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Some New Inequalities for the Incomplete Zeta Functions

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Abstract

Recently, we gave three inequalities involving the incomplete zeta functions. In this paper, we present some new inequalities for the incomplete zeta functions.

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1 Introduction

In [2], for $0 \leq a < b$, the incomplete zeta function $\xi_{a,b}$ is defined by

$$\xi_{a,b}(s) = \frac{1}{\Gamma(s)} \int_a^b \frac{t^{s-1}}{e^t - 1} dt,$$

for all $s > 1$, where Γ is the gamma function.

Moreover, Sulaiman [2] also presented two inequalities involving the incomplete zeta functions.

We define the function $h_{a,b}$ for $0 \leq a < b$ by

$$h_{a,b}(s) = \int_a^b \frac{t^{x-1}}{e^t - 1} dt$$

for all $s > 1$.

In [1], we gave three inequalities involving the incomplete zeta functions as follows.

Theorem 1.1. [1] *Let $0 \leq a < b$ and $\alpha \geq 1$. Then, for any $x, y > 1$,*

$$h_{a,b}(x^\alpha y) - h_{a,b}(xy) \geq y\left(h_{a,b}(x^\alpha) - h_{a,b}(x)\right).$$

Theorem 1.2. [1] *Let $0 \leq a < b$ and $0 < y < 1$. Then, for any $x > 1$,*

$$h_{a,b}(x+y) \geq h_{a,b}(x) - h_{a,b}(1-y).$$

Theorem 1.3. [1] *Let $0 \leq a < b$ and $x, y, p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$h_{a,b}\left(\frac{x}{p} + \frac{y}{q}\right) \leq h_{a,b}^{1/p}(x) h_{a,b}^{1/q}(y). \tag{1}$$

In this paper, we present some new inequalities for the incomplete zeta functions.

2 Results

First, we generalize the inequality (1).

Theorem 2.1. *Let $0 \leq a < b$ and $x_1, x_2, ..., x_n > 1$ and let $p_1, p_2, ..., p_n > 1$ be such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then*

$$h_{a,b}\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right) \leq \prod_{i=1}^{n} h_{a,b}^{1/p_i}(x_i). \tag{2}$$

Proof. By the definition of $h_{a,b}$ and the assumption,

$$\begin{aligned}
h_{a,b}\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right) &= \int_a^b \frac{t^{\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)-1}}{e^t - 1} dt \\
&= \int_a^b \frac{t^{\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)-\left(\sum_{i=1}^{n} \frac{1}{p_i}\right)}}{e^t - 1} dt \\
&= \int_a^b \frac{t^{\sum_{i=1}^{n} \frac{x_i-1}{p_i}}}{e^t - 1} dt \\
&= \int_a^b \frac{\prod_{i=1}^{n} t^{\frac{x_i-1}{p_i}}}{e^t - 1} dt
\end{aligned}$$

and then

$$
\begin{aligned}
h_{a,b}\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right) &= \int_a^b \frac{\prod_{i=1}^{n} t^{\frac{x_i-1}{p_i}}}{(e^t-1)^{\sum_{i=1}^{n}\frac{1}{p_i}}}dt \\
&= \int_a^b \frac{\prod_{i=1}^{n} t^{\frac{x_i-1}{p_i}}}{\prod_{i=1}^{n}(e^t-1)^{\frac{1}{p_i}}}dt \\
&= \int_a^b \prod_{i=1}^{n}\left(\frac{t^{\frac{x_i-1}{p_i}}}{(e^t-1)^{1/p_i}}\right)dt.
\end{aligned}
$$

By the generalized Hölder inequality,

$$
\begin{aligned}
h_{a,b}\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right) &\leq \prod_{i=1}^{n}\left(\int_a^b \frac{t^{x_i-1}}{e^t-1}dt\right)^{1/p_i} \\
&= \prod_{i=1}^{n} h_{a,b}^{1/p_i}(x_i).
\end{aligned}
$$

This implies the inequality (2). $\square$

Finally, we present a new inequality as follows.

Theorem 2.2. *Let $0 \leq a < b$ and $x_1, x_2, ..., x_n > 0$ and let $p_1, p_2, ..., p_n > 1$ be such that $\sum_{i=1}^{n}\frac{1}{p_i} = 1$. Then*

$$
h_{a,b}\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right) \leq \prod_{i=1}^{n} h_{a,b}^{1/p_i}(1+x_i). \tag{3}
$$

Proof. By the definition of $h_{a,b}$ and the assumption,

$$
\begin{aligned}
h_{a,b}\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right) &= \int_a^b \frac{t^{\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right)-1}}{e^t-1}dt \\
&= \int_a^b \frac{t^{\sum_{i=1}^{n}\frac{x_i}{p_i}}}{e^t-1}dt \\
&= \int_a^b \frac{\prod_{i=1}^{n} t^{\frac{x_i}{p_i}}}{e^t-1}dt
\end{aligned}
$$

and then

$$\begin{aligned} h_{a,b}\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right) &= \int_a^b \frac{\prod_{i=1}^{n} t^{\frac{x_i}{p_i}}}{(e^t-1)^{\sum_{i=1}^{n}\frac{1}{p_i}}}dt \\ &= \int_a^b \frac{\prod_{i=1}^{n} t^{\frac{x_i}{p_i}}}{\prod_{i=1}^{n}(e^t-1)^{\frac{1}{p_i}}}dt \\ &= \int_a^b \prod_{i=1}^{n}\left(\frac{t^{\frac{x_i}{p_i}}}{(e^t-1)^{1/p_i}}\right)dt. \end{aligned}$$

By the generalized Hölder inequality,

$$\begin{aligned} h_{a,b}\left(1+\sum_{i=1}^{n}\frac{x_i}{p_i}\right) &\leq \prod_{i=1}^{n}\left(\int_a^b \frac{t^{x_i}}{e^t-1}dt\right)^{1/p_i} \\ &= \prod_{i=1}^{n}\left(\int_a^b \frac{t^{1+x_i-1}}{e^t-1}dt\right)^{1/p_i} \\ &= \prod_{i=1}^{n} h_{a,b}^{1/p_i}(1+x_i). \end{aligned}$$

This implies the inequality (3). □

References

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