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Applications of method of integral equations and integral operators to Riemann-Hilbert problem for elliptic complex equations of first order

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Abstract

In this article, we first propose the Riemann-Hilbert problem for uniformly elliptic complex equations of first order and its well-posedness. Then we give the integral representation of solutions of Riemann-Hilbert problem for the complex equations. Moreover we shall obtain a priori estimates of solutions of the modified Riemann-Hilbert problem and verify its solvability by the method of integral equations. Finally the solvability results of the original boundary value problem can be obtained..

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1 Formulation of Riemann-Hilbert problem for elliptic complex equations of first order

First of all, we introduce the linear elliptic equations of first order

$$w_{\bar z}=F(z,w,w_z),\, F=Q_1(z)w_z+Q_2(z)\bar w_{\bar z}+A_1(z)w+A_2(z)\bar w+A_3(z),\, z\in D, \quad (1.1)$$

where $z = x+iy$, $w_{\bar{z}} = [w_x + iw_y]/2$ (see [4,9,11]). We assume that equation (1.1) satisfy the following conditions.

Condition C 1) $Q_j(z)$, $A_j(z)\,(j = 1, 2, 3)$ are measurable in $z \in D$, and satisfy

$$L_p[A_j, \overline{D}] \le k_0,\ j = 1, 2,\ L_p[A_3, \overline{D}] \le k_1, \tag{1.2}$$

where $p_0, p\,(2 < p_0 \le p), k_0, k_1$ are non-negative constants.

2) The complex equation (1.1) satisfies the uniform ellipticity condition

$$|Q_1(z)| + |Q_2(z)| \le q_0\,(< 1), \tag{1.3}$$

q_0 is a non-negative constant.

Let D be an $N + 1\,(N \ge 1)$-connected bounded domain in $\mathbb{C}$ with the boundary $\partial D = \Gamma = \cup_{j=0}^{N} \Gamma_j \in C^1_\mu\,(0 < \mu < 1)$. Without loss of generality, we assume that D is a circular domain in $|z| < 1$, bounded by the $(N + 1)$-circles $\Gamma_j : |z - z_j| = r_j$, $j = 0, 1, ..., N$ and $\Gamma_0 = \Gamma_{N+1} : |z| = 1$, $z = 0 \in D$. In this article, the notations are as the same in References [4-12]. Now we formulate the Riemann-Hilbert problem for equation (1.1) as follows.

Problem A The Riemann-Hilbert boundary value problem for (1.1) is to find a continuous solution $w(z)$ in $\overline{D}$ satisfying the boundary condition:

$$\mathrm{Re}[\overline{\lambda(z)}w(z)] = c(z),\ z \in \Gamma, \tag{1.4}$$

where $\lambda(z), c(z)$ satisfy the conditions

$$C_\alpha[\lambda(z), \Gamma] \le k_0,\ C_\alpha[c(z), \Gamma] \le k_2, \tag{1.5}$$

in which $\lambda(z) = a(z)+ib(z), |\lambda(z)| = 1$ on Γ, and $\alpha\,(1/2 < \alpha < 1)$ ia a positive constant. The index K of Problems A is defined as follows:

$$K = K_1 + \cdots + K_m = \sum_{j=0}^{N} \frac{1}{2\pi}\Delta_{\Gamma_j} \arg \lambda(z)\,(j = 0, 1, ..., N) \tag{1.6}$$

in which the partial indexes $K_j = \Delta_{\Gamma_j} \arg \lambda(z)/2\pi\,(j = 0, 1, ..., N)$ of $\lambda(z)$ are integers.

Due to when the index $K < 0$, Problem A may not be solvable, when $K \ge 0$, the solution of Problem A is not necessarily unique. Hence we put forward a well-posed-ness of Riemann–Hilbert problem with modified boundary conditions.

Problem B Find a continuous solution $w(z)$ of the complex equation (1.1) in $\overline{D}$ satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)}w(z)] = c(z) + h(z),\ z \in \Gamma, \tag{1.7}$$

where

$$h(z)=\begin{cases} \left.\begin{array}{l} 0,\ z\in\Gamma_0, \\ h_j,\ z\in\Gamma_j, j=1,...,N, \end{array}\right\} & \text{if } K\geq 0 \\ \left.\begin{array}{l} h_j,\ z\in\mathit{\Gamma}_j,\ j=1,...,N, \\ h_0+\mathrm{Re}\sum\limits_{m=1}^{-K-1}(h_m^{+}+\mathrm{i}h_m^{-})z^m, z\in\Gamma_0, \end{array}\right\} & \text{if } K<0, \end{cases} \tag{1.8}$$

in which $h_j(j=0,1,...,N+1)$, $h_m^{\pm}\,(m=1,...,-K-1)$ are unknown real constants to be determined appropriately. Moreover we assume that the solution $w(z)$ satisfies the following point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)]=b_j,\ j\in J=\{1,...,2K+1\},\ \text{if } K\geq 0, \tag{1.9}$$

where $a_j\in\Gamma_0\,(j=1,...,2K+1,\ \text{ if }\ K\ \geq\ 0)$ are distinct fixed points; and $b_j(j\in J)$ are all real constants satisfying the conditions

$$|\ b_j\ |\leq k_3,\ j\in J, \tag{1.10}$$

herein k_3 is a non-negative constant. Problem B with $\mathrm{A}_3(\mathrm{z},\mathrm{w})=0$ in D, $c(z)=0$ on $\mathit{\Gamma}$ and $b_j\ (j\in J)$ is called Problem B_0.

We mention that the undetermined real constants h_j, $h_m^{\pm}$ in (1.8) are for ensuring the existence of continuous solutions, and the point conditions in (1.9) are for ensuring the uniqueness of continuous solutions in $\overline{D}$. The advantages of the new well-posed-ness is simpler than others (see [4-6,11]).

2 Integral representation of solutions of Riemann-Hilbert problem for analytic functions

Now we transform the boundary condition (1.7) into the standard form and first find a solution $S(z)$ of the modified Dirichlet problem with the boundary condition

We first transform the boundary condition (1.7) into the standard form and first find a solution $S(z)$ of the modified Dirichlet problem with the boundary condition

$$\begin{gathered} \mathrm{Re}S(z)=S_1(z)-\theta(t),\ S_1(t)=\begin{cases}\arg\lambda(t)-K\arg t,\ t\in\Gamma_0,\\ \arg\lambda(t),\ t\in\Gamma_j, j=1,...,N,\end{cases} \\ \theta(t)=\begin{cases}0,\ t\in\Gamma_0,\\ \theta_j,\ t\in\Gamma_j,\ j=1,...,N,\end{cases}\quad \mathrm{Im}[S(1)]=0, \end{gathered} \tag{2.1}$$

where $\theta_j\,(j=1,...,N)$ are real constants. Thus the boundary condition (1.7) into the standard boundary condition

$$\begin{aligned}
&\mathrm{Re}[\overline{\lambda(t)}w(t)]=\mathrm{Re}[\overline{\Lambda(t)}\Psi(t)]=c(t)+h(t),\ t\in\Gamma,\\
&w(z)=e^{iS(z)}\Psi(z),\\
&\Lambda(t)=\lambda(t)\overline{e^{iS(t)}}=\begin{cases} t^K,\ t\in\Gamma_0,\\ e^{i\theta_j},t\in\Gamma_j, j=1,...,N,\end{cases}\\
&X(z)=\begin{cases} z^K e^{iS(z)}, z\in\Gamma_0,\\ e^{i\theta_j}e^{iS(z)}, z\in\Gamma_j, j=1,...,N,\end{cases}
\end{aligned}\tag{2.2}$$

where the index is also equal to K, and the point constant (1.9) is also equal to

$$\mathrm{Im}[\overline{\Lambda(a_j)}\Psi(a_j)]=b_j,\ j\in J,\tag{2.3}$$

and $\Psi(z)$ satisfies the complex equation

$$\begin{aligned}
&\Psi_{\bar z}=\{\Phi[z,\Psi(z)e^{iS(z)},[\Psi(z)e^{iS(z)}]_z\}e^{-iS(z)}-iS'\Psi(z)\},\\
&\Psi_{\bar z}=Q_1(z)\Psi_z+e^{-2i\mathrm{Re}S(z)}Q_2(z)\overline{\Psi}_{\bar z}+[A_1(z)+e^{-iS(z)}(e^{iS(z)})'Q_1]\Psi\ .\\
&\quad+[e^{-2i\mathrm{Re}S(z)}A_2+e^{-i\overline{S(z)}}(e^{-i\overline{S(z)}})_{\bar z}]\overline{\Psi}+e^{-iS(z)}A_3,\ z\in D.
\end{aligned}\tag{2.4}$$

The above boundary value problem will be called Problem B′. It is easy to see the equivalence of Problem B with the boundary conditions (1.7), (1.9) for (1.1) and Problem B′ with the boundary conditions (2.2), (2.3) for (2.4).

Theorem 2.1 *Under the above conditions, Problem* B *with the index* $K\geq 0$ *for analytic functions has a unique solution, which can be expressed by the integral as stated in* (2.10) *below.*

Proof In this case: the index $K\geq 0$, if there are two solutions $\Psi_1(z)$, $\Psi_2(z)$ two solutions of Problem B′ for analytic functions, then $\Psi(z)=\Psi_1(z)-\Psi_2(z)$ satisfies the boundary conditions

$$\begin{aligned}
&\mathrm{Re}[\overline{\Lambda(t)}\Psi(t)]=h(t),\ t\in\Gamma,\\
&\mathrm{Im}[\overline{\Lambda(a_j)}\Psi(a_j)]=0,\ j\in J,
\end{aligned}$$

thus we can derive the contraction inequality

$$2K+1\leq 2N_D+N_\Gamma=2K,$$

where N_D, N_Γ are denoted the zero numbers of $\Psi(z)$ in D and $\mathit{\Gamma}$ respectively, this zero points formula can be seen as in [1,5]. This contradiction verifies

$\Psi(z) \equiv 0$ in D, and then $\Psi_1(z) = \Psi_2(z)$ in D. Hence the solution of Problem B for analytic functions is unique.

Next we shall find the solution of Problem B′, and then obtain the solution of Problem B. We can introduce

$$P_0(z,t)=P_{N+1}(z,t)=\begin{cases}\dfrac{X(z)\lambda(z)c(t)(t+z)}{X(t)(t-z)\,t}, \; t\in\Gamma_0,\\ 0, \; t\in\Gamma_j, \; j=1,...,N,\end{cases}$$
$$P_j(z,t)=\begin{cases}\dfrac{e^{i\theta_j}X(z)\lambda(z)c(t)(t+z-2z_j)}{X(t)(t-z)(t-z_j)}, \; t\in\Gamma_j,\\ 0, \; t\in\Gamma\backslash\Gamma_j, \; j=1,...,N,\end{cases} \tag{2.5}$$

and find a solution of the boundary value problem with the boundary conditions

$$\begin{gathered}\mathrm{Re}[\overline{\Lambda(z)}P_*(z,t)]=-\mathrm{Re}[\overline{\Lambda(z)}Q(z,t)]+h(z,t), \; z\in\Gamma,\\ Q(z,t)=\sum_{m=1,m\neq j}^{N+1}P_m(z,t), \; z\in\Gamma_j, j=1,...,N+1,\\ \mathrm{Im}[\overline{\Lambda(a_j)}P_*(a_j,t)]=-\mathrm{Im}[\overline{\Lambda(a_j)}Q(a_j,t)], \; j\in J,\end{gathered} \tag{2.6}$$

and

$$P(z,t)=\sum_{j=1}^{N+1}P_j(z,t)+P_*(z,t), \; t\in\Gamma \tag{2.7}$$

is the Schwarz kernel of Problem B′. Thus we get the representation of solutions of Problem B as follows:

$$\Psi(z)=\frac{1}{2\pi i}\int_\Gamma P(z,t)c(t)dt+\Psi_0(z), \tag{2.8}$$

in which $\Psi_0(z)$ is the solution of corresponding homogeneous problem, which can be determined by some point conditions

$$\mathrm{Im}[\overline{\Lambda(a_j)}\Psi_0(a_j)]=b_j-\mathrm{Im}[\frac{\overline{\Lambda(a_j)}}{2\pi i}\int_\Gamma P(a_j,t)c(t)dt], \; j\in J. \tag{2.9}$$

Thus the solution of original boundary value problem (Problem B) can be expressed as

$$w(z)=\Phi(z)=\Psi(z)e^{iS(z)}=\frac{1}{2\pi i}\int_\Gamma T(z,t)c(t)dt+\Phi_0(z), \tag{2.10}$$

where $T(z,t)=P(z,t)e^{iS(z)}$, $T(z,t)$ is the Schwarz kernel, and $w_0(z)=\Phi_0(z)=\Psi_0(z)e^{iS(z)}$ is a solution of Problem B_0 with the point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}\Phi_0(a_j)]=b_j-\mathrm{Im}[\frac{\overline{\lambda(a_j)}}{2\pi i}\int_\Gamma T(a_j,t)c(t)dt], \; j\in J. \tag{2.11}$$

Theorem 2.2 *Under the above conditions, Problem* B *with the index* $K < 0$ *for analytic functions has a unique solution, which can be expressed a integral as stated in* (2.16) *below.*

Proof The unique of solutions of Problem B for analytic functions can seen as in [5]. Moreover similar to the proof of Theorem 2.1, we first find the solution of Problem B$'$. If $K < 0$, we introduce

$$\begin{aligned} &P_0(z,t) = P_{N+1}(z,t) = \begin{cases} \dfrac{2z^{|K|}e^{iS(z)}\lambda(t)c(t)}{e^{iS(t)}(t-z)t^{|K|}}, \ t\in\Gamma_0, \\ 0, \ t \in \Gamma_j, \ j = 1, ..., N, \end{cases} \\ &P_j(z,t) = \begin{cases} \dfrac{e^{i\theta_j}e^{iS(z)}\lambda(t)c(t)(t+z-2z_j)}{e^{iS(t)}(t-z)(t-z_j)}, \ t\in\Gamma_j, \\ 0, \ t \in \Gamma\backslash\Gamma_j, \ j=1, ..., N, \end{cases} \end{aligned} \tag{2.12}$$

Similarly to the proof of Theorem 2.1, we can find a solution of the boundary value problem with the boundary conditions

$$\begin{aligned} &\mathrm{Re}[\overline{\Lambda(z)}P_*(z,t)] = -\mathrm{Re}[\overline{\Lambda(z)}Q(z,t)] + h(z,t), \ z\in\Gamma, \\ &Q(z,t) = \sum_{m=1, m\neq j}^{N+1} P_m(z,t), \ z \in \Gamma_j, \ j=1, ..., N+1, \end{aligned} \tag{2.13}$$

and

$$P(z,t) = \sum_{j=1}^{N+1} P_j(z,t) + P_*(z,t), \ t \in \Gamma \tag{2.14}$$

is the Schwarz kernel of Problem B$'$. Thus we get the representation of solutions of Problem B$'$ as follows:

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma P(z,t)c(t)dt. \tag{2.15}$$

Thus the solution of original boundary value problem (Problem B) can be expressed as

$$w(z) = \Phi(z) = \Psi(z)e^{iS(z)} = \frac{1}{2\pi i}\int_\Gamma T(z,t)c(t)dt, \tag{2.16}$$

in which $T(z,t) = P(z,t)e^{iS(z)}$, $T(z,t)$ is the Schwaez kernel. In the above discussion, we have to use the $N - 2K - 1$ solvability conditions of Problem B, if $K < 0$.

3 Integral representation of solutions for homogeneous Riemann-Hilbert problem for elliptic complex equations

We first consider the homogeneous boundary problem (Problem B_0) for the complex equation (1.1), and give the integral representation of solutions of Problem B_0 for (1.1).

Introduce the two double integral operator of homogeneous Riemann-Hilbert problem in the simple connected domain D as follows

$$\begin{aligned} T_1 F &= -\frac{2}{\pi}\iint_D \Big[\frac{F(\zeta)}{\zeta - z} - \frac{z^{2K+1}\overline{F(\zeta)}}{1-\overline{\zeta}z}\Big] d\sigma_\zeta, \ \text{ if } K \geq 0, \\ T_2 F &= -\frac{2}{\pi}\iint_D \Big[\frac{F(\zeta)}{\zeta - z} - \frac{\overline{\zeta}^{-2K-1}\overline{F(\zeta)}}{1-\overline{\zeta}z}\Big] d\sigma_\zeta \ \text{ if } \ K < 0. \end{aligned} \tag{3.1}$$

It is easy to see that

$$\begin{aligned} &\mathrm{Re}[\overline{z}^K T_1 F(z)] = 0 \ \text{ on } \Gamma = \{|z| = 1\}, \ \text{ if } K \geq 0, \\ &\mathrm{Re}[\overline{z}^K T_2 F(z)] = 0 \ \text{ on } \Gamma = \{|z| = 1\}, \ \text{ if } K < 0, \end{aligned} \tag{3.2}$$

if there are $-2K-1$ solvability conditions hold, namely $-2K-1$ real equalities hold, i.e.

$$\begin{aligned} &ic_0 - \frac{2}{\pi}\iint_D \zeta^{-K-1} F(\zeta) d\sigma_\zeta, \ c_0 \text{ is a real constant}, \\ -\frac{2}{\pi}&\iint_D [\zeta^{-K-m-1}F(\zeta) + \overline{\zeta}^{-K+m-1}\overline{F(\zeta)}] d\sigma_\zeta = 0, \ m = 1, ..., -K-1. \end{aligned} \tag{3.3}$$

(see (1.33), Chapter II, [6]).

For the $N+1$-connected domain $D\,(N > 0)$, the solution of homogeneous Riemman-Hilbert boundary value problem (Problem B_0) can be similarly represented by

$$\begin{aligned} \tilde{T}F &= -\frac{2}{\pi}\iint_D [P(z,\zeta)F(\zeta) + Q(z,\zeta)\overline{F(\zeta)}] d\sigma_\zeta = TF + \sum_{j=1}^{N+1} + T_* F, \\ P(z,\zeta) &= \frac{1}{2}[G_1(z,\zeta) + G_2(z,\zeta) + H_1(z,\zeta) - H_2(z,\zeta)], \ z,\zeta \in \overline{D}, \\ Q(z,\zeta) &= \frac{1}{2}[G_1(z,\zeta) - G_2(z,\zeta) + H_1(z,\zeta) + H_2(z,\zeta)], \ z,\zeta \in \overline{D}, \end{aligned}$$

$$
\begin{aligned}
&G_1(z,\zeta)=\frac{1}{\zeta-z}+\sum_{j=1}^{N+1}g_j(z,\zeta),\ G_2(z,\zeta)=\frac{1}{\zeta-z}-\sum_{j=1}^{N+1}g_j(z,\zeta),\ z,\zeta\in D,\\
&g_0(z,\zeta)=g_{N+1}(z,\zeta)=\frac{z^{2K+1}}{1-\overline{\zeta}z}\ \text{if}\ K\ge 0\ \text{and}\ \frac{\overline{\zeta}^{-2K-1}}{1-\overline{\zeta}z}\ \text{if}\ K<0,\\
&g_j(z,\zeta)=\frac{e^{2i\theta_j}(z-z_j)}{r_j^2-(\overline{\zeta-z_j})(z-z_j)},\ \ j=1,...,N,
\end{aligned}
\tag{3.4}
$$

where $H_1(t,\zeta)$, $H_2(t,\zeta)$ are the solution with the boundary conditions

$$
\begin{aligned}
&\mathrm{Re}[\overline{\Lambda(t)}H_1(t,\zeta)]+\mathrm{Re}[\overline{\Lambda(t)}\sum_{m=1,m\neq j}^{N+1}g_m(t,\zeta)]=h(t),\ t\in\Gamma,\\
&\mathrm{Im}[\overline{\Lambda(a_j)}H_1(a_j,\zeta)]+\mathrm{Im}[\overline{\Lambda(a_j)}\sum_{m=1,m\neq j}^{N+1}g_m(a_j,\zeta)]=0,\ j\in J,\\
&\mathrm{Re}[\overline{\Lambda(t)}iH_2(t,\zeta)]+\mathrm{Re}[\overline{\Lambda(t)}i\sum_{m=1,m\neq j}^{N+1}g_m(t,\zeta)]=h(t),\ t\in\Gamma,\\
&\mathrm{Im}[\overline{\Lambda(a_j)}iH_2(a_j,\zeta)]+\mathrm{Im}[\overline{\Lambda(a_j)}i\sum_{m=1,m\neq j}^{N+1}g_m(a_j,\zeta)]=0,\ j\in J,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&TF=-\frac{1}{\pi}\iint_D\frac{F(\zeta)}{\zeta-z}d\sigma_\zeta,\\
&T_jF=-\frac{1}{\pi}\iint_D\frac{e^{2i\theta_j}(z-z_j)\overline{F(\zeta)}}{r_j^2-(\overline{\zeta-z_j})(z-z_j)}d\sigma_\zeta,\ j=1,...,N,\\
&T_0F=T_{N+1}F=-\frac{1}{\pi}\iint_D g_0(z,\zeta)\overline{F(\zeta)}d\sigma_\zeta,\\
&T_*F=\frac{1}{2\pi}\iint_D[(H_1-H_2)F(\zeta)+(H_1+H_2)\overline{F(\zeta)}]d\sigma_\zeta.
\end{aligned}
\tag{3.6}
$$

In fact we only use the integral representation of Problem B_0 form equation (1.1) later on.

Theorem 3.1 *Let the complex equation* (1.1) *satisfy Condition* C. *Then any solution* $w(z)$ $(w_{\bar z}\in L_{p_0}(\bar D), 2<p_0\le p)$ *of Problem* B *for* (1.1) *possesses the representation*

$$w(z)=\Phi(z)+\tilde{T}\rho, \tag{3.7}$$

where $\rho(z)=w_{\bar z}$, $\Phi(z)$ *is an analytic function as stated in* (2.5) *or* (2.12) *in* D, *and* $\tilde{T}\rho$ *is as stated in* (3.6), *and* $\Phi(z)$ *satisfies the estimates*

$$C_\beta[\Phi(z),\overline{D}]\le M_1,\ L_{p_0}[\Phi'(z),\overline{D}]\le M_2, \tag{3.8}$$

in which $\beta = 1 - 2/P_0$, $M_j = M_j(p_0, \beta, k, D)), j = 1, 2$, $k = k(k_0, k_1, k_2, k_3)$. *Moreover* $\tilde{T}\rho$ *satisfies the homogeneous boundary condition of Problem* B, *and* $\tilde{S}F = \tilde{F}_z$ *possesses the properties*

$$||\tilde{S}F||_{L_{p_0}(\overline{D})} \le \tilde{\Lambda}||F||_{L_{p_0}(\overline{D})},\ \tilde{\Lambda} \le 1,\ \text{if } K \le 0. \tag{3.9}$$

and for a positive number $q_0 < 1$ *there exists a constant* $2 < p_0 \le P$ *such that*

$$q_0 \tilde{\Lambda}_{p_0} < 1. \tag{3.10}$$

By using (3.6), Chapter I, [1], Theorems 2.1 and 2.2, we can get (3.10), and (3.11),(3.12) can be obtained by the method of Theorem 3.5, Chapter I, [4] and Lemma 2.7, Chapter II, [6].

4 The method of integral equations for solving Riemann-Hilbert problem for elliptic complex equations in multiply connected domains

By using the method as in Theorems 4.6-4.7, Chapter II, [4] and Section 4, Chapter III, [11], we can derive the solvability results about Problem B for equation (1.1) with Condition C. First of all, we give the estimates of solutions of Problem B for the equation (1.1).

Theorem 4.1 *Suppose that the first order complex equation* (1.1) *satisfies Condition* C. *Then any solution* $w(z)$ *of Problem* B *for the complex equation* (1.1) *satisfies the conditions*

$$C_\beta[w(z), \overline{D}] < M_3, ,\ L_{p_0}[|w_{\bar z}| + |w_z|, \overline{D}] \le M_4, \tag{4.1}$$

in which $\beta = 1 - 2/p_0$, $k = k(k_0, k_1, k_2, k_3)$, $M_j = M_j(q_0, p_0, \beta, k, D), j = 3, 4)$ *are positive constants.*

Proof Due to the solution $w(z)$ of Problem B for the complex equation (1.1) can be expressed as (3.9), and the analytic function $\Phi(z)$ possesses the properties in (3.10), it is necessary to consider any solution $W(z)$ of the complex equation of $W(z) = \tilde{T}\omega$:

$$\left.\begin{aligned} &W_{\bar z} = Q_1(z)W_z + Q_2(z)\overline{W}_{\bar z} + A_1(z)W + A_2(z)\bar{W} + A(z),\\ &A = Q_1(z)\Phi'(z) + Q_2(z)\overline{\Phi'(z)} + A_1(z)\Phi(z) + A_2(z)\overline{\Phi(z)} + A_3(z), \end{aligned}\right\} z \in D, \tag{4.2}$$

where $A(z) \in L_{p_0}(\overline{D})$.

We first verify the uniqueness of solutions of the homogeneous problem B_0 with the index $K \geq 0$, i.e. the solution $W(z) \equiv 0$ of the homogeneous problem B_0 for the homogeneous equation

$$W_{\bar{z}}=Q_1(z)W_z + Q_2(z)\overline{W}_{\bar{z}} + A_1(z)W + A_2(z)\overline{W} \ \text{ in } D \tag{4.3}$$

with the index $K \geq 0$. The solution $W(z)$ of (4.3) can be expressed as

$$W(z) = \Psi[\zeta(z)]e^{\phi(z)} \ \text{ in } D, \tag{4.4}$$

where $\zeta(z) = \eta(\chi(z))$ is a homeomorphism in $\overline{D}$, which quasiconformally maps D onto the $N+1$-connected circular domain G with boundary $L = \zeta(\Gamma)$ in $\{|\zeta| < 1\}$, such that three points on Γ onto three points on L respectively, $\Psi(\zeta)$ is an analytic function in G, $\phi(z) = i\tilde{T}_1 g(z)$, $\chi(z) = z+Th$ are the solutions of the complex equations

$$\phi(z) = i\tilde{T}_1 g, \ \chi(z) = z + Th \tag{4.5}$$

of the complex equations

$$\begin{gathered}\phi_{\bar{z}} = [Q_1 + Q_2\overline{W_z}/W_z]\phi_z + A_1 + A_2\overline{W}/W, \ z \in D,\\ \chi_{\bar{z}} = [Q_1 + Q_2\overline{W_z}/W_z]\chi_z, \ z \in D,\end{gathered} \tag{4.6}$$

respectively, $\tilde{T}_1 g$ is a double integral satisfying the modified Dirichlet boundary condition in D, $\chi(z)$ is a homeomorphism in $\overline{D}$, $\zeta = \eta(\chi)$ is a univalent analytic function, which conformally maps $E = \chi(D)$ onto the domain G, $\zeta(z) = \eta[\chi(z)]$ in D, and $\Psi(\zeta)$ is an analytic function in G. Due to $\tilde{S}h = [\tilde{T}h]_z$ possesses the properties in (3.9) and (3.10), and $\Pi h = [Th]_z$ has the similar properties, we can get

$$\begin{aligned}L_{p_0}[g(z), \overline{D}] &\leq L_{p_0}[|A_1|+|A_2|, \overline{D}]/(1 - q_0\tilde{\Lambda}_{p_0}),\\ L_{p_0}[h(z), \overline{D}] &\leq L_{p_0}[|A_1|+|A_2|, \overline{D}]/(1 - q_0\Lambda_{p_0}),\end{aligned}$$

by the principle of contract mapping, we can obtain that $\psi(z)$, $\chi(z)$ of the equations in (4.6), and $\psi(z)$, $\chi(z), \zeta(z)$ satisfy the estimates

$$\begin{gathered}C_\beta[\phi, \overline{D}] \leq k_4, \ L_{p_0}[|\phi_{\bar{z}}| + |\phi_z|, \overline{D}] \leq k_4, \ L_{p_0}[|\chi_{\bar{z}}| + |\chi_z|, \overline{D}] \leq k_5,\\ C_\beta[\zeta(z), \overline{D}] \leq k_4, \ C_\beta[z(\zeta), \overline{G}] \leq k_4,\end{gathered} \tag{4.7}$$

in which $\beta = 1-2/p_0$, $p_0\,(2 < p_0 \leq p)$, $k_j = k_j(q_0, p_0, \beta, k_0, k_1, D)\,(j = 4, 5)$ are non-negative constants dependent on $q_0, p_0, \beta, k_0, k_1, D$, and $\Psi[\zeta(z)] = \tilde{T}\omega$ satisfies the the boundary condition

$$\text{Re}[\overline{\lambda(z(\zeta))}\Psi(\zeta)] = h(z(\zeta)) \ \text{ in } L \tag{4.8}$$

of homogeneous Problem B for analytic functions. If $\Phi(\zeta)\not\equiv 0$, thus we can derive the contraction inequality

$$2K+1\le 2N_D+N_\Gamma=2K,$$

where N_G, N_L are denoted the zero numbers of $\Psi(\zeta)$ in G and L respectively, this zero points formula is the same as in the proof of Theorem 2.1. This contradiction verifies $\Psi(\zeta)\equiv 0$ in G, and then $W(z)\equiv 0$ in D. Hence the solution of Problem B_0 for (4.2) is unique.

If $K<0$, we can use Theorem 4.12, [1], i.e. Problem B_0 has a solution $W(z)=\Psi[\zeta(z)]e^{i\phi(z)}$ in D if and only if $h[z(\zeta)]$ satisfies the conditions

$$\int_L [\overline{\Lambda(\zeta)}]^{-1}\Phi_n(\zeta)h[z(\zeta)]\zeta'(s)ds=0,\ n=1,...,N-2K-1, \tag{4.10}$$

where $\Lambda(\zeta)=\lambda(z(\zeta))$, $\Phi_n(\zeta)\,(n=1,...,N-2K-1)$ are linearly independent solutions of the corresponding conjugate homogeneous problem B_0' with the boundary condition

$$\mathrm{Re}[[\overline{\Lambda(\zeta)}]^{-1}\Phi_n(\zeta)\zeta'(s)]=0,\ \zeta\in L,\ n=1,...,N-2K-1, \tag{4.11}$$

whose index is $K'=N-K-1$. Thus

$$\begin{aligned}&\sum_{j=0}^{N}h_j\int_{L_j}\mathrm{Im}[\overline{\Lambda(\zeta)}]^{-1}\Phi_n(\zeta)\zeta'(s)ds\\&+\sum_{m=1}^{-K-1}\int_{L_0}\mathrm{Re}[(h_m^+ +ih_m^-)[z(\zeta)]^m]\mathrm{Im}[\overline{\Lambda(\zeta)}]^{-1}\Phi_n(\zeta)\zeta'(s)]ds=0,\\&\qquad n=1,...,N-2K-1.\end{aligned}$$

If $h_j=0\,(1,...,N)$, $h_m^{\pm}\,(m=1,...,-K-1)$ are not all equal to zero, then the coefficients determinant of the above algebraic system certainly equals zero. Therefore we can find real constants $c_1,...,c_{N-2K-1}$, which are not all equal to zero, such that

$$\begin{aligned}&\int_{L_j}\mathrm{Im}[\overline{\Lambda(\zeta)}]^{-1}\Phi(\zeta)\zeta'(s)ds=0,\ j=0,1,...,N,\\&\int_{L_j}\mathrm{Im}[[\overline{\Lambda(\zeta)}]^{-1}\Phi(\zeta)\zeta'(s)\begin{Bmatrix}\cos[m\arg z(\zeta)]\\ \sin[m\arg z(\zeta)]\end{Bmatrix}]ds=0,\ m=1,...,-K-1.\end{aligned}$$

where $\Phi(z)=\sum_{n=1}^{N-2K-1}c_n\Phi_n(z)$ is a solution of Problem B_0' and $\Phi(z)\neq 0$ in $G=\zeta(D)$. From the first formula in (4.13), there exists points $a_j^*\in L_j\,(j=1,...,N)$ so that

$$\overline{\Lambda(\zeta)}]^{-1}\Phi(\zeta)\zeta'(s)|_{\zeta=a_j^*}=0,\ \text{i.e. } \Phi^*(a_j^*)=0,\ j=1,...,N.$$

In addition, let $U(z)$ be a harmonic function in $D_0=\{|z|<1\}$, which satisfies the boundary condition

$$U(z)=\mathrm{Im}[[\overline{\Lambda(\zeta)}]^{-1}\Phi(\zeta)\zeta'(s)]|_{\zeta=\zeta(z)} \text{ on } |z|=1.$$

Let $s=s(\theta)$ denote the corresponding relation between s and θ in $\zeta=e^{is}=\zeta(e^{i\theta})$. Then we have

$$U(z)=\frac{1}{2\pi}\int_0^{2\pi}U(t)\mathrm{Re}\frac{t+z}{t-z}ds(\theta)=\mathrm{Re}\frac{z^{|K|}}{\pi}\int_0^{2\pi}\frac{U(t)ds(\theta)}{t^{|K|-1}(t-z)} \text{ in } |z|<1. \tag{4.12}$$

From the above formula and the second formula in (4.7), we can see that there exists the points $a_j^*\in L_0\,(j=N+1,...,N-2K)$, so that

$$\overline{[\Lambda(\zeta)]}^{-1}\Phi(\zeta)\zeta'(s)]|_{\zeta=a_j^*}=0, \text{ i.e. } \Phi(a_j^*)=0,\, j=N+1,...N-2K.$$

Thus we get the absurd inequality

$$2N-2K\le 2N_G+N_L=2(N-K-1)=2N-2K-2$$

This contradiction proves that

$$h_j=0\,(j=0,1,...,N),\, h_m^{\pm}=0\,(m=1,...,-K-1)$$

in (4.7), and so $\Phi(\zeta)=0$ in G. Consequently $W(z)=0$ in D, and then $W_1(z)=W_2(z)$ in D. This prove the uniqueness of solutions of Problem B_0 with the index $K<0$.

Denote $4d=\min_{z\in\Gamma}|z|$ and $D_1=\{|z|\le d\}$, $D_2=\{d<|z|\le 2d\}$, $D_3=\{2d<|z|\le 3d\}$, $D_4=\{3d<|z|\le 4d\}$, and construct two continuously differential functions

$$\tau_1(z)=\begin{cases}0 \text{ in } D_1,\\ 1 \text{ in } \overline{D}\backslash\{D_1\cup D_2\},\\ \tau_1(z) \text{ in } D_2,\end{cases}\quad \tau_2(z)=\begin{cases}1 \text{ in } D_1\cup D_2,\\ 0 \text{ in } \overline{D}\backslash\{D_1\cup D_2\cup D_3\},\\ \tau_2(z) \text{ in } D_3,\end{cases}$$

where $0\le\tau_1(z)\le 1$ in D_2 and $0\le\tau_2(z)\le 1$ in D_3. From (4.2), we see that two functions $\tilde{W}(z)=\tau_1(z)z^{-K}W(z)$ and $\hat{W}(z)=\tau_2(z)W(z)$ are the solutions of following complex equations

$$\begin{aligned}&\tilde{W}_{\bar z}=Q_1\tilde{W}_z+Q_2\overline{\tilde{W}}_{\bar z}+A_1(z)\tilde{W}+[A_2(z)\tau_1z^{-K}/\overline{\tau_1z^{-K}}]\overline{\tilde{W}}+\tilde{A},\\ &\tilde{A}=[(\tau_1z^{-K})_{\bar z}-Q_1(\tau_1z^{-K})_z]W-Q_2\overline{(\tau_1z^{-K})_z}\overline{W}+\tau_1z^{-K}A(z) \text{ in } D,\\ &\hat{W}_{\bar z}=Q_1\hat{W}_z+Q_2\overline{\hat{W}}_{\bar z}+A_1(z)\hat{W}+[A_2(z)\tau_2/\overline{\tau_2(z)}]\overline{\hat{W}}+\hat{A},\\ &\hat{A}=[\tau_{1\bar z}-Q_1\tau_{1z}]W-Q_2\overline{\tau_{1z}}\overline{W}+\tau_2A(z) \text{ in } D,\end{aligned}\tag{4.13}$$

and satisfy the boundary conditions

$$\begin{aligned} &\mathrm{Re}[\overline{\Lambda(z)}\tilde{W}(z)] = h(z) \ \text{ on } \Gamma, \\ &\mathrm{Re}[\overline{\Lambda(z)}\hat{W}(z)] = 0 \ \text{ on } \Gamma, \end{aligned} \tag{4.14}$$

respectively, the indexes of above boundary value problems are equal to $\kappa = 0$, and the function $W(z)$ is bounded in $\overline{D}$ from (3.8),(3.9),(4.4),(4.7). Moreover by using Theorem 4.3, Chapter II, [4], or the the reduction to absurdity as stated in the second method below, we can obtain the estimates

$$\begin{aligned} &C_\beta[\tilde{W}(z), \overline{D}] \le M_5, \ L_{p_0}[|\tilde{W}_{\bar{z}}| + |\tilde{W}_z|, \overline{D}] \le M_6, \\ &C_\beta[\hat{W}(z), \overline{D}] \le M_7, \ L_{p_0}[|\hat{W}_{\bar{z}}| + |\hat{W}_z|, \overline{D}] \le M_8, \end{aligned}$$

where $M_j = M_j(q_0, p, \beta, k, D)\,(j = 5, 6, 7, 8)$ are positive constants. In particular we have

$$\begin{aligned} &C_\beta[W(z), \overline{D}\backslash\{D_1 \cup D_2\}] \le M_5, \ L_{p_0}[|W_{\bar{z}}| + |W_z|, \overline{D}\backslash\{D_1 \cup D_2\}] \le M_6, \\ &C_\beta[W(z), D_1 \cup D_2] \le M_7, \ L_{p_0}[|W_{\bar{z}}| + |W_z|, D_1 \cup D_2] \le M_8. \end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned} C_\beta[W(z), \overline{D}] &\le M_9 = M_9(M_5, M_7, \tau_1, \tau_2, K), \\ L_{p_0}[|W_{\bar{z}}| + |W_z|, \overline{D}] &\le M_{10} = M_{10}(M_6, M_8, \tau_1, \tau_2, K). \end{aligned} \tag{4.15}$$

Next we prove the solvability of Problem B for the equation (1.1).

Theorem 4.2 *Under the conditions in Theorem* 4.1, *Problem* B *for* (1.1) *is solvable.*

Proof We use the Fredholm theorem of integral equation

$$\omega(z) = Q_1(z)\tilde{S}w + Q_2(z)\overline{\tilde{S}\omega} + A_1(z)\tilde{T}\omega + A_2(z)\overline{\tilde{T}\omega(z)} + A(z), \ \omega(z) \in L_{p_0}(\overline{D}), \tag{4.16}$$

which is corresponding to the complex equation (4.2) trough the relation $W(z) = \tilde{T}\omega$. Because $\tilde{T}\omega$ is a complete continuous operator, the inverse operator of homogeneous integral equation

$$\omega(z) = Q_1(z)\tilde{S}w + Q_2(z)\overline{\tilde{S}\omega} + A_1(z)\tilde{T}\omega + A_2(z)\overline{\tilde{T}\omega(z)} \tag{4.17}$$

is also complete continuous. Provided we verify that the homogeneous integral equation only has the trivial solution, then the above nonhomogeneous integral equation has a unique solution. In the following we shall prove that the above Problem B_0 has no non-zero solution. Let $W(z)$ be any solution of Problem

B_0, we shall verify $W(z)\equiv 0$ in D. In fact from the proof of Theorem 4.1, $W(z)$ can be repressed as

$$W(z)=[\Psi(\zeta(z))+\psi(z)]e^{\phi(z)} \text{ in } D,$$

where $\phi(z)$, $\Psi[\zeta(z)]$ are as sated as in (4.5), $\psi(z)=Th=0$ is the solution of the equation

$$\psi_{\bar z}=[Q_1+Q_2\overline{W_z}/W_z]\psi_z \text{ in } D,$$

and $\Psi(\zeta)$ is an analytic function in G satisfying the homogeneous boundary condition of Problem B_0, hence by the proof of Theorem 4.1, $\Psi(\zeta)\equiv 0$ in D and then $W(z)\equiv 0$ in D. This show that the above homogeneous integral equation only has zero solution, and then the nonhomogeneous integral equation has a unique solution.

Theorem 4.3 *Let the system* (1.1) *satisfies Condition* C. *Then Problem* A $(K\le 0)$ *has* $-2K+N-1$ *solvability conditions and its solution* $w(z)$ *can be written in the form* $w(z)=\Phi(z)+\tilde T\rho$, *where* $\Phi(z)$, $\tilde T\rho$ *are as stated in* (3.9). *Moreover if* $K\ge 0$, *under* N *solvability conditions, the general solution of Problem* A *can be written as*

$$w(z)=w_0(z)+\sum_{k=1}^{2K+1}d_kw_k(z), \tag{4.18}$$

where $w_0(z)$ *is a solution of nonhomogeneous boundary value problem (Problem* A*) forb* (1.1)*, and* $d_k\,(k=1,...,2K+1)$ *are the arbitrary real constants,* $w_k(z)\,(k=1,...,2K+1)$ *are linearly independent solutions of homogeneous boundary value problem (Problem* A_0*) for* (1.1).

Proof The above theorem shows that the general solution of Problem B for (1.1) includes the number of arbitrary real constants as stated in the above theorem. In fact, for the linear case of the complex equation (1.1) satisfying Condition C, under N solvability conditions, its general solution of Problem A with the index $K\ge 0$ can be written as (4.18), where $w_0(z)$ is a solution of nonhomogeneous boundary value problem (Problem A), and $d_k\,(k=1,...,2K+1)$ are the arbitrary real constants, $w_k(z)\,(k=1,...,2K+1)$ are linearly independent solutions of homogeneous boundary value problem (Problem A_0), which can be satisfied the point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}w_k(a_j)]=\delta_{jk},\ j,k=1,...,2K+1,$$

where $\delta_{jk}=1,$ if $j=k=1,...,g$ and $\delta_{jk}=0,$ if $j\ne k,\ 1\le j,k\le 2K+1.$

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