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Properties of composition operators on spaces of hyperbolic type

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Abstract

In this paper, we study boundedness and compactness of the composition operators C_ϕ between the hyperbolic Bloch and general hyperbolic Besov-type classes.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$. Let $H(\mathbb{D})$ denote the classes of functions holomorphic in the unit disc $\mathbb{D}$, the hyperbolic function classes are subsets of the class $B(\mathbb{D})$ of all analytic functions f in the unit disc $\mathbb{D}$ such that $|f(z)| < 1$. If (X, d) is a metric space, we denote the open and closed balls with center x and radius $r > 0$ by

$$B(x, r) := \{y \in X : d(y, x) < r\} \text{ and } \bar{B}(x, r) := \{y \in X : d(x, y) \leq r\},$$

respectively.
Hyperbolic function classes are usually defined by using either the hyperbolic derivative $f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}$ of $f \in B(\mathbb{D})$, or the hyperbolic distance $\rho(f(z), 0) := \frac{1}{2}\log(\frac{1+|f(z)|}{1-|f(z)|})$ between $f(z)$ and zero.

The hyperbolic $\mathcal{B}^*_\alpha$ (see [6]) is defined as the sets of $f \in B(\mathbb{D})$ for which

$$\mathcal{B}^*_\alpha = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha f^*(z) < \infty\}.$$

The little hyperbolic Bloch space $\mathcal{B}^*_{\alpha,0}$ is a subspace of $\mathcal{B}^*_\alpha$ consisting of all $f \in \mathcal{B}^*_\alpha$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)^\alpha f^*(z) = 0.$$

Quite recently, the first author in [6] gave the following definitions for (p,α)-Bloch spaces $\mathcal{B}_{p,\alpha}$ and $\mathcal{B}_{p,\alpha,0}$ for $f \in H(\mathbb{D})$

$$\|f\|_{\mathcal{B}_{p,\alpha}} = \frac{p}{2}\sup_{z\in\mathbb{D}}|f(z)|^{\frac{p}{2}-1}|f'(z)|(1-|z|^2)^\alpha < \infty,$$

and

$$\lim_{|z|\to 1}|f(z)|^{\frac{p}{2}-1}|f'(z)|(1-|z|^2)^\alpha = 0,$$

where $2 \le p < \infty$ and $0 < \alpha < 1$.
Also in [6], the first author gave the following generalized hyperbolic derivative:

$$f^*_p(z) = \frac{p}{2}\frac{|f(z)|^{\frac{p}{2}-1}|f'(z)|}{1-|f(z)|^p}, \quad f(z) \in H(\mathbb{D}),$$

when $p = 2$ we obtain the usual hyperbolic derivative as defined above.
A function $f \in B(\mathbb{D})$ is said to belong to the generalized (p,α) hyperbolic Bloch-type class $\mathcal{B}^*_{p,\alpha}$ if

$$\|f\|_{\mathcal{B}^*_{p,\alpha}} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha f^*_p(z) < \infty,$$

the little generalized (p,α) hyperbolic Bloch-type class $\mathcal{B}^*_{p,\alpha,0}$ consists of all $f \in \mathcal{B}^*_{p,\alpha}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha f^*_p(z) = 0.$$

Remark 1.1 *It should be remarked that, the Schwarz-Pick lemma implies* $\mathcal{B}^*_{p,\alpha} \equiv B(\mathbb{D})$ *for all* $1 \le \alpha < \infty$ *with* $\|f\|_{\mathcal{B}^*_{p,\alpha}} \le 1$, *hence the class* $\mathcal{B}^*_{p,\alpha}$ *is of interest only when* $0 < \alpha < 1$.

Denote by

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|}$$

the Green's function of $\mathbb{D}$ with logarithmic singularity at $a \in \Delta$. Now, we define the hyperbolic $F_p(p,q,s;\omega)$ type class $F^*_p(p,q,s;\omega)$. Let $2 \le p < \infty, 0 < s < \infty$ and $-2 < q < \infty$, for given a reasonable function $\omega : (0,1] \to (0,\infty)$, the hyperbolic class $F^*_p(p,q,s;\omega)$ consists of those functions $f \in B(\mathbb{D})$ for which

$$\|f\|^p_{F^*_p(p,q,s)} = \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(f^*_p(z))^p(1-|z|^2)^q\frac{g^s(z,a)}{\omega(1-|z|)}dA(z) < \infty.$$

Moreover, we say that $f \in F_p^*(p,q,s)$ belongs to the class $F_{p,0}^*(p,q,s;\omega)$ if

$$\lim_{|a|\to 1}\int_{\mathbb{D}} (f_p^*(z))^p(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z) = 0.$$

Note that hyperbolic classes are not linear spaces, since they consist of functions that are self-maps of $\mathbb{D}$. Thus, the result is a generalization of the recent results of Pérez-González, Rättyä and Taskinen [31].
For any holomorphic self-mapping ϕ of $\mathbb{D}$. The symbol ϕ induces a linear composition operator $C_\phi(f) = f \circ \phi$ from $H(\mathbb{D})$ or $B(\mathbb{D})$ into itself. The study of composition operator C_ϕ acting on spaces of differen function classes has engaged many analysts for many years (see e.g. [10, 11, 16, 17, 21, 26, 27, 28, 29, 30] and others).

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \le C\|f\|_X$ for all maps $f \in X$. By elementary functional analysis, it is well-known that a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y contained in $B(\mathbb{D})$ or $H(\mathbb{D})$, $T : X \to Y$ is compact if and only if for each bounded sequence $(x_n) \in X$, the sequence $(Tx_n) \in Y$ contains a subsequence converging to a function $f \in Y$.

Two quantities A and B are said to be equivalent if there exist two finite positive constants C_1 and C_2 such that $C_1B \le A \le C_2B$, written as $A \approx B$. Throughout this paper, the letter C denotes different positive constants which are not necessarily the same from line to line.
Now, we introduce the following definitions:

Definition 1.1 *A composition operator* $C_\phi : \mathcal{B}_{p,\alpha}^* \to F_p^*(p,q,s;\omega)$ *is said to be bounded, if there is a positive constant* C *such that* $\|C_\phi f\|_{F_p^*(p,q,s;\omega)} \le C\|f\|_{\mathcal{B}_{p,\alpha}^*}$ *for all* $f \in \mathcal{B}_{p,\alpha}^*$.

Definition 1.2 *A composition operator* $C_\phi : \mathcal{B}_{p,\alpha}^* \to F_p^*(p,q,s;\omega)$ *is said to be compact, if it maps any ball in* $\mathcal{B}_{p,\alpha}^*$ *onto a pre-compact set in* $F_p^*(p,q,s;\omega)$.

We can find a natural metric on the generalized hyperbolic (p,α)-Bloch class $\mathcal{B}_{p,\alpha}^*$ and the class $F_p^*(p,q,s;\omega)$. Let $2 \le p < \infty, 0 < s < \infty, -2 < q < \infty$, and $0 < \alpha < 1$. First we can find a natural metric in $\mathcal{B}_{p,\alpha}^*$ [6] by defining

$$d(f,g;\mathcal{B}_{p,\alpha}^*) := d_{\mathcal{B}_{p,\alpha}^*}(f,g) + \|f-g\|_{\mathcal{B}_{p,\alpha}} + |f(0)-g(0)|^{\frac{p}{2}},$$

where

$$d_{\mathcal{B}_{p,\alpha}^*}(f,g) := \sup_{a\in\mathbb{D}}\Big|\frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p}\Big|(1-|z|^2)^\alpha.$$

For $f,g \in F^*_p(p,q,s;\omega)$, define their distance by

$$d(f,g;F^*_p(p,q,s;\omega)) := d_{F^*_p(p,q,s;\omega)}(f,g) + \|f-g\|_{F_p(p,q,s;\omega)} + |f(0)-g(0)|,$$

where

$$d_{F^*_p(p,q,s;\omega)}(f,g) := \Big(\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f^*_p(z)-g^*_p(z)|^p(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z)\Big)^{\frac{1}{p}}.$$

The following result of the complete metric spaces $d(.,.;\mathcal{B}^*_{p,\alpha})$ is proved in ([6]). Now we prove the following proposition:

Proposition 1.1 *The class $\mathcal{B}^*_{p,\alpha}$ equipped with the metric $d(.,.;\mathcal{B}^*_{p,\alpha})$ is a complete metric space. Moreover,$\mathcal{B}^*_{p,\alpha,0}$ is a closed (and therefore complete) subspace of $\mathcal{B}^*_{p,\alpha}$.*

Proposition 1.2 *The class $F^*_p(p,q,s)$ equipped with the metric $d(.,.;F^*_p(p,q,s;\omega))$ is a complete metric space. Moreover, $F^*_{p,0}(p,q,s;\omega)$ is a closed (and therefore complete) subspace of $F^*_p(p,q,s;\omega)$.*

The proof is very similar to the corresponding result in [7], so it will be omitted.

The following lemma follows by standard arguments similar to those outline in [34]. Hence we omit the proof.

Lemma 1.3 *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself and let $2 \le p < \infty$, $0<\alpha<1$, $0<s<\infty$, and $-2<q<\infty$. Then the composition operator $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s;\omega)$ is compact if and only if for any bounded sequence $(f_n)_{n\in N} \in \mathcal{B}^*_{p,\alpha}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$ we have*

$$\lim_{n\to\infty} \|C_\phi f_n\|_{F^*_p(p,q,s;\omega)} = 0.$$

There are some papers used the weight function ω to study some classes of function spaces, for more details, we refer to [7, 8, 13, 14, 15, 18, 32, 33].

2 Boundedness of composition operator

For $0<\alpha<1$ $2\le p<\infty$. Let $f,g\in\mathcal{B}^*_{p,\alpha}$, we will suppose that

$$(|f^*_p(z)| + |g^*_p(z)|) \ge \frac{C}{(1-|z|^2)^\alpha} > 0, \tag{1}$$

for some constant C and for each $z\in\mathbb{D}$.

Now, we give the following result.

Theorem 2.1 *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself and let $0 < \alpha < 1$, $2 \le p < \infty, 0 \le s < \infty, -2 < q < \infty$. Suppose that (1) is satisfied. Then the following statements are equivalent:*
(i) $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s;\omega)$ is bounded;
(ii) $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s;\omega)$ is Lipschitz continuous;

(iii)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^p)^{p\alpha}}(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z) < \infty.$$

Proof: To prove (i)$\Leftrightarrow$ (iii), first assume that (iii) holds and that $f \in \mathcal{B}^*_{p,\alpha}$, then, we obtain

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} ((f_p\circ\phi)^*(z))^p(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z)$$
$$= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^*_p(\phi(z)))^p|\phi'(z)|^p(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z)$$
$$\le \|f\|^p_{\mathcal{B}^*_{p,\alpha}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)^p|)^{p\alpha}}(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z).$$

Hence, it follows that (i) holds.
Conversely, assuming that (i) holds, then there exists a constant C such that

$$\|C_\phi f\|_{F^*_p(p,q,s;\omega)} \le C\|f\|_{\mathcal{B}^*_{p,\alpha}}.$$

For giving $f \in \mathcal{B}^*_{p,\alpha}$, the function $f_t(z) = f(tz)$, where $0 < t < 1$, belongs to $\mathcal{B}^*_{p,\alpha}$ with the property $\|f_t\|_{\mathcal{B}^*_{p,\alpha}} \le \|f\|_{\mathcal{B}^*_{p,\alpha}}$. Let $f,\ g$ be the functions from (1), we have

$$|f^*_p(z)| + |g^*_p(z)| \ge \frac{C}{(1-|z|^2)^\alpha} > 0$$

for all $z \in \mathbb{D}$, then

$$\frac{|\phi'(z)|}{(1-|\phi(z)|)^\alpha} \le (f_p\circ\phi)^*(z) + (g_p\circ\phi)^*(z),$$

thus,

$$\int_{\mathbb{D}} \frac{|t\phi'(z)|^p}{(1-|t\phi(z)^p|)^{p\alpha}}(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z)$$
$$\le \int_{\mathbb{D}} \Big(((f_p\circ\phi)^*(z))^p + ((g_p\circ\phi)^*(z))^p\Big)(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z)$$
$$\le C(\|C_\phi f\|^p_{F^*_p(p,q,s;\omega)} + \|C_\phi g\|^p_{F^*_p(p,q,s)})$$
$$\le C\|C_\phi\|^p(\|f\|^p_{\mathcal{B}^*_{p,\alpha}} + \|g\|^p_{\mathcal{B}^*_{p,\alpha}}),$$

this estimate together with the Fatou's lemma, implies that C_ϕ is bounded, so (iii) is satisfied.

To prove (ii)$\Leftrightarrow$ (iii), assume first that $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s;\omega)$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d(f\circ\phi, g\circ\phi; F^*_p(p,q,s;\omega)) \le C d(f,g;\mathcal{B}^*_{p,\alpha}), \quad \text{for all } f,g \in \mathcal{B}^*_{p,\alpha}.$$

Taking $g=0$, this implies

$$\|f\circ\phi\|_{F^*_p(p,q,s;\omega)} \le C(\|f\|_{\mathcal{B}^*_{p,\alpha}} + \|f\|_{\mathcal{B}_{p,\alpha}} + |f(0)|^{\frac{p}{2}}), \quad \text{for all } f \in \mathcal{B}^*_{p,\alpha}. \tag{2}$$

The assertion (iii) for $\alpha = 1$, follows by choosing $f(z) = z$ in (2).
If $0 < \alpha < 1$, then

$$\begin{aligned} |f(z)|^{\frac{p}{2}} &\le C\Big|\int_0^z |f(s)|^{\frac{p}{2}-1} f'(s)ds + |f(0)^{\frac{p}{2}}\Big| \\ &\le C\|f\|_{\mathcal{B}_{p,\alpha}} \int_0^{|z|} \frac{ds}{(1-s^2)^\alpha} + |f(0)|^{\frac{p}{2}} \\ &\le C\frac{\|f\|_{\mathcal{B}_{p,\alpha}}}{1-\alpha} + |f(0)|^{\frac{p}{2}}, \end{aligned}$$

this yields

$$|f(\phi(0)) - g(\phi(0))|^{\frac{p}{2}} \;\le\; C\frac{\|f-g\|_{\mathcal{B}_{p,\alpha}}}{(1-\alpha)} + \frac{2}{p}|f(0)-g(0)|^{\frac{p}{2}}$$

Moreover, from (1), for $f, g \in \mathcal{B}^*_{p,\alpha}$, we deduce that

$$(|f^*_p(z)| + |g^*_p(z)|)(1-|z|^2)^\alpha \ge C > 0, \quad \text{for all } z \in \mathbb{D}.$$

Therefore,

$$\begin{aligned} &\|f\|_{\mathcal{B}^*_{p,\alpha}} + \|g\|_{\mathcal{B}^*_{p,\alpha}} + \|f\|_{\mathcal{B}_{p,\alpha}} + \|g\|_{\mathcal{B}_{p,\alpha}} + |f(0)|^{\frac{p}{2}} + |g(0)|^{\frac{p}{2}} \\ \ge\;& C\int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)^p|)^{p\alpha}}(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z). \end{aligned}$$

For which the assertion (iii) follows .
Assume now that (iii) is satisfied, we have

$$\begin{aligned} & d(f\circ\phi, g\circ\phi; F^*_p(p,q,s;\omega)) = d_{F^*_p(p,q,s;\omega)}(f\circ\phi, g\circ\phi) \\ & + \|f\circ\phi - g\circ\phi\|_{F(p,q,s;\omega)} + |f(\phi(0)) - g(\phi(0))^{\frac{p}{2}}| \\ & \le d_{\mathcal{B}^*_{p,\alpha}}(f,g)\Big(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^p(1-|z|^2)^q}{(1-(\phi(z))^p)^{p,\alpha}} \frac{g^s(z,a)}{\omega(1-|z|)} dA(z)\Big)^{\frac{1}{p}} \\ & + \|f-g\|_{\mathcal{B}_{p,\alpha}}\Big(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^p(1-|z|^2)^q}{(1-(\phi(z))^p)^{p,\alpha}} \frac{g^s(z,a)}{\omega(1-|z|)} dA(z)\Big)^{\frac{1}{p}} \\ & + \frac{\|f-g\|_{\mathcal{B}_{p,\alpha}}}{1-\alpha} + |f(0)-g(0)|^{\frac{p}{2}} \le C\, d(f,g;\mathcal{B}^*_{p,\alpha}). \end{aligned}$$

Thus $C_\phi : \mathcal{B}^*_{p,\alpha} \to F_p(p,q,s;\omega)$ is Lipschitz continuous and the proof is established.

Remark 2.1 *We know that a composition operator* $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s;\omega)$ *is said to be bounded if there is a positive constant C such that* $\|C_\phi f\|_{F^*_p(p,q,s;\omega)} \leq C\|f\|_{\mathcal{B}^*_{p,\alpha}}$; *for all* $f \in \mathcal{B}^*_{p,\alpha}$. *Theorem 2.1 shows that* $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s;\omega)$ *is bounded if and only if it is Lipschitz continuous, that is, if there exists a positive constant C such that* $d(f\circ\phi, g\circ\phi; F^*_p(p,q,s;\omega)) \leq Cd(f,g;\mathcal{B}^*_{p,\alpha}), \quad$ *for all* $f, g \in \mathcal{B}^*_{p,\alpha}$.

By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, Since the boundedness is trivially also equivalent to the Lipschitz-continuity. Our result for composition operators in hyperbolic spaces is the correct and natural generalization of the linear operator theory.

3 Compactness of $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s;\omega)$

Recall that a composition operator $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s\omega)$ is said to be compact, if it maps any ball in $\mathcal{B}^*_{p,\alpha}$ onto a pre-compact set in $F^*_p(p,q,s;\omega)$. Now, we give the following important results.

Proposition 3.1 *Assume* ϕ *is a holomorphic mapping from* $\mathbb{D}$ *into itself. Let* $2 \leq p < \infty$, $-2 < q < \infty$, $0 < \alpha < 1$ *and* $0 \leq s < \infty$. *If* $C_\phi : \mathcal{B}^*_{p,\alpha} \to F_p(p,q,s;\omega)$ *is compact, it maps closed balls onto compact sets.*

Proof: If $B \subset \mathcal{B}^*_{p,\alpha}$ is a closed ball and $g \in F^*_p(p,q,s;\omega)$ belongs to the closure of $C_\phi(B)$, we can find a sequence $(f_n)_{n=1}^\infty \subset B$ such that $f_n \circ \phi$ converges to $g \in F^*_p(p,q,s;\omega)$ as $n \to \infty$. But $(f_n)_{n=1}^\infty$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^\infty$ converging uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. A s in earlier arguments of Proposition 2.1 in [31], we get a positive estimate which shows that f must belong to the closed ball B. On the other hand, also the sequence $(f_{n_j} \circ \phi)_{j=1}^\infty$ converges uniformly on compact subsets to an analytic function, which is $g \in F^*_p(p,q,s;\omega)$. We get $g = f \circ \phi$, i.e. g belongs to $C_\phi(B)$. Thus, this set is closed and also compact.

Compactness of composition operators acting between $\mathcal{B}^*_{p,\alpha}$ and $F^*_p(p,q,s;\omega)$ classes can be characterized in the following result.

Theorem 3.1 *Assume* ϕ *is a holomorphic mapping from* $\mathbb{D}$ *into itself. Let* $2 \leq p < \infty$, $-2 < q < \infty$, $0 < \alpha < 1$ *and* $0 \leq s < \infty$. *Then the following statements are equivalent:*

(i) $C_\phi : \mathcal{B}^*_{p,\alpha} \to F^*_p(p,q,s;\omega)$ *is compact.*

(ii) $G(\phi, g) = 0$, *where*

$$G(\phi, g) = \lim_{r\to 1^-} \sup_{a\in\mathbb{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^p)^{p\alpha}}(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z) = 0.$$

Proof: We first assume that (ii) holds. Let $B := \bar{B}(g,\delta) \subset \mathcal{B}^*_{p,\alpha}$, $g \in \mathcal{B}^*_{p,\alpha}$ and $\delta > 0$, be a closed ball, and let $(f_n)_{n=1}^\infty \subset B$ be any sequence. We show that its image has a convergent subsequence in $F^*_p(p,q,s;\omega)$, which proves the compactness of C_ϕ by definition.

Again, $(f_n)_{n=1}^\infty \subset B(\mathbb{D})$ is normal, hence, there is a subsequence $(f_{n_j})_{j=1}^\infty$ which converges uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. By Cauchy formula for the derivative of an analytic function, also the sequence $(f'_{n_j})_{j=1}^\infty$ converges uniformly on the compact subsets of $\mathbb{D}$ to f'. It follows that also the sequences $(f_{n_j}\circ\phi)_{j=1}^\infty$ and $(f'_{n_j}\circ\phi)_{j=1}^\infty$ converge uniformly on the compact subsets of $\mathbb{D}$ to $f\circ\phi$ and $f'\circ\phi$, respectively. Moreover, $f \in B \subset \mathcal{B}^*_{p,\alpha}$ since for any fixed $R, 0 < R < 1$, the uniform convergence yield

$$\sup_{|z|\le R}\Big|\frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p}\Big|(1-|z|^2)^\alpha$$
$$+ \sup_{|z|\le R} |f'(z)-g'(z)||f(z)-g(z)|^{\frac{p}{2}-1}(1-|z|^2)^\alpha + |f(0)-g(0)|^{\frac{p}{2}-1}$$

$$= \lim_{j\to\infty}\sup_{|z|\le R}\Big|\frac{f'_{n_j}(z)|f_{n_j}(z)|^{\frac{p}{2}-1}}{1-|f_{n_j}(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p}\Big|(1-|z|^2)^\alpha$$
$$+ \lim_{j\to\infty}(\sup_{|z|\le R} |f'_{n_j}(z)-g'(z)||f_{n_j}(z)-g_(z)|^{\frac{p}{2}-1}(1-|z|^2)^\alpha + |f_{n_j}(0)-g(0)|^{\frac{p}{2}-1})$$
$$< \delta.$$

Hence, $d(f,g;\mathcal{B}^*_{p,\alpha}) \le \delta$.

Let $\varepsilon > 0$. Since (ii) is satisfied, we may fix $r, 0 < r < 1$, such that

$$\sup_{a\in\mathbb{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^p)^{p\alpha}}(1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z) \le \varepsilon.$$

By the uniform convergence, we may fix $N_1 \in \mathbb{N}$ such that

$$|f_{n_j}\circ\phi(0) - f\circ\phi(0)| \le \varepsilon, \quad \text{for all } j \ge N_1. \tag{3}$$

The condition (ii) is known to imply the compactness of $C_\phi : \mathcal{B}_{p,\alpha} \to F_p(p,q,s;\omega)$, hence possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$\|f_{n_j}\circ\phi - f\circ\phi\|_{F_p(p,q,s)} \le \varepsilon, \quad \text{for all } j \ge N_2;\ \ N_2 \in \mathbb{N}. \tag{4}$$

Now let

$$I_1(a,r)=\sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r}[(f_{p,n_j}\circ\phi)^*(z)-(g_p\circ\phi)^*(z)]^p(1-|z|^2)^q\frac{g^s(z,a)}{\omega(1-|z|)}dA(z),$$

and

$$I_2(a,r)=\sup_{a\in\mathbb{D}}\int_{|\phi(z)|\le r}[(f_{p,n_j}\circ\phi)*(z)-(g_p\circ\phi)^*(z)]^p(1-|z|^2)^q\frac{g^s(z,a)}{\omega(1-|z|)}dA(z).$$

Since $(f_{n_j})_{j=1}^{\infty}\subset B$ and $f\in B$, it follows that

$$\begin{aligned}
I_1(a,r) &= \sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r}[(f_{p,n_j}\circ\phi)^*(z)-(g_p\circ\phi)^*(z)]^p(1-|z|^2)^q\frac{g^s(z,a)}{\omega(1-|z|)}dA(z)\\
&\le \frac{p}{2}\sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r}Ł(f_{n_j},g,\phi)(1-|z|^2)^q\frac{g^s(z,a)}{\omega(1-|z|)}dA(z)\\
&\le d_{\mathcal{B}^*_{p,\alpha}}(f_{n_j},g)\sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p(1-|z|^2)^q}{1-(|\phi(z)|^p)^{\alpha p}}\frac{g^s(z,a)}{\omega(1-|z|)}dA(z),
\end{aligned}$$

where

$$Ł(f_{n_j},g,\phi)=\Big|\frac{((f_{n_j}\circ\phi)'(z))|((f_{n_j}\circ\phi)(z)))|^{\frac{p}{2}-1}}{1-|(f_{n_j}\circ\phi)(z)|^p}-\frac{(g\circ\phi)'(z)|((g_{n_j}\circ\phi)(z)))|^{\frac{p}{2}-1}}{1-|(g\circ\phi)(z)|^p}\Big|^p$$

hence,

$$I_1(a,r)\le C\varepsilon. \tag{5}$$

On the other hand, by the uniform convergence on the compact disc $\mathbb{D}$, we can find an $N_3\in\mathbb{N}$ such that for all $j\ge N_3$,

$$Ł_1(f_{n_j},g,\phi)=\Big|\frac{(f'_{n_j}(\phi(z))|((f_{n_j}\circ\phi)(z)))|^{\frac{p}{2}-1}}{1-|(f_{n_j}\circ\phi)(z)|^p}-\frac{g'_{n_j}(\phi(z))|((g_{n_j}\circ\phi)(z))|^{\frac{p}{2}-1}}{1-|(g\circ\phi)(z)|^p}\Big|\le\varepsilon.$$

For all z with $|\phi(z)|\le r$. Hence, for such j,

$$\begin{aligned}
I_2(a,r) &= \sup_{a\in\mathbb{D}}\int_{|\phi(z)|\le r}[(f_{p,n_j}\circ\phi)^*(z)-(g_p\circ\phi)^*(z)]^p(1-|z|^2)^q\frac{g^s(z,a)}{\omega(1-|z|)}dA(z)\\
&\le \sup_{a\in\mathbb{D}}\int_{|\phi(z)|\le r}Ł_1(f_{n_j},g,\phi)|\phi'(z)|^p(1-|z|^2)^q\frac{g^s(z,a)}{\omega(1-|z|)}dA(z)\\
&\le \varepsilon\Big(\sup_{a\in\mathbb{D}}\int_{|\phi(z)|\le r}\frac{|\phi'(z)|^p(1-|z|^2)^q}{1-(|\phi(z)|^p)^{\alpha p}}\frac{g^s(z,a)}{\omega(1-|z|)}dA(z)\Big)^{\frac{1}{p}}\le C\varepsilon,
\end{aligned}$$

hence,

$$I_2(a,r)\le C\,\varepsilon. \tag{6}$$

where C is bounded which is obtained from (iii) of Theorem 3.1. Combining (4), (5), (6) and (7) we deduce that $f_{n_j} \to f$ in $F_p^*(p,q,s)$.
For the converse direction, let $f_n(z) := \frac{1}{2}n^{\alpha-1}z^n$ for all $n \in \mathbb{N}$, $n \geq 2$.

$$\|f\|_{\mathcal{B}^*_{p,\alpha}} = \frac{p}{2}\sup_{a\in\mathbb{D}} \frac{n^{\frac{\alpha p}{2}}|z|^{\frac{\alpha p}{2}-1}(1-|z|^2)^\alpha}{1-2^{-p}n^{p(\alpha-1)}|z|^{np}} \leq (2^{p-1}+1)\sup_{a\in\mathbb{D}} n^{\frac{\alpha p}{2}}|z|^{\frac{\alpha p}{2}-1}(1-|z|^2)^\alpha$$

Then the sequence $(f_n)_{n=1}^\infty$ belongs to the ball $\overline{B}(0;(2^{p-1}+1)) \subset \mathcal{B}^*_{p,\alpha}$ [6]. We are assuming that C_ϕ maps the closed ball $\overline{B}(0;(2^{p-1}+1)) \subset \mathcal{B}^*_{p,\alpha}$ into a compact subset of $F_p^*(p,q,s;\omega)$, hence, there exists an unbounded increasing subsequence $(n_j)_{j=1}^\infty$ such that the image subsequence $(C_\phi f_{n_j})_{n=1}^\infty$ converges with respect to the norm. Since, both $(f_n)_{n=1}^\infty$ and $(C_\phi f_{n_j})_{n=1}^\infty$ converge to the zero function uniformly on compact subsets of $\mathbb{D}$, the limit of the latter sequence must be 0. Hence,

$$\lim_{j\to\infty} \|n_j^{\alpha-1}\phi^{n_j}\|_{F_p^*(p,q,s;\omega)} = 0. \tag{7}$$

Now let $r_j = 1 - \frac{1}{n_j}$. For all numbers a, $r_j \leq a < 1$, (see [6]) we have the estimate

$$\frac{n_j^\alpha a^{n_j-1}}{1-a^{n_j}} \geq \frac{1}{e(1-a)^\alpha}. \tag{8}$$

Using (8), we deduce

$$\begin{aligned}
&\|n_j^{\alpha-1}\phi^{n_j}\|_{F_p^*(p,q,s;\omega)} \\
&\geq \frac{p}{2}\sup_{a\in\mathbb{D}}\int_{|\phi(z)|\geq r_j} \Big|\frac{n_j^\alpha(\phi(z))^{n_j-1}|\phi^{n_j}(z)|^{\frac{p}{2}-1}|\phi'(z)|}{1-|\phi^{n_j}(z)|^p}\Big|^p \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z) \\
&\geq \frac{Cp}{2(2e)^p}\sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r_j} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^p)^{p\alpha}}\frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z). \qquad (9)
\end{aligned}$$

From (8) and (10), the condition (ii) follows. The proof is therefore completed.

Remark 3.1 *It is still an open problem to study composition operators in Clifford analysis. For more details on some classes of quaternion function spaces, we refer to ([1, 2, 3, 4, 5, 9, 19, 20, 23, 24, 25]) and others.*

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