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A remark on the ass and mule problem

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Abstract

In this paper, we present the existence of integral solutions on some cases for the Doslic's problem.

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1 Introduction and Preliminaries

The classical ass and mule problem seeks for the total number of sacks satisfying the two conditions: (1) if the ass give a sack into the mule then the mule has as many as the ass, and (2) if the mule give a sack into the ass then the ass has twice as many as the mule.

The general version of the classical ass and mule problem seeks for the total number of sacks satisfying the two conditions: (1) if I had a from you then I'd have b times you, and (2) if you had c from me then you'd have d times me. We consider on the system of equations

$$x + a = b(y - a) \qquad \text{and} \qquad y + c = d(x - c).$$

The solutions of the system are

$$x = c + \frac{(b+1)(a+c)}{bd-1} \qquad \text{and} \qquad y = a + \frac{(d+1)(a+c)}{bd-1}.$$

Moreover, x and y are integers if and only if

$$\frac{bd-1}{\gcd(b+1, d+1)} \quad \text{divides} \quad a + c.$$

In 2003, Doslic [1] focus on the problem where $c = b$ and $d = a$, i.e.,

$$x + a = b(y - a) \qquad \text{and} \qquad y + b = a(x - b).$$

The solutions of the system are

$$x = b + \frac{(b+1)(a+b)}{ba-1} \qquad \text{and} \qquad y = a + \frac{(a+1)(a+b)}{ba-1}.$$

Moreover, x and y are integers if and only if

$$\frac{ba-1}{\gcd(b+1, a+1)} \quad \text{divides} \quad a + b.$$

However, he [1] gave only positive solutions. In 2005, Singmaster [2] gave non-positive solutions for the Doslic's problem by setting $B = \dfrac{1 + b + b^2 + a}{ab - 1}$. He [2] presented that if $a < -1$ and $b \geq 1$ then $B \leq -1$ is impossible. In this paper, we will show that if $a < -1$ and $b \geq 1$ then $B \leq -1$ is possible; moreover, we also give the integer solutions where $B \leq -1$.

2 Results

In this section, a and b are integers. Now, we set $B = \dfrac{1+b+b^2+a}{ab-1}$.

Theorem 2.1. *Let $a < -1$ and $b \geq 1$. Then $B \leq -1$ if and only if $a+b \geq 0$.*

Proof. We obtain that

$$\begin{aligned} B \leq -1 \;&\Leftrightarrow\; \frac{1+b+b^2+a}{ab-1} \leq -1 \\ &\Leftrightarrow\; 1+b+b^2+a \geq (-1)(ab-1) \\ &\Leftrightarrow\; ab+b^2+a+b \geq 0 \\ &\Leftrightarrow\; b(a+b)+(a+b) \geq 0 \\ &\Leftrightarrow\; (b+1)(a+b) \geq 0 \\ &\Leftrightarrow\; a+b \geq 0. \end{aligned}$$

□

Corollary 2.2. *If $a < -1$ and $b \geq 1$ then $B \leq -1$ is possible.*

Theorem 2.3. *Assume that $B = -t$ where t is a positive integer. Then*

$$x = b+1-t, \quad y = 1 - \frac{(b+1)^2+(t-1)^2}{1+bt} \quad \textit{and} \quad a = \frac{t-1-b^2-b}{1+bt}.$$

lead to some integer solutions for Doslic's problem.

Proof. We note that

$$\begin{aligned} B = -t \;&\Leftrightarrow\; \frac{1+b+b^2+a}{ab-1} = -t \\ &\Leftrightarrow\; 1+b+b^2+a = (-t)(ab-1) \\ &\Leftrightarrow\; a+abt = t-1-b^2-b \\ &\Leftrightarrow\; a(1+bt) = t-1-b^2-b \\ &\Leftrightarrow\; a = \frac{t-1-b^2-b}{1+bt}. \end{aligned}$$

We recall on any solutions for Doslic's problem that $x = b + \dfrac{(b+1)(a+b)}{ba-1}$.
Then $x = b + \dfrac{ab-1+1+b+b^2+a}{ab-1} = b+1+B = b+1-t$.

Since $y+b = a(x-b)$, it follows that

$$
\begin{aligned}
y = a(x-b) - b \;&\Leftrightarrow\; y = \left(\frac{t-1-b^2-b}{1+bt}\right)(b+1-t-b) - b \\
&\Leftrightarrow\; y = \frac{(b^2+b+1-t)(t-1)}{1+bt} - b \\
&\Leftrightarrow\; y = \frac{b^2t+bt+2t-t^2-b^2-b-1}{1+bt} - b \\
&\Leftrightarrow\; y = \frac{b^2t+bt+2t-t^2-b^2-b-1-b(1+bt)}{1+bt} \\
&\Leftrightarrow\; y = \frac{bt+2t-t^2-b^2-2b-1}{1+bt} \\
&\Leftrightarrow\; y = \frac{1+bt-(b^2+2b+1)-(t^2-2t+1)}{1+bt} \\
&\Leftrightarrow\; y = 1 - \frac{(b+1)^2+(t-1)^2}{1+bt}.
\end{aligned}
$$

□

3 Open Problems

We pose a question that what's happen if we seek for the rational solutions of the general version of the classical ass and mule problem with the system of equations $x + a = b(y - a)$ and $y + c = d(x - c)$.

References

[1] T. Doslic, Fibonacci in Hogwarts, Math. Gaz., 2003, **87**(November), 432–436.

[2] D. Slngmaster, Integral solutions of ass and mule problems, Math. Gaz., 2005, **89**(November), 365–370.

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