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A New Inequalities for the Riemann Zeta Functions

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Abstract

Recently, we gave two inequalities involving the Riemann zeta functions. In this paper, we present a new inequalities for the Riemann zeta functions.

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1 Introduction

The Riemann zeta function is defined by

$$\xi(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt,$$

for all $s > 1$, where Γ is the gamma function.

In [1], Laforgia and Natalini showed that

$$(s+1)\frac{\xi(s)}{\xi(s+1)} \geq s\frac{\xi(s+1)}{\xi(s+2)},$$

for all $s > 1$.

In [3], Sulaiman proved that

$$\xi(x^\alpha y)\Gamma(x^\alpha y) - \xi(xy)\Gamma(xy) \geq y\left(\xi(x^\alpha)\Gamma(x^\alpha) - \xi(x)\Gamma(x)\right)$$

and

$$\xi\left(\frac{x}{p}+\frac{y}{q}\right) \le \frac{\Gamma^{1/p}(x)\,\Gamma^{1/q}(y)}{\Gamma\left(\frac{x}{p}+\frac{y}{q}\right)}\xi^{1/p}(x)\xi^{1/q}(y)$$

for all $x, y, \alpha, p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

In [2], we presented the generalizations for above inequalities as follows.

Theorem 1.1. [2] *Let $\alpha \ge 1$ and let f, g be functions such that $f, g > 1$ and $f' \ge 0$. Then, for any $x > 1$ and for any y,*

$$\begin{aligned}\xi(f(x^\alpha)g(y))\Gamma(f(x^\alpha)g(y)) - \xi(f(x)g(y))\Gamma(f(x)g(y))\\ \ge g(y)\left(\xi(f(x^\alpha))\Gamma(f(x^\alpha)) - \xi(f(x))\Gamma(f(x))\right).\end{aligned}$$

Theorem 1.2. [2] *Let $x_1, x_2, ..., x_n > 1$ and $p_1, p_2, ..., p_n > 1$ be such that $\sum_{i=1}^{n}\frac{1}{p_i} = 1$. Then*

$$\xi\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right) \le \frac{\prod_{i=1}^{n}\Gamma^{1/p_i}(x_i)\,\xi^{1/p_i}(x_i)}{\Gamma\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}.$$

In this paper, we present a new inequalities for the Riemann zeta functions.

2 Results

Theorem 2.1. *Let $x_1, x_2, ..., x_n > 0$ and $p_1, p_2, ..., p_n > 1$ be such that $\sum_{i=1}^{n}\frac{1}{p_i} = 1$. Then*

$$\xi\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right) \le \frac{\prod_{i=1}^{n}\Gamma^{1/p_i}(1+x_i)\,\xi^{1/p_i}(1+x_i)}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}. \tag{1}$$

Proof. By the definition of ξ and the assumption,

$$\begin{aligned}\xi\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right) &= \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\int_0^\infty \frac{t^{\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)-1}}{e^t-1}dt \\ &= \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\int_0^\infty \frac{t^{\sum_{i=1}^{n}\frac{x_i}{p_i}}}{e^t-1}dt \\ &= \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\int_0^\infty \frac{\prod_{i=1}^{n} t^{\frac{x_i}{p_i}}}{e^t-1}dt.\end{aligned}$$

and then

$$\begin{aligned}\xi\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right) &= \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\int_0^\infty \frac{\prod_{i=1}^{n} t^{\frac{x_i}{p_i}}}{(e^t-1)^{\sum_{i=1}^{n}\frac{1}{p_i}}}dt \\ &= \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\int_0^\infty \frac{\prod_{i=1}^{n} t^{\frac{x_i}{p_i}}}{\prod_{i=1}^{n}(e^t-1)^{\frac{1}{p_i}}}dt \\ &= \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\int_0^\infty \prod_{i=1}^{n}\left(\frac{t^{\frac{x_i}{p_i}}}{(e^t-1)^{\frac{1}{p_i}}}\right)dt.\end{aligned}$$

By the generalized Hölder inequality,

$$\xi\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right) \le \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\prod_{i=1}^{n}\left(\int_0^\infty \frac{t^{x_i}}{e^t-1}dt\right)^{1/p_i}$$
$$= \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\prod_{i=1}^{n}\left(\int_0^\infty \frac{t^{1+x_i-1}}{e^t-1}dt\right)^{1/p_i}$$
$$= \frac{1}{\Gamma\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\prod_{i=1}^{n}\left(\Gamma\left(1+x_i\right)\xi(1+x_i)\right)^{1/p_i}.$$

This implies the inequality (1). □

Corollary 2.2. *Let* $x, y > 0$ *and* $p, q > 1$ *be such that* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\xi\left(1+\frac{x}{p}+\frac{y}{q}\right) \le \frac{\Gamma^{1/p}\left(1+x\right)\Gamma^{1/q}\left(1+y\right)}{\Gamma\left(1+\frac{x}{p}+\frac{y}{q}\right)}\xi^{1/p}(1+x)\xi^{1/q}(1+y).$$

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