant Boussinesq (VCVB) Model [28], which is the general form of CBEVC. However, to our knowledge, no work has been done on CBEVC by the (G'/G)-expansion method and solutions with variable separation forms.

The outline of the paper is as follows. In Section 2, we will describe the (G'/G)-expansion method and provide the main steps of the method. In Section 3, we will discuss the (1+1)-dimensional Boussinesq equation with variable coefficients, and construct the traveling wave solutions (TWS) and two different kinds of non-traveling wave solutions (NTWS) for the CBEVC (1) via the (G'/G)-expansion method. Some conclusions and prospects will be given in the last section.

2 $(G^{'}/G)$-expansion method

The fundamental steps of the $(G^{'}/G)$-expansion method can be introduced briefly as follows.

By supposing the given (1+1)-dimensional nonlinear evolution equation for $u(x,t)$ as:

$$F\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\right) = 0, \tag{2.1}$$

where F is a polynomial of $u(x,t)$ and its partial derivatives.
Step 1. Taking $u(x,t) = u(\xi)$, $\xi = \xi(x,t)$, transform partial differential equation(2.1) to the ordinary differential equation:

$$F(u, u', u^{''}, ...) = 0, \tag{2.2}$$

Step 2. Supposing the solution of (2.1) can be expressed in $(G^{'}/G)$ as follows:

$$u(\xi) = \sum_{i=1}^{N} a_i \left(\frac{G^{'}(\xi)}{G(\xi)}\right)^i, \tag{2.3}$$

where a_i are real constants with $a_N \neq 0$ and N is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation:

$$G^{''}\left(\xi\right) + \lambda G^{'}\left(\xi\right) + \mu G\left(\xi\right) = 0, \tag{2.4}$$

where λ and μ are real constants to be determined.
Step 3. Determining N. Considering homogenous balance between the highest order derivatives with the highest order nonlinear terms in (2.2).
Step 4. Substituting the general solution of (2.4) together with (2.3) into Eq.(2.2) yields an algebraic equation involving powers of $(\frac{G^{'}(\xi)}{G(\xi)})$. Collecting the coefficients of same power of $(\frac{G^{'}(\xi)}{G(\xi)})$ and setting them to zero, we can obtain a system of algebraic equations for a_i, λ, μ. Solving the system by Matlab, Maple or Mathematica to determine these constants. Finally, we obtain solutions of Eq.(2.2) by depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$. Then we will find the exact solutions of Eq.(2.1).

In this paper, when solving the traveling wave solution, we expand $\xi(x,t) = x - Vt$ to a more general form $\xi(x,t) = f(\eta)$, $\eta = x - Vt$. Considering the non-traveling wave solution, in general, we suppose that $\xi(x,t)$ is in variable separation forms, such as $\xi(x,t) = f(x)+g(t)$ or $\xi(x,t) = f(x)g(t)$. It is worthy pointing out that, to our knowledge, the solutions of the (1+1)-dimensional Boussinesq equations with variable coefficients listed in this paper are not found in the other papers.

3 The traveling wave solution

We suppose that the solution is in form of travelling wave solution as follows:

$$\xi(x,t) = f(\eta),\ \eta = x - Vt, \tag{3.1}$$

where $f(\eta)$ is an arbitrary function of the indicated variables.

Considering the homogenous balance between the highest order derivatives and the non-linear terms, we can obtain

$$m = 1, n = 2. \tag{3.2}$$

So (2.3) can be rewritten as follows:

$$\begin{cases} u(x,t) = a_0 + a_1(\frac{G'}{G}) \\ v(x,t) = b_0 + b_1(\frac{G'}{G}) + b_2(\frac{G'}{G})^2 \end{cases}, \tag{3.3}$$

Substituting (3.3), (2.3) into (1.1), collecting the terms of $\frac{G'(\xi)}{G(\xi)}$ with the same power, then letting each coefficient equals to zero, we can derive a set of over-determined algebraic equations for a_i and b_i(which is Eqs.(3.4)):

$$b_1V\mu f'(\eta) - a_1\mu B(t)f'(\eta) - a_1b_0\mu B(t)f'(\eta) - a_0b_1\mu B(t)f'(\eta) - \frac{1}{3}a_1\mu B(t)f''(\eta)$$

$$+\frac{2}{3}a_1\lambda\mu B(t)f'(\eta)f''(\eta) - \frac{1}{3}\mu B(t)f'(\eta)[a_1\lambda^2 f'(\eta)^2 + 2a_1\mu f'(\eta)^2 - a_1\lambda f'(\eta)] = 0,$$

$$b_1V\mu f'(\eta) - a_1\mu B(t)f'(\eta) - a_1b_0\mu B(t)f'(\eta) - a_0b_1\mu B(t)f'(\eta) - \frac{1}{3}a_1\mu B(t)f''(\eta)$$

$$+\frac{2}{3}a_1\lambda\mu B(t)f'(\eta)f''(\eta) - \frac{1}{3}\mu B(t)f'(\eta)[a_1\lambda^2 f'(\eta)^2 + 2a_1\mu f'(\eta)^2 - a_1\lambda f'(\eta)] = 0,$$

$$-3a_1b_2B(t)f'(\eta) - 2a_1B(t)f'(\eta)^3 = 0,$$

$$2b_2Vf'(\eta) - 2a_1b_1B(t)f'(\eta) - 2a_0b_2B(t)f'(\eta) - 3a_1b_2\lambda B(t)f'(\eta)$$

$$-2a_1\lambda B(t)f'(\eta)^3 + \frac{4}{3}a_1B(t)f'(\eta)f''(\eta) - \frac{2}{3}B(t)f'(\eta)[3a_1\lambda f'(\eta)^2 - a_1f''(\eta)] = 0,$$

$$b_1Vf'(\eta) + 2b_2V\lambda f'(\eta) - a_1B(t)f'(\eta) - a_1b_0B(t)f'(\eta) - a_0b_1B(t)f'(\eta)$$

$$-2a_1b_1\lambda B(t)f'(\eta) - 2a_0b_2\lambda B(t)f'(\eta) - 3a_1b_2\mu B(t)f'(\eta)$$

$$-2a_1\mu B(t)f'(\eta)^3 - \frac{2}{3}\lambda B(t)f'(\eta)[3a_1\lambda f'(\eta)^2 - a_1f''(\eta)] - \frac{1}{3}B(t)f'(\eta)[a_1\lambda^2 f'(\eta)^2$$

$$+2a_1\mu f'(\eta)^2 - a_1\lambda f''(\eta)] + \frac{1}{3}B(t)[6a_1\lambda f'(\eta)f''(\eta) - a_1f^{(3)}(\eta)] = 0,$$

$$b_1V\lambda f'(\eta) + 2b_2V\mu f'(\eta) - a_1\lambda B(t)f'(\eta) - a_1b_0\lambda B(t)f'(\eta)$$

$$-a_0b_1\lambda B(t)f^{'}(\eta)-2a_1b_1\mu B(t)f^{'}(\eta)-2a_0b_2\mu B(t)f^{'}(\eta)$$
$$-\frac{2}{3}\mu B(t)f^{'}(\eta)[3a_1\lambda f^{'}(\eta)^2-a_1f^{''}(\eta)]-\frac{1}{3}\lambda B(t)f^{'}(\eta)[a_1\lambda^2f^{'}(\eta)^2+2a_1\mu f^{'}(\eta)^2$$
$$-a_1\lambda f^{''}(\eta)]+\frac{1}{3}B(t)[2a_1\lambda^2f^{'}(\eta)f^{''}(\eta)+4a_1\mu f^{'}(\eta)f^{''}(\eta)-a_1\lambda f^{(3)}(\eta)]=0,$$
$$a_1V\mu f^{'}(\eta)-a_0a_1\mu B(t)f^{'}(\eta)-b_1\mu B(t)f^{'}(\eta)=0,$$
$$-a_1{}^2B(t)f^{'}(\eta)-2b_2B(t)f^{'}(\eta)=0,$$
$$a_1Vf^{'}(\eta)-a_0a_1B(t)f^{'}(\eta)-b_1B(t)f^{'}(\eta)-a_1{}^2\lambda B(t)f^{'}(\eta)-2b_2\lambda B(t)f^{'}(\eta)=0,$$
$$a_1V\lambda f^{'}(\eta)-a_0a_1\lambda B(t)f^{'}(\eta)-b_1\lambda B(t)f^{'}(\eta)-a_1{}^2\mu B(t)f^{'}(\eta)-2b_2\mu B(t)f^{'}(\eta)=0.$$

Let us take $f^{(3)}(\eta)=f^{''}(\eta)$, then $f(\eta)=l_1e^{\eta}+l_2+l_3\eta$, where $\eta=x-Vt$. Using Mathematica to solve Eqs.(3.4), we would end up with the explicit expressions of the constants a_0, a_1, a_2, b_0, b_1:

Case 1:

$$a_0=\frac{\sqrt{3}f^{''}(\eta)+\frac{3Vf^{'}(\eta)}{B(t)}-\sqrt{3}\lambda f^{'}(\eta)^2}{3f^{'}(\eta)},a_1=-\frac{2f^{'}(\eta)}{\sqrt{3}},$$
$$b_0=\frac{1}{3f^{'}(\eta)^2}[f^{''}(\eta)^2-f^{''}(\eta)f^{'}(\eta)-3f^{'}(\eta)^2+\lambda f^{''}(\eta)f^{'}(\eta)^2$$
$$-2\mu f^{'}(\eta)^4],b_1=\frac{2}{3}f^{''}(\eta)-\frac{2}{3}\lambda f^{'}(\eta)^2,b_2=-\frac{2}{3}f^{'}(\eta)^2.$$

It is well known that the general solutions of Eq. (2.4) are as follows:

$$\frac{G^{'}(\xi)}{G(\xi)}=\begin{cases}-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1f)+C_2\cosh(\delta_1f)}{C_1\cosh(\delta_1f)+C_2\sinh(\delta_1f)}, & \lambda^2-4\mu>0,\\ -\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2f)+C_2\cos(\delta_2f)}{C_1\cos(\delta_2f)+C_2\sin(\delta_2f)}, & \lambda^2-4\mu<0,\\ -\frac{\lambda}{2}+\frac{C_2}{C_1+C_2f(\eta)}, & \lambda^2-4\mu=0.\end{cases}$$

where $\delta_1=\frac{\sqrt{\lambda^2-4\mu}}{2}$, $\delta_2=\frac{\sqrt{4\mu-\lambda^2}}{2}$, C_1,C_2 are arbitrary constants. The exact solutions are expressed by three types of functions, which are hyperbolic, trigonometric and rational function solutions, respectively.

Therefore we can get:

$$\begin{cases}u_{11}=\frac{\sqrt{3}f^{''}(\eta)+\frac{3Vf^{'}(\eta)}{B(t)}-\sqrt{3}\lambda f^{'}(\eta)^2}{3f^{'}(\eta)}\\ \qquad -\frac{2f^{'}(\eta)[-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1f)+C_2\cosh(\delta_1f)}{C_1\cosh(\delta_1f)+C_2\sinh(\delta_1f)}]}{\sqrt{3}},\\ v_{11}=\frac{1}{3f^{'}(\eta)^2}[f^{''}(\eta)^2-f^{''}(\eta)f^{'}(\eta)-3f^{'}(\eta)^2+\lambda f^{''}(\eta)f^{'}(\eta)^2-2\mu f^{'}(\eta)^4]\\ \qquad +[\frac{2}{3}f^{''}(\eta)-\frac{2}{3}f^{'}(\eta)^2]\{-\frac{\lambda}{2}+\frac{\delta_1[C_2\cosh(f\delta_1)+C_1\sinh(f\delta_1)]}{C_1\cosh(f\delta_1)+C_2\sinh(f\delta_1)}\}\\ \qquad -\frac{2}{3}f^{'}(\eta)^2\{-\frac{\lambda}{2}+\frac{\delta_1[C_2\cosh(f\delta_1)+C_1\sinh(f\delta_1)]}{C_1\cosh(f\delta_1)+C_2\sinh(f\delta_1)}\}^2\end{cases}$$

$$\begin{cases} u_{12}=\dfrac{\sqrt{3}f''(\eta)+\frac{3Vf'(\eta)}{B(t)}-\sqrt{3}\lambda f'(\eta)^2}{3f'(\eta)} \\ \quad -\dfrac{2f'(\eta)\{-\frac{\lambda}{2}+\frac{\delta_2[C_2\cos(f\delta_2)-C_1\sin(f\delta_2)]}{C_1\cos(f\delta_2)+C_2\sin(f\delta_2)}\}}{\sqrt{3}}, \\ v_{12}=\dfrac{1}{3f'(\eta)^2}[f''(\eta)^2-f''(\eta)f'(\eta)-3f'(\eta)^2+\lambda f''(\eta)f'(\eta)^2-2\mu f'(\eta)^4] \\ \quad +[\dfrac{2}{3}f''(\eta)-\dfrac{2}{3}\lambda f'(\eta)^2]\{-\dfrac{\lambda}{2}+\dfrac{\delta_2[C_2\cos(f\delta_2)-C_1\sin(f\delta_2)]}{C_1\cos(f\delta_2)+C_2\sin(f\delta_2)}\} \\ \quad -\dfrac{2}{3}f'(\eta)^2\{-\dfrac{\lambda}{2}+\dfrac{\delta_2[C_2\cos(f\delta_2)-C_1\sin(f\delta_2)]}{C_1\cos(f\delta_2)+C_2\sin(f\delta_2)}\}^2; \end{cases}$$

$$\begin{cases} u_{13}=-\dfrac{2(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2f(\eta)})f'(\eta)}{\sqrt{3}}+\dfrac{\sqrt{3}f''(\eta)+\frac{3Vf'(\eta)}{B(t)}-\sqrt{3}\lambda f'(\eta)^2}{3f'(\eta)}, \\ v_{13}=-\dfrac{2}{3}(-\dfrac{\lambda}{2}+\dfrac{C_2}{C_1+C_2f(\eta)})^2f'(\eta)^2 \\ \quad +(-\dfrac{\lambda}{2}+\dfrac{C_2}{C_1+C_2f(\eta)})[\dfrac{2}{3}f''(\eta)-\dfrac{2}{3}\lambda f'(\eta)^2] \\ \quad +\dfrac{1}{3f'(\eta)^2}[f''(\eta)^2-f''(\eta)f'(\eta)-3f'(\eta)^2+\lambda f''(\eta)f'(\eta)^2-2\mu f'(\eta)^4]. \end{cases}$$

Case 2:

$$a_0=\frac{-\sqrt{3}f''(\eta)+\frac{3Vf'(\eta)}{B(t)}+\sqrt{3}\lambda f'(\eta)^2}{3f'(\eta)}, a_1=\frac{2f'(\eta)}{\sqrt{3}},$$

$$b_0=\frac{1}{3f'(\eta)^2}[f''(\eta)^2-f''(\eta)f'(\eta)-3f'(\eta)^2+\lambda f''(\eta)f'(\eta)^2$$
$$-2\mu f'(\eta)^4], b_1=\frac{2}{3}f''(\eta)-\frac{2}{3}\lambda f'(\eta)^2, b_2=-\frac{2}{3}f'(\eta)^2.$$

Using the same method mentioned in Result 1, we will get three types of solutions as follows:

$$\begin{cases} u_{21}=\frac{-\sqrt{3}f''(\eta)+\frac{3Vf'(\eta)}{B(t)}+\sqrt{3}\lambda f'(\eta)^2}{3f'(\eta)} \\ \quad +\dfrac{2f'(\eta)\{-\frac{\lambda}{2}+\frac{\delta_1[C_2\cosh(f\delta_1)+C_1\sinh(f\delta_1)]}{C_1\cosh(f\delta_1)+C_2\sinh(f\delta_1)}\}}{\sqrt{3}}, \\ v_{21}=\frac{1}{3f'(\eta)^2}[f''(\eta)^2-f''(\eta)f'(\eta)-3f'(\eta)^2+\lambda f''(\eta)f'(\eta)^2 \\ \quad -2\mu f'(\eta)^4]+[\dfrac{2}{3}f''(\eta)-\dfrac{2}{3}\lambda f'(\eta)^2]\{-\dfrac{\lambda}{2}+\dfrac{\delta_1[C_2\cosh(f\delta_1)+C_1\sinh(f\delta_1)]}{C_1\cosh(f\delta_1)+C_2\sinh(f\delta_1)}\} \\ \quad -\dfrac{2}{3}\lambda f'(\eta)^2\{-\dfrac{\lambda}{2}+\dfrac{\delta_1[C_2\cosh(f\delta_1)+C_1\sinh(f\delta_1)]}{C_1\cosh(f\delta_1)+C_2\sinh(f\delta_1)}\}^2; \end{cases}$$

$$\begin{cases} u_{22} = \dfrac{-\sqrt{3}f''(\eta) + \frac{3Vf'(\eta)}{B(t)} + \sqrt{3}\lambda f'(\eta)^2}{3f'(\eta)} \\ \quad + \dfrac{2f'(\eta)\{-\frac{\lambda}{2} + \frac{\delta_2[C_2\cos(f\delta_2) - C_1\sin(f\delta_2)]}{C_1\cos(f\delta_2)+C_2\sin(f\delta_2)}\}}{\sqrt{3}}, \\ v_{22} = \dfrac{1}{3f'(\eta)^2}[f''(\eta)^2 - f''(\eta)f'(\eta) - 3f'(\eta)^2 + \lambda f''(\eta)f'(\eta)^2 \\ \quad -2\mu f'(\eta)^4] + [\dfrac{2}{3}f''(\eta) - \dfrac{2}{3}\lambda f'(\eta)^2]\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cos(f\delta_2) - C_1\sin(f\delta_2)]}{C_1\cos(f\delta_2) + C_2\sin(f\delta_2)}\} \\ \quad -\dfrac{2}{3}\lambda f'(\eta)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cos(f\delta_2) - C_1\sin(f\delta_2)]}{C_1\cos(f\delta_2) + C_2\sin(f\delta_2)}\}^2; \end{cases}$$

$$\begin{cases} u_{23} = \dfrac{2(-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2f(\eta)})f'(\eta)}{\sqrt{3}} + \dfrac{-\sqrt{3}f''(\eta) + \frac{3Vf'(\eta)}{B(t)} + \sqrt{3}\lambda f'(\eta)^2}{3f'(\eta)}, \\ v_{23} = -\dfrac{2}{3}(-\dfrac{\lambda}{2} + \dfrac{C_2}{C_1 + C_2f(\eta)})^2 f'(\eta)^2 + (-\dfrac{\lambda}{2} + \dfrac{C_2}{C_1 + C_2f(\eta)})[\dfrac{2}{3}f''(\eta) \\ \quad -\dfrac{2}{3}\lambda f'(\eta)^2] + \dfrac{1}{3f'(\eta)^2}[f''(\eta)^2 - f''(\eta)f'(\eta) \\ \quad -3f'(\eta)^2 + \lambda f''(\eta)f'(\eta)^2 - 2\mu f'(\eta)^4]. \end{cases}$$

4 Non-traveling wave solution

In this section, we will study two kinds of non-traveling wave solutions and then use the variable separation approach to get some of their solutions.

4.1 $\xi(x,t) = f(x)g(t)$

Assuming (1.1) has the following form of non-traveling wave solution:

$$\xi(x,t) = f(x)g(t), \tag{4.1}$$

where $f(x)$ and $g(t)$ are arbitrary functions of the indicated variables.

Hereby the general solutions of Eq.(2.4) are as follows:

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} -\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 fg)+C_2\cosh(\delta_1 fg)}{C_1\cosh(\delta_1 fg)+C_2\sinh(\delta_1 fg)}, & \lambda^2 - 4\mu > 0, \\ -\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 fg)+C_2\cos(\delta_2 fg)}{C_1\cos(\delta_2 fg)+C_2\sin(\delta_2 fg)}, & \lambda^2 - 4\mu < 0, \\ -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2f(\eta)}, & \lambda^2 - 4\mu = 0. \end{cases}$$

where $\delta_1 = \frac{\sqrt{\lambda^2-4\mu}}{2}$, $\delta_2 = \frac{\sqrt{4\mu-\lambda^2}}{2}$, C_1, C_2 are arbitrary constants.

Considering the homogenous balance between the highest order derivatives and the non-linear terms we have:

$$m = 1, n = 2, \tag{4.2}$$

So (2.3) can be written as follows (where a_0, a_1, b_0, b_1, b_2 are all constants):

$$\begin{cases} u(x,t) = a_0 + a_1(\frac{G'}{G}) \\ v(x,t) = b_0 + b_1(\frac{G'}{G}) + b_2(\frac{G'}{G})^2 \end{cases}, \tag{4.3}$$

Substituting (4.3), (2.3) into equations (1.1) we have: (Eqs.(4.4))

$$-a_1\mu B(t)g(t)f'(x) - a_1b_0\mu B(t)g(t)f'(x) - a_0b_1\mu B(t)g(t)f'(x) - b_1\mu f(x)g'(t)$$
$$+\frac{2}{3}a_1\lambda\mu B(t)g(t)^2f'(x)f''(x) - \frac{1}{3}\mu B(t)g(t)f'(x)[a_1\lambda^2g(t)^2f'(x)^2 + 2a_1\mu g(t)^2f'(x)^2$$
$$-a_1\lambda g(t)f''(x)] - \frac{1}{3}a_1\mu B(t)g(t)f^{(3)}(x) = 0,$$
$$-3a_1b_2B(t)g(t)f'(x) - 2a_1B(t)g(t)^3f'(x)^3 = 0,$$
$$-2a_1b_1B(t)g(t)f'(x) - 2a_0b_2B(t)g(t)f'(x) - 3a_1b_2\lambda B(t)g(t)f'(x)$$
$$-2a_1\lambda B(t)g(t)^3f'(x)^3 - 2b_2f(x)g'(t) + \frac{4}{3}a_1g(t)^2B(t)f'(x)f''(x)$$
$$-\frac{2}{3}B(t)g(t)f'(x)[4a_1\lambda g(t)^2f'(x)^2 - a_1g(t)f''(x)] = 0,$$
$$-a_1B(t)g(t)f'(x) - a_1b_0B(t)g(t)f'(x) - a_0b_1B(t)g(t)f'(x) - 2a_1b_1\lambda g(t)f'(x)$$
$$-2a_0b_2\lambda B(t)g(t)f'(x) - 3a_1b_2\mu B(t)g(t)f'(x) - 2a_1\mu B(t)g(t)^3f'(x)^3 - b_1f(x)g'(t)$$
$$-2b_2\lambda f(x)g'(t) - \frac{2}{3}\lambda B(t)g(t)f'(x)[4a_1\lambda g(t)^2f'(x)^2 - a_1g(t)f''(x)]$$
$$-\frac{1}{3}B(t)g(t)f'(x)(a_1\lambda^2g(t)^2f'(x)^2 + 2a_1\mu g(t)^2f'(x)^2 - a_1\lambda g(t)f''(x))$$
$$+\frac{1}{3}B(t)(6a_1\lambda g(t)^2f'(x)f''(x) - a_1g(t)f^{(3)}(x)) = 0,$$
$$-a_1\lambda B(t)g(t)f'(x) - a_1b_0\lambda B(t)g(t)f'(x) - a_0b_1\lambda B(t)g(t)f'(x) - 2a_1b_1\mu B(t)g(t)f'(x)$$
$$-2a_0b_2\mu B(t)g(t)f'(x) - b_1\lambda f(x)g'(t) - 2b_2\mu f(x)g'(t)$$
$$-\frac{2}{3}\mu B(t)g(t)f'(x)[4a_1\lambda g(t)^2f'(x)^2 - a_1g(t)f''(x)]$$
$$-\frac{1}{3}\lambda B(t)g(t)f'(x)[a_1\lambda^2g(t)^2f'(x)^2 + 2a_1\mu g(t)^2f'(x)^2 - a_1\lambda g(t)f''(x)]$$
$$+\frac{1}{3}B(t)[-a_1\lambda g(t)f''(x) + 2a_1\lambda^2g(t)^2f'(x)f''(x) + 4a_1\mu^2g(t)^2f'(x)f''(x)] = 0,$$
$$-a_0a_1\mu B(t)g(t)f'(x) - b_1\mu B(t)g(t)f'(x) - a_1\mu f(x)g'(t) = 0,$$
$$-{a_1}^2B(t)g(t)f'(x) - 2b_2B(t)g(t)f'(x) = 0,$$
$$-a_0a_1B(t)g(t)f'(x) - b_1B(t)g(t)f'(x) - {a_1}^2\lambda B(t)g(t)f'(x)$$

$$-2b_2\lambda B(t)g(t)f^{'}(x) - a_1 f(x)g^{'}(t) = 0,$$

$$-a_0a_1\lambda B(t)g(t)f^{'}(x) - b_1\lambda B(t)g(t)f^{'}(x) - {a_1}^2\mu B(t)g(t)f^{'}(x)$$

$$-2b_2\mu B(t)g(t)f^{'}(x) - a_1\lambda f(x)g^{'}(t) = 0,$$

By using the Mathematica software, Eqs.(4.4) can be solved. Since $g(t) \neq 0$, letting:

$$-\frac{8\mu B(t)g(t)^3 f^{'}(x)^2 f^{''}(x)}{3\sqrt{3}} + \frac{8\mu^2 B(t)g(t)^3 f^{'}(x)^2 f^{''}(x)}{3\sqrt{3}} = 0,$$

which can be reduced as follows:

$$(\mu^2 - \mu)f^{''}(x) = 0.$$

Then

$$f^{''}(x) = 0 \ \ or \ \ \mu = 0 \ \ or \ \ \mu = 1.$$

When $\mu = 0$, to ensure solutions of the equations were existed,

$$\lambda = 0 \ \ or \ \ f^{''}(x) - f^{(3)}(x) = 0.$$

while $\mu = 1$, we have

$$\lambda = 0 \ \ or \ \ f^{''}(x) - f^{(3)}(x) = 0.$$

Therefore, this section is partitioned into the five different parts to discuss solutions in different types:

I. $f^{''}(x) = 0$

II. $\mu = 0, \lambda = 0$

III. $\mu = 0, f^{''}(x) - f^{(3)}(x) = 0$

IV. $\mu = 1, \lambda = 0$

V. $\mu = 1, f^{''}(x) - f^{(3)}(x) = 0$

Type I: When $f^{''}(x) = 0$, then $f(x) = C_3 x + b$, we get:

Case 1:

$$a_0 = \frac{-4\sqrt{3}\lambda B(t)g(t)^2{C_3}^2 - 9(C_3x + b)g^{'}(t)}{9B(t)g(t)C_3}, a_1 = -\frac{2g(t)C_3}{\sqrt{3}},$$

$$b_0 = \frac{1}{27}(-27+7\lambda^2 g(t)^2{C_3}^2 - 18\mu g(t)^2{C_3}^2), b_1 = -\frac{8}{9}\lambda g(t)^2{C_3}^2, b_2 = -\frac{2}{3}g(t)^2{C_3}^2.$$

By means of the same method as above, novel solutions can be constructed as follows:

$$\begin{cases} u_{11} = -\dfrac{2C_3 g(t)\{-\frac{\lambda}{2} + \frac{\delta_1[C_2\cosh(\delta_1 fg)+C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg)+C_2\sinh(\delta_1 fg)}\}}{\sqrt{3}} \\ \qquad + \dfrac{-4\sqrt{3}{C_3}^2\lambda B(t)g(t)^2 - 9(b+C_3x)g'(t)}{9C_3B(t)g(t)}, \\ v_{11} = \dfrac{1}{27}[-27 + 7{C_3}^2\lambda^2 g(t)^2 - 18{C_3}^2\mu g(t)^2] \\ \qquad -\dfrac{8}{9}{C_3}^2\lambda g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_1[C_2\cosh(\delta_1 fg) + C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg) + C_2\sinh(\delta_1 fg)}\} \\ \qquad -\dfrac{2}{3}{C_3}^2 g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_1[C_2\cosh(\delta_1 fg) + C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg) + C_2\sinh(\delta_1 fg)}\}^2; \end{cases}$$

$$\begin{cases} u_{12} = -\dfrac{2C_3 g(t)\{-\frac{\lambda}{2} + \frac{\delta_2[C_2\cos(\delta_2 fg)-C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg)+C_2\sin(\delta_2 fg)}\}}{\sqrt{3}} \\ \qquad + \dfrac{-4\sqrt{3}{C_3}^2\lambda B(t)g(t)^2 - 9(b+C_3x)g'(t)}{9C_3B(t)g(t)}, \\ v_{12} = \dfrac{1}{27}[-27 + 7{C_3}^2\lambda^2 g(t)^2 - 18{C_3}^2\mu g(t)^2] \\ \qquad -\dfrac{8}{9}{C_3}^2\lambda g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cos(\delta_2 fg) - C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg) + C_2\sin(\delta_2 fg)}\} \\ \qquad -\dfrac{2}{3}{C_3}^2 g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cos(\delta_2 fg) - C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg) + C_2\sin(\delta_2 fg)}\}^2; \end{cases}$$

$$\begin{cases} u_{13} = -\dfrac{2C_3[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2f(x)g(t)}g(t)]}{\sqrt{3}} \\ \qquad + \dfrac{-4\sqrt{3}{C_3}^2\lambda B(t)g(t)^2 - 9(b+C_3x)g'(t)}{9C_3B(t)g(t)}, \\ v_{13} = -\frac{8}{9}{C_3}^2\lambda[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2f(x)g(t)}]g(t)^2 - \dfrac{2}{3}{C_3}^2[-\dfrac{\lambda}{2} + \dfrac{C_2}{C_1+C_2f(x)g(t)}]^2 g(t)^2 \\ \qquad + \dfrac{1}{27}[-27 + 7{C_3}^2\lambda^2 g(t)^2 - 18{C_3}^2\mu g(t)^2]. \end{cases}$$

Case 2:

$$a_0 = \frac{4\sqrt{3}\lambda B(t)g(t)^2{C_3}^2 - 9(C_3x+b)g'(t)}{9B(t)g(t)C_3}, a_1 = \frac{2g(t)C_3}{\sqrt{3}},$$

$$b_0 = \frac{1}{27}[-27+7\lambda^2 g(t)^2{C_3}^2 - 18\mu g(t)^2{C_3}^2], b_1 = -\frac{8}{9}\lambda g(t)^2{C_3}^2, b_2 = -\frac{2}{3}g(t)^2{C_3}^2.$$

Calculating Case 2 with aforementioned method, we get:

$$\begin{cases} u_{21} = \dfrac{2C_3 g(t)\{-\frac{\lambda}{2} + \frac{\delta_1[C_2\cosh(\delta_1 fg)+C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg)+C_2\sinh(\delta_1 fg)}\}}{\sqrt{3}} \\ \qquad + \dfrac{4\sqrt{3}{C_3}^2\lambda B(t)g(t)^2 - 9(b+C_3x)g'(t)}{9C_3B(t)g(t)}, \\ v_{12} = \dfrac{1}{27}[-27+7{C_3}^2\lambda^2 g(t)^2 - 18{C_3}^2\mu g(t)^2] \\ \qquad -\dfrac{8}{9}{C_3}^2\lambda g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_1[C_2\cosh(\delta_1 fg)+C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg)+C_2\sinh(\delta_1 fg)}\} \\ \qquad -\dfrac{2}{3}{C_3}^2 g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_1[C_2\cosh(\delta_1 fg)+C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg)+C_2\sinh(\delta_1 fg)}\}^2; \end{cases}$$

$$\begin{cases} u_{22} = \dfrac{2C_3 g(t)\{-\frac{\lambda}{2} + \frac{\delta_2[C_2\cos(\delta_2 fg)-C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg)+C_2\sin(\delta_2 fg)}\}}{\sqrt{3}} \\ \qquad + \dfrac{4\sqrt{3}{C_3}^2\lambda B(t)g(t)^2 - 9(b+C_3x)g'(t)}{9C_3B(t)g(t)}, \\ v_{22} = \frac{1}{27}[-27+7{C_3}^2\lambda^2 g(t)^2 - 18{C_3}^2\mu g(t)^2] \\ \qquad -\dfrac{8}{9}{C_3}^2\lambda g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cos(\delta_2 fg)-C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg)+C_2\sin(\delta_2 fg)}\} \\ \qquad -\dfrac{2}{3}{C_3}^2 g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cos(\delta_2 fg)-C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg)+C_2\sin(\delta_2 fg)}\}^2; \end{cases}$$

$$\begin{cases} u_{23} = \dfrac{2C_3 g(t)[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2f(x)g(t)}]}{\sqrt{3}} \\ \qquad + \dfrac{4\sqrt{3}{C_3}^2\lambda B(t)g(t)^2 - 9(b+C_3x)g'(t)}{9C_3B(t)g(t)}, \\ v_{23} = -\dfrac{8}{9}{C_3}^2\lambda g(t)^2[-\dfrac{\lambda}{2} + \dfrac{C_2}{C_1+C_2f(x)g(t)}] - \dfrac{2}{3}{C_3}^2 g(t)^2[-\dfrac{\lambda}{2} + \dfrac{C_2}{C_1+C_2f(x)g(t)}]^2 \\ \qquad + \dfrac{1}{27}[-27+7{C_3}^2\lambda^2 g(t)^2 - 18{C_3}^2\mu g(t)^2]. \end{cases}$$

Type II: If $\lambda = 0, \mu = 0$, we can yield:
Case 1:

$$a_0 = -\frac{f(x)g'(t)}{B(t)g(t)f'(x)} + \frac{f''(x)}{\sqrt{3}f'(x)}, a_1 = -\frac{2g(t)f'(x)}{\sqrt{3}},$$

$$b_0 = \frac{-27f'(x)^2 + 9f''(x)^2 - 9f'(x)f^{(3)}(x)}{27f'(x)^2}, b_1 = \frac{2}{3}g(t)f''(x), b_2 = -\frac{2}{3}g(t)^2f'(x)^2.$$

In this situation, we can obtain the NTWS for rational functions:

$$\begin{cases} u_{11} = -\dfrac{2C_2 g(t) f'(x)}{\sqrt{3}[C_1 + C_2 f(x) g(t)]} - \dfrac{f(x) g'(t)}{B(t) g(t) f'(x)} + \dfrac{f''(x)}{\sqrt{3} f'(x)}, \\ v_{11} = -\dfrac{2{C_2}^2 g(t)^2 f'(x)^2}{3[C_1 + C_2 f(x) g(t)]^2} + \dfrac{2C_2 g(t) f''(x)}{3[C_1 + C_2 f(x) g(t)]} \\ \quad + \dfrac{-27 f'(x)^2 + 9 f''(x)^2 - 9 f'(x) f^{(3)}(x)}{27 f'(x)^2}. \end{cases}$$

Case 2:

$$a_0 = -\frac{f(x) g'(t)}{B(t) g(t) f'(x)} - \frac{f''(x)}{\sqrt{3} f'(x)}, a_1 = \frac{2g(t) f'(x)}{\sqrt{3}},$$

$$b_0 = \frac{-27 f'(x)^2 + 9 f''(x)^2 - 9 f'(x) f^{(3)}(x)}{27 f'(x)^2}, b_1 = \frac{2}{3} g(t) f''(x), b_2 = -\frac{2}{3} g(t)^2 f'(x)^2.$$

In this situation, $\lambda^2 - 4\mu = 0$, we can obtain the NTWS for rational functions:

$$\begin{cases} u_{21} = \dfrac{2C_2 g(t) f'(x)}{\sqrt{3}[C_1 + C_2 f(x) g(t)]} - \dfrac{f(x) g'(t)}{B(t) g(t) f'(x)} - \dfrac{f''(x)}{\sqrt{3} f'(x)}, \\ v_{21} = -\dfrac{2{C_2}^2 g(t)^2 f'(x)^2}{3[C_1 + C_2 f(x) g(t)]^2} + \dfrac{2C_2 g(t) f''(x)}{3[C_1 + C_2 f(x) g(t)]} \\ \quad + \dfrac{-27 f'(x)^2 + 9 f''(x)^2 - 9 f'(x) f^{(3)}(x)}{27 f'(x)^2}. \end{cases}$$

Type III: When $\mu = 0$, $f''(x) = f^{(3)}(x)$, we have:
Case 1:

$$a_0 = -\frac{-3 f''(x) + 4\lambda f'(x)^2 g(t)}{3\sqrt{3} f'(x)} - \frac{f(x) g'(t)}{B(t) f'(x) g(t)}, a_1 = -\frac{2 f'(x) g(t)}{\sqrt{3}},$$

$$b_0 = \frac{1}{27 f'(x)^2}[9 f''(x)^2 - 9 f''(x) f'(x) - 27 f'(x)^2$$

$$+3\lambda f''(x) f'(x)^2 g(t) + 7\lambda^2 f'(x)^4 g(t)^2],$$

$$b_1 = -\frac{2}{9} g(t)[-3 f''(x) + 4\lambda f'(x)^2 g(t)], b_2 = -\frac{2}{3} f'(x)^2 g(t)^2.$$

$$\begin{cases} u_{11} = -\dfrac{-3f''(x)+4\lambda f'(x)^2 g(t)}{3\sqrt{3}f'(x)} - \dfrac{f(x)g'(t)}{B(t)f'(x)g(t)} \\ \quad -\dfrac{2f'(x)g(t)\{-\frac{\lambda}{2}+\frac{\frac{|\lambda|}{2}[C_2\cosh(\frac{|\lambda|}{2}fg)+C_1\sinh(\frac{|\lambda|}{2}fg)]}{C_1\cosh(\frac{|\lambda|}{2}fg)+C_2\sinh(\frac{|\lambda|}{2}fg)}\}}{\sqrt{3}}, \\ v_{11} = \dfrac{1}{27f'(x)^2}[9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2 + 3\lambda f''(x)f'(x)^2 g(t) \\ \quad +7\lambda^2 f'(x)^4 g(t)^2] - \dfrac{2}{9}g(t)[-3f''(x)+4\lambda f'(x)^2 g(t)] \\ \quad \{-\dfrac{\lambda}{2} + \dfrac{\frac{|\lambda|}{2}[C_2\cosh(\frac{|\lambda|}{2}fg)+C_1\sinh(\frac{|\lambda|}{2}fg)]}{C_1\cosh(\frac{|\lambda|}{2}fg)+C_2\cosh(\frac{|\lambda|}{2}fg)}\} \\ \quad -\dfrac{2}{3}f'(x)^2 g(t)^2(-\dfrac{\lambda}{2}+\dfrac{\frac{|\lambda|}{2}(C_1\cosh(\frac{|\lambda|}{2}fg)+C_2\sinh(\frac{|\lambda|}{2}fg))}{C_1\cosh(\frac{|\lambda|}{2}fg)+C_2\sinh(\frac{|\lambda|}{2}fg)})^2; \end{cases}$$

$$\begin{cases} u_{12} = -\dfrac{2[\frac{C_2}{C_1+C_2f(x)g(t)}]f'(x)g(t)}{\sqrt{3}} - \dfrac{-3f''(x)}{3\sqrt{3}f'(x)} - \dfrac{f(x)g'(t)}{B(t)f'(x)g(t)}, \\ v_{12} = -\dfrac{2}{3}[\dfrac{C_2}{C_1+C_2f(x)g(t)}]^2 f'(x)^2 g(t)^2 + \dfrac{2}{3}[\dfrac{f''(x)C_2 g(t)}{C_1+C_2f(x)g(t)}] \\ \quad +\dfrac{1}{27f'(x)^2}[9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2]^2. \end{cases}$$

Case 2:

$$a_0 = \frac{-3f''(x)+4\lambda f'(x)^2 g(t)}{3\sqrt{3}f'(x)} - \frac{f(x)g'(t)}{B(t)f'(x)g(t)}, a_1 = \frac{2f'(x)g(t)}{\sqrt{3}},$$

$$b_0 = \frac{1}{27f'(x)^2}[9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2$$

$$+3\lambda f''(x)f'(x)^2 g(t) + 7\lambda^2 f'(x)^4 g(t)^2],$$

$$b_1 = -\frac{2}{9}g(t)[-3f''(x)+4\lambda f'(x)^2 g(t)], b_2 = -\frac{2}{3}f'(x)^2 g(t)^2.$$

Using the same procedure as Result 1, we have,

$$\begin{cases} u_{21} = \dfrac{-3f''(x)+4\lambda f'(x)^2 g(t)}{3\sqrt{3}f'(x)} - \dfrac{f(x)g'(t)}{B(t)f'(x)g(t)} \\ \quad + \dfrac{2f'(x)g(t)\{-\frac{\lambda}{2}+\frac{\frac{|\lambda|}{2}[C_2\cosh(\frac{|\lambda|}{2}fg)+C_1\sinh(\frac{|\lambda|}{2}fg)]}{C_1\cosh(\frac{|\lambda|}{2}fg)+C_2\sinh(\frac{|\lambda|}{2}fg)}\}}{\sqrt{3}}, \\ v_{21} = \dfrac{1}{27f'(x)^2}[9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2 + 3\lambda f''(x)f'(x)^2 g(t) \\ \quad +7\lambda^2 f'(x)^4 g(t)^2] - \dfrac{2}{9}g(t)[-3f''(x)+4\lambda f'(x)^2 g(t)] \\ \quad \{-\dfrac{\lambda}{2}+\dfrac{\frac{|\lambda|}{2}[C_2\cosh(\frac{|\lambda|}{2}fg)+C_1\sinh(\frac{|\lambda|}{2}fg)]}{C_1\cosh(\frac{|\lambda|}{2}fg)+C_2\sinh(\frac{|\lambda|}{2}fg)}\} \\ \quad -\dfrac{2}{3}f'(x)^2 g(t)^2\{-\dfrac{\lambda}{2}+\dfrac{\frac{|\lambda|}{2}[C_1\cosh(\frac{|\lambda|}{2}fg)+C_2\sinh(\frac{|\lambda|}{2}fg)]}{C_1\cosh(\frac{|\lambda|}{2}fg)+C_2\sinh(\frac{|\lambda|}{2}fg)}\}^2; \end{cases}$$

$$\begin{cases} u_{22} = \dfrac{2[\frac{C_2}{C_1+C_2 f(x)g(t)}]f'(x)g(t)}{\sqrt{3}} + \dfrac{-3f''(x)}{3\sqrt{3}f'(x)} - \dfrac{f(x)g'(t)}{B(t)f'(x)g(t)}, \\ v_{22} = -\dfrac{2}{3}[\dfrac{C_2}{C_1+C_2 f(x)g(t)}]^2 f'(x)^2 g(t)^2 + \dfrac{2}{3}[\dfrac{f''(x)C_2 g(t)}{C_1+C_2 f(x)g(t)}] \\ \quad +\dfrac{1}{27f'(x)^2}[9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2]^2. \end{cases}$$

Type IV: Substituting $\mu = 1, \lambda = 0$ into Eq. (4.4), we have:
Case 1:

$$a_0 = -\frac{f(x)g'(t)}{B(t)g(t)f'(x)} + \frac{f''(x)}{\sqrt{3}f'(x)}, a_1 = -\frac{2g(t)f'(x)}{\sqrt{3}},$$

$$b_0 = \frac{-3f'(x)^2 - 2g(t)^2 f'(x)^4 + f''(x)^2 - f'(x)f^{(3)}(x)}{3f'(x)^2},$$

$$b_1 = \frac{2}{3}g(t)f''(x), b_2 = -\frac{2}{3}g(t)^2 f'(x)^2.$$

In this condition, $\lambda^2 - 4\mu = -4 < 0$, we can obtain the NTWS for trigonometric functions:

$$\begin{cases} u_{11} = -\dfrac{2g(t)[C_1\cos(fg) - C_1\sin(fg)]f'(x)}{\sqrt{3}[C_1\cos(fg)+C_2\sin(fg)]} - \dfrac{f(x)g'(t)}{B(t)g(t)f'(x)} + \dfrac{f''(x)}{\sqrt{3}f'(x)}, \\ v_{11} = -\dfrac{2g(t)^2[C_2\cos(fg) - C_1\sin(fg)]^2 f'(x)^2}{3[C_1\cos(fg)+C_2\sin(fg)]^2} + \dfrac{2g(t)[C_2\cos(fg) - C_2\sin(fg)]f''(x)}{3[C_1\cos(fg)+C_2\sin(fg)]} \\ \quad +\dfrac{-3f'(x)^2 - 2g(t)^2 f'(x)^4 + f''(x)^2 - f'(x)f^{(3)}(x)}{3f'(x)^2}. \end{cases}$$

Case 2:

$$a_0 = -\frac{f(x)g'(t)}{B(t)g(t)f'(x)} - \frac{f''(x)}{\sqrt{3}f'(x)}, a_1 = \frac{2g(t)f'(x)}{\sqrt{3}},$$

$$b_0 = \frac{-3f'(x)^2 - 2g(t)^2 f'(x)^4 + f''(x)^2 - f'(x)f^{(3)}(x)}{3f'(x)^2},$$

$$b_1 = \frac{2}{3}g(t)f''(x), b_2 = -\frac{2}{3}g(t)^2 f'(x)^2.$$

Since $\lambda^2 - 4\mu = -4 < 0$, we can obtain the NTWS for trigonometric functions:

$$\begin{cases} u_{12} = \dfrac{2g(t)[C_1\cos(fg) - C_1\sin(fg)]f'(x)}{\sqrt{3}[C_1\cos(fg) + C_2\sin(fg)]} - \dfrac{f(x)g'(t)}{B(t)g(t)f'(x)} - \dfrac{f''(x)}{\sqrt{3}f'(x)}, \\ v_{12} = -\dfrac{2g(t)[C_2\cos(fg) - C_1\sin(fg)]^2 f'(x)^2}{3[C_1\cos(fg) + C_2\sin(fg)]^2} + \dfrac{2g(t)[C_2\cos(fg) - C_2\sin(fg)]f''(x)}{3[C_1\cos(fg) + C_2\sin(fg)]} \\ \quad + \dfrac{-3f'(x)^2 - 2g(t)^2 f'(x)^4 + f''(x)^2 - f'(x)f^{(3)}(x)}{3f'(x)^2}. \end{cases}$$

Type V: When $\mu = 1, f''(x) = f^{(3)}(x)$, we have:

$$a_0 = -\frac{f''(x)}{\sqrt{3}f'(x)} + \frac{4\lambda f'(x)g(t)}{3\sqrt{3}} - \frac{f(x)g'(t)}{B(t)f'(x)g(t)},$$

$$a_1 = \frac{2f'(x)g(t)}{\sqrt{3}}, b_0 = \frac{1}{9}\lambda f''(x)g(t) + \frac{7}{27}\lambda^2 f'(x)^2 g(t)^2$$

$$+[9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2 - 18f'(x)^4 g(t)^2]/[27f'(x)^2],$$

$$b_1 = \frac{2}{3}f''(x)g(t) - \frac{8}{9}\lambda f'(x)^2 g(t)^2, b_2 = -\frac{2}{3}f'(x)^2 g(t)^2.$$

Similarly, we have:

$$\begin{cases} u_{21} = \dfrac{-3f''(x) + 4\lambda f'(x)^2 g(t)}{3\sqrt{3}f'(x)} - \dfrac{f(x)g'(t)}{B(t)f'(x)g(t)} \\ \quad + \dfrac{2f'(x)g(t)\{-\frac{\lambda}{2} + \frac{\delta_1[C_2\cosh(\delta_1 fg) + C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg) + C_2\sinh(\delta_1 fg)}\}}{\sqrt{3}}, \\ v_{21} = [9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2 + 3\lambda f''(x)f'(x)^2 g(t) \\ \quad -18f'(x)^4 g(t)^2 + 7\lambda^2 f'(x)^4 g(t)^2]/[27f'(x)^2] \\ \quad -\dfrac{2}{9}g(t)[-3f''(x) + 4\lambda f'(x)^2 g(t)]\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cosh(\delta_1 fg) + C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg) + C_2\sinh(\delta_1 fg)}\} \\ \quad -\dfrac{2}{3}f'(x)^2 g(t)^2\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cosh(\delta_1 fg) + C_1\sinh(\delta_1 fg)]}{C_1\cosh(\delta_1 fg) + C_2\sinh(\delta_1 fg)}\}^2; \end{cases}$$

$$\begin{cases} u_{22} = \dfrac{-3f''(x) + 4\lambda f'(x)^2 g(t)}{3\sqrt{3}f'(x)} - \dfrac{f(x)g'(t)}{B(t)f'(x)g(t)} \\ \quad + \dfrac{2f'(x)g(t)\{-\frac{\lambda}{2} + \frac{\delta_2[C_2\cos(\delta_2 fg) - C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg) + C_2\sin(\delta_2 fg)}\}}{\sqrt{3}}, \\ v_{22} = [9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2 + 3\lambda f''(x)f'(x)^2 g(t) \\ \quad -18f'(x)^4 g(t)^2 + 7\lambda^2 f'(x)^4 g(t)^2]/[27f'(x)^2] \\ \quad -\dfrac{2}{9}g(t)[-3f''(x) + 4\lambda f'(x)^2 g(t)]\{-\dfrac{\lambda}{2} + \dfrac{\delta_2[C_2\cos(\delta_2 fg) - C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg) + C_2\sin(\delta_2 fg)}\} \\ \quad -\dfrac{2}{3}f'(x)^2 g(t)^2\{-\dfrac{\lambda}{2} + \delta_2\dfrac{[C_2\cos(\delta_2 fg) - C_1\sin(\delta_2 fg)]}{C_1\cos(\delta_2 fg) + C_2\sin(\delta_2 fg)}\}^2; \end{cases}$$

When $\lambda = \pm 2$,
(i)$\lambda = 2$:

$$\begin{cases} u_{23} = \dfrac{2[-1 + \frac{C_2}{C_1 + C_2 f(x)g(t)}]f'(x)g(t)}{\sqrt{3}} + \dfrac{-3f''(x) + 8f'(x)^2 g(t)}{3\sqrt{3}f'(x)} - \dfrac{f(x)g'(t)}{B(t)f'(x)g(t)}, \\ v_{23} = -\dfrac{2}{3}[-1 + \dfrac{C_2}{C_1 + C_2 f(x)g(t)}]^2 f'(x)^2 g(t)^2 \\ \quad -\dfrac{2}{9}[-1 + \dfrac{C_2}{C_1 + C_2 f(x)g(t)}]g(t)[-3f''(x) + 8f'(x)^2 g(t)] \\ \quad +[9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2 \\ \quad +6f''(x)f'(x)^2 g(t) + 10f'(x)^4 g(t)^2]/[27f'(x)^2]; \end{cases}$$

(ii)$\lambda = -2$:

$$\begin{cases} u_{24} = \dfrac{2[1 + \frac{C_2}{C_1 + C_2 f(x)g(t)}]f'(x)g(t)}{\sqrt{3}} + \dfrac{-3f''(x) - 8f'(x)^2 g(t)}{3\sqrt{3}f'(x)} - \dfrac{f(x)g'(t)}{B(t)f'(x)g(t)}, \\ v_{24} = -\dfrac{2}{3}[1 + \dfrac{C_2}{C_1 + C_2 f(x)g(t)}]^2 f'(x)^2 g(t)^2 \\ \quad -\dfrac{2}{9}[1 + \dfrac{C_2}{C_1 + C_2 f(x)g(t)}]g(t)[-3f''(x) - 8f'(x)^2 g(t)] \\ \quad +[9f''(x)^2 - 9f''(x)f'(x) - 27f'(x)^2 \\ \quad -6f''(x)f'(x)^2 g(t) + 10f'(x)^4 g(t)^2]/[27f'(x)^2] \end{cases}$$

4.2 $\xi(x,t) = f(x) + g(t)$

We can assume that (1.1) has the solution in the form of NTWS:

$$\xi(x,t) = f(x) + g(t), \tag{4.5}$$

where $f(x)$ and $g(t)$ are arbitrary functions of the indicated variables.

Therefore the general solutions of Eq. (2.4) are as follows:

$$\frac{G'(\xi)}{G(\xi)}=\begin{cases}-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1(f+g))+C_2\cosh(\delta_1(f+g))}{C_1\cosh(\delta_1(f+g))+C_2\sinh(\delta_1(f+g))}, & \lambda^2-4\mu>0,\\ -\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2(f+g))+C_2\cos(\delta_2(f+g))}{C_1\cos(\delta_2(f+g))+C_2\sin(\delta_2(f+g))}, & \lambda^2-4\mu<0,\\ -\frac{\lambda}{2}+\frac{C_2}{C_1+C_2f(\eta)}, & \lambda^2-4\mu=0.\end{cases}$$

where $\delta_1=\frac{\sqrt{\lambda^2-4\mu}}{2}$, $\delta_2=\frac{\sqrt{4\mu-\lambda^2}}{2}$, C_1, C_2 are arbitrary constants.

Substituting (4.5), (2.3) into equations (1.1), we have (Eqs.(4.6)):

$$-b_1g'(t)\mu-a_1f'(x)\mu B(t)-a_1b_0f'(x)\mu B(t)-a_0b_1f(x)\mu B(t)$$
$$+\frac{1}{3}a_1f''(x)\lambda\mu B(t)-\frac{1}{3}a_1f(x)^2\lambda^2\mu B(t)-\frac{2}{3}a_1f'(x)^2\mu^2B(t)=0,$$
$$-3a_1b_2f(x)B(t)-2a_1f'(x)^2B(t)=0,$$
$$-2b_2g'(t)-2a_1b_1f(x)B(t)-2a_0b_2f(x)B(t)+\frac{2}{3}a_1f''(x)B(t)$$
$$-3a_1b_2f'(x)\lambda B(t)-4a_1f'(x)^2\lambda B(t)=0,$$
$$-2b_2g'(t)-2a_1b_1f'(x)B(t)-2a_0b_2f'(x)B(t)+\frac{2}{3}a_1f''(x)B(t)$$
$$-3a_1b_2f'(x)\lambda B(t)-4a_1f'(x)^2\lambda B(t)=0,$$
$$-b_1g'(t)\lambda-2b_2g'(t)\mu-a_1f'(x)\lambda B(t)-a_1b_0f'(x)\lambda B(t)$$
$$-a_0b_1f'(x)\lambda B(t)+\frac{1}{3}a_1f''(x)\lambda^2B(t)-\frac{1}{3}a_1f'(x)^2\lambda^3B(t)-2a_1b_1f'(x)\mu B(t)$$
$$-2a_0b_2f'(x)\mu B(t)+\frac{2}{3}a_1f''(x)\mu B(t)-\frac{8}{3}a_1f'(x)^2\lambda\mu B(t)=0,$$
$$-a_1g'(t)\mu-a_0a_1f'(x)\mu B(t)-b_1f'(x)\mu B(t)=0,$$
$$-{a_1}^2f'(x)B(t)-2b_2f'(x)B(t)=0,$$
$$-a_1g'(t)-a_0a_1f'(x)B(t)-b_1f'(x)B(t)-{a_1}^2f'(x)\lambda B(t)$$
$$-2b_2f'(x)\lambda B(t)=0,$$
$$-a_1g'(t)\lambda-a_0a_1f'(x)\lambda B(t)-b_1f'(x)\lambda B(t)-{a_1}^2f'(x)\mu B(t)-2b_2f'(x)\mu B(t)=0.$$

Eqs.(4.6) have solutions if and only if it satisfies the following:

$$-\lambda[\frac{8B(t)f'(x)^3}{\sqrt{3f(x)}}-\frac{4B(t)f'(x)^4}{\sqrt{3}f(x)^{3/2}}]=\frac{4f'(x)^2g'(t)}{3f(x)}-\frac{4B(t)f'(x)f''(x)}{3\sqrt{3}\sqrt{f(x)}}$$

$$\lambda^2[-\frac{2f(x)^{3/2}f'(x)}{3\sqrt{3}}+\frac{2f'(x)^3}{3\sqrt{3}\sqrt{f(x)}}]+\mu[-\frac{4f'(x)^3}{\sqrt{3}\sqrt{f(x)}}+\frac{4f'(x)^4}{\sqrt{3}f(x)^{3/2}}]=0,$$

which is:

$$\frac{2\lambda B(t)f'(x)^2}{\sqrt{3}} + \frac{\lambda B(t)f'(x)^3}{\sqrt{3}f(x)} - \frac{f'(x)g'(t)}{3f(x)} + \frac{B(t)f''(x)}{3\sqrt{3}\sqrt{f(x)}} = 0, \qquad (*)$$

$$\lambda^2[-\frac{f(x)^2}{3} + \frac{f'(x)^2}{3}] + \mu[-2f'(x)^2 + \frac{2f'(x)^3}{f(x)}] = 0, \qquad (**)$$

In order to make sure $g'(t)$ and $B(t)$ must be only for t, the ratios of the $B(t)$, $g'(t)$ must be functions of t.

$$\frac{B(t)}{g'(t)}[6\lambda f'(x)f(x) + 3\lambda f'(x)^3 + f'(x)] = \sqrt{3}f'(x)$$

That's to say, the following constrained conditions are satisfied:

$$6\lambda f'(x)f(x) + 3\lambda f'(x)^2 + 1 = C$$

Hereby we get, $B(t) = \frac{\sqrt{3}}{C}g'(t)$.

Substituting $B(t)$ into Eq.(**), we have three different solutions:

$$f(x) = C_3e^x, f(x) = e^{\frac{x(-\lambda^2-\lambda\sqrt{\lambda^2-24\mu})}{12\mu}}C_3, f(x) = e^{\frac{x(-\lambda^2+\lambda\sqrt{\lambda^2-24\mu})}{12\mu}}C_3.$$

(1) Substituting $f(x) = e^{\frac{x(-\lambda^2-\lambda\sqrt{\lambda^2-24\mu})}{12\mu}}C_3$, $f(x) = e^{\frac{x(-\lambda^2+\lambda\sqrt{\lambda^2-24\mu})}{12\mu}}C_3$ into the Eq.(4.6), we yield $b_2 = 0$, which is not valid according to the constraints above.
(2) Substituting $f(x) = C_3e^x$ into the equations(4.6), we get:

$$9\lambda {C_3}^2e^{2x} + 1 = C.$$

The equation is tenable only if $\lambda = 0$. Therefore,

$$g'(t) = \frac{\sqrt{3}}{3}B(t).$$

Solving the Eq.(4.6), we can derive the following result:
Case 1:

$$a_0 = \frac{e^{-x}(\sqrt{3}\sqrt{C}e^{x/2} - 3\sqrt{3})}{9C}, a_1 = -\frac{2\sqrt{C}e^{x/2}}{\sqrt{3}},$$

$$b_0 = \frac{-27C + e^{-x} - 18C^2e^x\mu}{27C}, b_1 = \frac{2}{9}, b_2 = -\frac{2Ce^x}{3}.$$

we can obtain the following solutions:

$$\begin{cases} u_{11} = -\dfrac{2\sqrt{C}e^{x/2}\{\frac{\sqrt{\mu}[C_2\cos(\sqrt{\mu}(f+g))-C_1\sin(\sqrt{\mu}(f+g))]}{C_1\cos(\sqrt{\mu}(f+g))+C_2\sin(\sqrt{\mu}(f+g))}\}}{\sqrt{3}} \\ \quad +\dfrac{e^{-x}(\sqrt{3}\sqrt{C}e^{x/2}-3\sqrt{3})}{9C}, \\ v_{11} = \dfrac{-27C+e^{-x}-18C^2e^x\mu}{27C_1} + \dfrac{2}{9}\{\dfrac{\sqrt{\mu}[C_2\cos(\sqrt{\mu}(f+g))-C_1\sin(\sqrt{\mu}(f+g))]}{C_1\cos(\sqrt{\mu}(f+g))+C_2\sin(\sqrt{\mu}(f+g))}\} \\ \quad -\dfrac{2}{3}Ce^x\{\dfrac{\sqrt{\mu}[C_2\cos(\sqrt{\mu}(f+g))-C_1\sin(\sqrt{\mu}(f+g))]}{C_1\cos(\sqrt{\mu}(f+g))+C_2\sin(\sqrt{\mu}(f+g))}\}^2; \end{cases}$$

$$\begin{cases} u_{12} = -\dfrac{2\sqrt{C}e^{x/2}\{\frac{C_2}{C_1+C_2[f(x)+g(t)]}\}}{\sqrt{3}} + \dfrac{e^{-x}(\sqrt{3}\sqrt{C}e^{x/2}-3\sqrt{3})}{9C}, \\ v_{12} = \dfrac{-27C+e^{-x}-18C^2e^x\mu}{27C_1} + \dfrac{2}{9}\{\dfrac{C_2}{C_1+C_2[f(x)+g(t)]}\} \\ \quad -\dfrac{2}{3}Ce^x\{\dfrac{C_2}{C_1+C_2[f(x)+g(t)]}\}^2. \end{cases}$$

Case 2:

$$a_0 = -\frac{e^{-x}(\sqrt{3}\sqrt{C}e^{x/2}+3\sqrt{3})}{9C}, a_1 = \frac{2\sqrt{C}e^{x/2}}{\sqrt{3}},$$

$$b_0 = \frac{-27C+e^{-x}-18C^2e^x\mu}{27C}, b_1 = \frac{2}{9}, b_2 = -\frac{2Ce^x}{3}.$$

Similarly, we have

$$\begin{cases} u_{21} = \dfrac{2\sqrt{C}e^{x/2}\{\frac{\sqrt{\mu}[C_2\cos(\sqrt{\mu}(f+g))-C_1\sin(\sqrt{\mu}(f+g))]}{C_1\cos(\sqrt{\mu}(f+g))+C_2\sin(\sqrt{\mu}(f+g))}\}}{\sqrt{3}} \\ \quad +\dfrac{e^{-x}(\sqrt{3}\sqrt{C}e^{x/2}+3\sqrt{3})}{9C}, \\ v_{21} = \dfrac{-27C+e^{-x}-18C^2e^x\mu}{27C_1} + \dfrac{2}{9}\{\dfrac{\sqrt{\mu}[C_2\cos(\sqrt{\mu}(f+g))-C_1\sin(\sqrt{\mu}(f+g))]}{C_1\cos(\sqrt{\mu}(f+g))+C_2\sin(\sqrt{\mu}(f+g))}\} \\ \quad -\frac{2}{3}Ce^x\{\frac{\sqrt{\mu}[C_2\cos(\sqrt{\mu}(f+g))-C_1\sin(\sqrt{\mu}(f+g))]}{C_1\cos(\sqrt{\mu}(f+g))+C_2\sin(\sqrt{\mu}(f+g))}\}^2 \end{cases}$$

$$\begin{cases} u_{22} = \dfrac{2\sqrt{C}e^{x/2}\{\frac{C_2}{C_1+C_2[f(x)+g(t)]}\}}{\sqrt{3}} - \dfrac{e^{-x}(\sqrt{3}\sqrt{C}e^{x/2}+3\sqrt{3})}{9C}, \\ v_{22} = \dfrac{-27C+e^{-x}-18C^2e^x\mu}{27C_1} + \dfrac{2}{9}\{\dfrac{C_2}{C_1+C_2[f(x)+g(t)]}\} \\ \quad -\dfrac{2}{3}Ce^x\{\dfrac{C_2}{C_1+C_2[f(x)+g(t)]}\}^2. \end{cases}$$

5 Conclusion

With (G'/G)-expansion method, we have successfully constructed three types of traveling wave solutions in terms of hyperbolic, trigometric, and rational functions for the (1+1)-dimensional Boussinesq equations with variable coefficients. Moreover, the non-traveling wave solutions in variable separation form are also successfully established. Especially, the solutions in variable separation form we constructed have many potential applications in physics and engineering. These methods can also be applied in obtaining exact solutions of other kinds of equations.

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