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More on the Product of the Gamma Function and the Riemann Zeta Function

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Abstract

Recently, we generalized Sulaiman's inequalities involving the product of the gamma function and the Riemann zeta function. In this paper, we present a new inequality for the product of the gamma function and the Riemann zeta function.

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1 Introduction

The Riemann zeta function ξ is defined by

$$\xi(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt,$$

for all $s > 1$, where Γ is the gamma function.

We denote the product of the gamma function and the Riemann zeta function by h. Then $h(x) = \Gamma(x)\xi(x)$ for all $x > 1$.

In [2], Sulaiman showed that

$$h(1 + x + y) \leq h^{1/p}(1 + p(x + 1))h^{1/q}(1 + q(y - 1))$$

for all $x > -1, y > 1, p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In [1], we presented the generalization for above inequality as follows.

Theorem 1.1. [1] *Let* $x_1, x_2, ..., x_n > -1$, $y_1, y_2, ..., y_n > 1$, $p_1, p_2, ..., p_n > 1$ *and* $q_1, q_2, ..., q_n > 1$ *be such that* $\sum_{i=1}^{n}\left(\frac{1}{p_i}+\frac{1}{q_i}\right) = 1$. *Then*

$$h(1+\sum_{i=1}^{n}(x_i+y_i)) \leq \prod_{i=1}^{n} h^{1/p_i}(1+p_i(x_i+1))h^{1/q_i}(1+q_i(y_i-1)).$$

Next, we denote the n-th derivative of h by h_n where n is a non-negative integer.

In [2], Sulaiman showed that

$$h_{m+n}^2\left(\frac{x+y}{2}\right) \leq h_{2m}(x)h_{2n}(y)$$

and

$$h_{m+n+r}^3\left(\frac{x+y+z}{3}\right) \leq h_{3m}(x)h_{3n}(y)h_{3r}(z)$$

for all $x, y, z > 1$ and non-negative even integers n, m, r.

In [1], we presented the generalization for above inequalities as follows.

Theorem 1.2. [1] *Let* $x_1, x_2, ..., x_n > 1$ *and let* $k_1, k_2, ..., k_n$ *be non-negative even integers and let* $k = \sum_{i=1}^{n} k_i$. *Then*

$$h_k^n\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) \leq \prod_{i=1}^{n} h_{nk_i}(x_i).$$

In this paper, we present a new inequality for the product of the gamma function and the Riemann zeta function.

2 Results

We note that

$$h^{(k)}(x) = \int_0^\infty \frac{(\log_e t)^k t^{x-1}}{e^t-1}dt.$$

for all $x > 1$.

Theorem 2.1. *Let $x_1, x_2, ..., x_n > 0$ and let $k_1, k_2, ..., k_n$ be non-negative even integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$h_k^n\left(1+\sum_{i=1}^{n}\frac{x_i}{n}\right) \le \prod_{i=1}^{n} h_{nk_i}(1+x_i). \tag{1}$$

Proof. By the assumption,

$$\begin{aligned}
h_k\left(1+\sum_{i=1}^{n}\frac{x_i}{n}\right) &= h^{(k)}\left(1+\sum_{i=1}^{n}\frac{x_i}{n}\right)\\
&= \int_0^\infty \frac{(\log_e t)^k t^{\left(1+\sum_{i=1}^{n}\frac{x_i}{n}\right)-1}}{e^t-1}dt\\
&= \int_0^\infty \frac{(\log_e t)^k t^{\sum_{i=1}^{n}\frac{x_i}{n}}}{e^t-1}dt\\
&= \int_0^\infty \prod_{i=1}^{n}\frac{(\log_e t)^{k_i} t^{\frac{x_i}{n}}}{(e^t-1)^{1/n}}dt\\
&= \int_0^\infty \prod_{i=1}^{n}\left(\frac{(\log_e t)^{nk_i} t^{x_i}}{e^t-1}\right)^{1/n} dt.
\end{aligned}$$

By the generalized Hölder inequality,

$$\begin{aligned}
h_k\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) &\le \prod_{i=1}^{n}\left(\int_0^\infty \frac{(\log_e t)^{nk_i} t^{x_i}}{e^t-1}dt\right)^{1/n}\\
&= \prod_{i=1}^{n}\left(\int_0^\infty \frac{(\log_e t)^{nk_i} t^{1+x_i-1}}{e^t-1}dt\right)^{1/n}\\
&= \prod_{i=1}^{n}\left(h^{(nk_i)}(1+x_i)\right)^{1/n}\\
&= \prod_{i=1}^{n}\left(h_{nk_i}(1+x_i)\right)^{1/n}\\
&= \left(\prod_{i=1}^{n} h_{nk_i}(1+x_i)\right)^{1/n}.
\end{aligned}$$

This implies the inequality (1). □

Corollary 2.2. *Let $x > 0$ and let $k_1, k_2, ..., k_n$ be non-negative even integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$h_k^n (1 + x) \leq \prod_{i=1}^{n} h_{nk_i}(1 + x).$$

Proof. This follows from Theorem 2.1 in case $x_1 = x_2 = ... = x_n$. □

Corollary 2.3. *Let $x, y, z > 0$ and let n, m, r be non-negative even integers. Then*

$$h_{m+n}^2 \left(1 + \frac{x+y}{2}\right) \leq h_{2m}(1 + x)h_{2n}(1 + y)$$

and

$$h_{m+n+r}^3 \left(1 + \frac{x+y+z}{3}\right) \leq h_{3m}(1 + x)h_{3n}(1 + y)h_{3r}(1 + z).$$

Proof. This follows from Theorem 2.1. □

References

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