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LORENTZIAN PYTHAGOREAN TRIPLES and LORENTZIAN UNIT CIRCLE

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Abstract

Pythagorean triples on Lorentzian plane is that the differences of the squares of the sides of a right triangle equals the square of the hypotenuse. The edges of the right triangles are called Pythagorean triples when they are whole numbers. The study of these Pythagorean triples [1] began long before the time of Pythagoras. Babylonians and ancient Egypt also used these triples.In this study, we observe the relationship between Pythagorean triples and rational coordinates on Lorentzian unit circle.

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1 Preliminaries

In this chapter we give some definitions and theorems which we will use throughout this paper.

Definition 1.1. *Let V be a vector space. The scalar product*

$$\begin{array}{lccc} \langle,\rangle : & V \times V & \to & R \\ & \left(\overrightarrow{X}, \overrightarrow{Y}\right) & \to & \langle \overrightarrow{X}, \overrightarrow{Y} \rangle = \sum_{i=1}^{n-1} x_i y_i - x_n y_n \end{array} \quad (1)$$

is called a scalar product with index 1 in the sense of Lorentzian, and the pair $\{V, \langle,\rangle\}$ is called a Lorentzian vector space.In particular, if $V = R^2$, the pair, $\{R^2, \langle,\rangle\}$ is called the 2-dimensional Lorentzian vector space and it is denoted by L_1^2[2].

Definition 1.2. *A non-zero vector $\overrightarrow{X}$ in is called a time-like vector if $\langle \overrightarrow{X}, \overrightarrow{X} \rangle < 0$, null-like or light-like vector if $\langle \overrightarrow{X}, \overrightarrow{X} \rangle = 0$, space-like vector if $\langle \overrightarrow{X}, \overrightarrow{X} \rangle > 0$.*

Definition 1.3. *The norm of the vector$\overrightarrow{X}$ is defined by*

$$\| \overrightarrow{X} \| = \sqrt{| \langle \overrightarrow{X}, \overrightarrow{X} \rangle |} \quad (2)$$

[3].

Definition 1.4. *For Lorentzian vector ,*
i. $\| \overrightarrow{X} \| > 0$,
ii. $\| \overrightarrow{X} \| = 0 \Leftrightarrow \overrightarrow{X}$ *is a null vector,*
iii. If $\overrightarrow{X}$ *is a time-like vector then* $\| \overrightarrow{X} \|^2 = -\langle \overrightarrow{X}, \overrightarrow{X} \rangle$,
iv. If $\overrightarrow{X}$ *is a space–like vector then* $\| \overrightarrow{X} \|^2 = \langle \overrightarrow{X}, \overrightarrow{X} \rangle$.

Definition 1.5. *If $\langle \overrightarrow{X}, \overrightarrow{Y} \rangle = 0$ then the vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ are called vertical in Lorentzian mean.*

Definition 1.6. *Let $\overrightarrow{X} \in L_1^2$ be a time-like vector where$\overrightarrow{e} = (0, 1)$*
If $\langle \overrightarrow{X}, \overrightarrow{e} \rangle < 0$ then the vector $\overrightarrow{X}$ called a future-pointing time-like vector,
If $\langle \overrightarrow{X}, \overrightarrow{e} \rangle > 0$ then the vector $\overrightarrow{X}$ called a past-pointing time-like vector.

Definition 1.7. *Let τ be a set of all V time-like vectors a Lorentzian vector space. For $\overrightarrow{X} \in \tau$ the set $\left\{\overrightarrow{Y} \in \tau \mid \langle \overrightarrow{X}, \overrightarrow{Y} \rangle < 0\right\}$called time cone of $\overrightarrow{Y}$ which includes $\overrightarrow{X}$.*

Theorem 1.8. *Let $\overrightarrow{X}$ and $\overrightarrow{Y}$ be time-like vectors on Lorentzian vector space. These two vectors are on the same cone iff $\langle \overrightarrow{X}, \overrightarrow{Y} \rangle < 0$.*

Lemma 1.9. *Between two vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ on the same time cone, there is a unique angle $\varphi \geq 0$, called the hyperbolic angle, and it satisfies*

$$\langle \overrightarrow{X}, \overrightarrow{Y} \rangle = - \| \overrightarrow{X} \| \| \overrightarrow{Y} \| \, ch\varphi \quad (3)$$

[3].

2 Introduction

In this paper we assume $\overrightarrow{X} \in L_1^2$ is the set of future pointing time-like vectors according to $(+,-)$ metric sing.

Definition 2.1. *The Lorentzian circle is the set of*

$$L_{C_X} = \left\{ \overrightarrow{X} \in L_1^2 \mid \ \| \overrightarrow{OX} \| = r, r = const. \right\} \tag{4}$$

where $\overrightarrow{X} \in L_1^2$ is a time-like vector. The equation of Lorentzian circle is

$$\| \overrightarrow{OX} \| = \sqrt{| \langle \overrightarrow{OX}, \overrightarrow{OX} \rangle |} = r \tag{5}$$

or

$$\| \overrightarrow{OX} \|^2 = -r^2, r = const. \tag{6}$$

or

$$-x^2 + y^2 = r^2, r = const. \tag{7}$$

where $\overrightarrow{OX} = (x, y)$ (Figure 1).

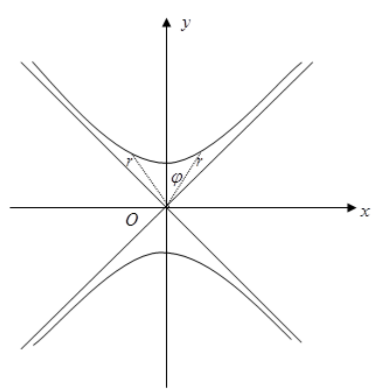

Figure 1: Lorentzian Circle

Lemma 2.2. *Let A, B, C be three points not on a same line, the length of $\overrightarrow{CB}$ space-like vector, $\overrightarrow{AC}$ and $\overrightarrow{AB}$ future-pointing time-like vectors are in order a, b, c and $\langle \overrightarrow{AC}, \overrightarrow{CB} \rangle = 0$ then*

$$ch\varphi = \frac{b}{c}, \ sh\varphi = \frac{a}{c}. \tag{8}$$

Thus the equation is

$$b^2 - a^2 = c^2 \tag{9}$$

[2] (Figure 2).

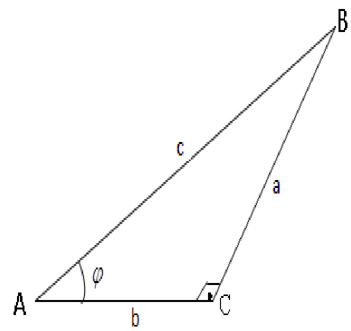

Figure 2: Orthogonal Triangle in L_1^2

Definition 2.3. *A Lorentzian primitive Pythagorean triple is a triple of numbers* (a, b, c) *so that* a, b *and* c *have no common factors and satisfy*

$$-a^2 + b^2 = c^2 \tag{10}$$

(Figure 2).

Conclusion 2.4. *In (10)* a *and* c *can be always swiched, so the problem now is to find all solutions in whole numbers to the equation,*

$$a^2 = b^2 - c^2 \longrightarrow \begin{cases} a \ odd, \\ c \ even, \\ a, b, c, \quad having\ no\ common\ factors. \end{cases}$$

Theorem 2.5. *Every Lorentzian primitive Pythagorean triple* (a, b, c) *with* a *odd and* c *even can be obtained by using the formulas*

$$a = st, \qquad b = \frac{s^2 \ + t^2}{2}, \qquad c = \frac{s^2 \ - t^2}{2} \tag{11}$$

where $s > t \geq 1$ *are chosen to be any odd integers with no common factors.*

Proof. *Our first observation is that if* (a, b, c) *is a Lorentzian primitive Pythagorean triple, then the equation (10) can be factorized as*

$$a^2 = b^2 - c^2 \tag{12}$$

or

$$a^2 = (b - c)(b + c) \,. \tag{13}$$

$b - c$ *and* $b + c$ *seems to have no common factors. Suppose that* d *is a common factor of* $b - c$ *and* $b + c$*; that is,* d *divides both* $b - c$ *and* $b + c$*. Than* d *also divides*

$$(b - c) + (b + c) = 2b \tag{14}$$

and

$$(b - c) - (b + c) = -2c. \tag{15}$$

Thus,

$$d \mid 2b \tag{16}$$

and

$$d \mid -2c. \tag{17}$$

But b and c have no common factor because we are assuming that (a, b, c) is a Lorentzian primitive Pyhtagorean triple. So d must equal to 1 or 2. But d also divides $(b - c)(b + c) = a^2$, and a is odd, so d must be 1. In other words, the only number dividing both $b - c$ and $b + c$ is 1, so $b - c$ and $b + c$ have no common factor. It is known that the product is a square since $(b - c)(b + c) = a^2$. The only way that this can happen is if $b - c$ and $b + c$ are themselves squares. So we can write

$$b + c = s^2 \tag{18}$$

and

$$b - c = t^2 \tag{19}$$

where $s > t \geq 1$ are odd integers with no common factors. Solving these two equations for b and c yields

$$b = \frac{s^2 + t^2}{2} \tag{20}$$

and

$$c = \frac{s^2 - t^2}{2} \tag{21}$$

and then

$$a = \sqrt{(b - c)(b + c)} \tag{22}$$
$$= st. \tag{23}$$

All possible Lorentizian primitive Pyhtagorean triples with are given on the following Table 1. □

s	t	$a = st,$	$b = \frac{s^2+t^2}{2}$	$c = \frac{s^2-t^2}{2}$
3	1	3	5	4
5	1	5	13	12
7	1	7	25	24
9	1	9	41	40
5	3	15	17	8
7	3	21	29	20
7	5	35	37	12
9	5	45	53	28
9	7	63	65	16

Table 1: Lorentzian Primitive Pyhtagorean Triples with $s \leq 9$

3 LORENTZIAN PRIMITIVE PYHTAGOREAN TRIPLES and LORENTZIAN UNIT CIRCLE

Theorem 3.1. *Every point on the Lorentzian unit circle*

$$-x^2+y^2=1 \tag{24}$$

whose coordinates are rational numbers can be obtained from the formula

$$(x,y)=\left(\frac{2m}{1-m^2},\frac{m^2+1}{1-m^2}\right) \tag{25}$$

by substituting in rational numbers for $m\,(m\neq\mp1)$.

Proof. *We described all solutions to*

$$-a^2+b^2=c^2 \tag{26}$$

in whole numbers a,b,c. *If we divide this equation by* c^2, *we obtain*

$$-\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1. \tag{27}$$

So the pair of rational numbers $\left(\frac{a}{b},\frac{b}{c}\right)$ *is a solution to the equation*

$$-x^2+y^2=1. \tag{28}$$

This equation is a Lorentzian unit circle of radius 1. *The Lorentz unit circle has two obvious points with rational coordinates* $(0,1)$ *and* $(0,-1)$. *Suppose that we take any (rational) number* m *and look at the line* L *going through the point* $(0,1)$ *and having slope* m. *The line* L *is given by the equation*

$$L: y=mx+1 \tag{29}$$

[4]. To find the intersection of the circle and line, we need to solve the equations $-x^2+y^2=1$ *and* $y=mx+1$ *for* x *and* y. *Substituting the second equation into the first and simplifying, we solve*

$$\left(m^2-1\right)x^2+2mx=0. \tag{30}$$

This is just a quadratic equation, so we could see the quadratic formula to solve for x. *Then,*

$$\begin{aligned} x_1 &= 0 \\ x_2 &= \tfrac{2m}{1-m^2} \end{aligned} \tag{31}$$

roots are found. If we substitute these values of x into the equation $y = mx+1$ of the line L to find the y coordinates.

$$y = 1 \tag{32}$$

and

$$y = \frac{m^2+1}{1-m^2}. \tag{33}$$

Thus, for every rational number $m\,(m \neq \mp 1)$ we get a solution in rational coordinates

$$\left(\frac{2m}{1-m^2}, \frac{m^2+1}{1-m^2}\right) \tag{34}$$

to the equation $-x^2 + y^2 = 1$ (Figure 3).

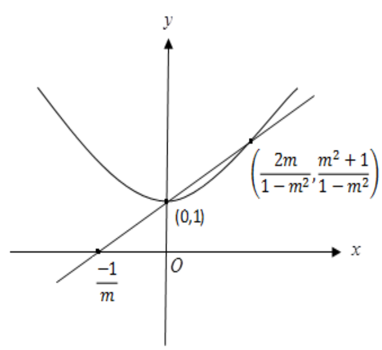

Figure 3: Rational Coordinates on Lorentzian Unit Circle

Conclusion 3.2. *For the point $(0, 1)$, which is the limiting value as $m \to \infty$, Lorentzian unit circle has no solution in rational coordinates.*

Conclusion 3.3. *If we write the rational number m as a fraction $\frac{v}{u}$ on the rational coordinates of Lorentzian unit circle*

$$\left(\frac{2m}{1-m^2}, \frac{m^2+1}{1-m^2}\right).$$

Where $u, v \in Z$, $u \neq 0$ then our formula becomes

$$(x, y) = \left(\frac{2uv}{u^2-v^2}, \frac{u^2+v^2}{u^2-v^2}\right) \tag{36}$$

and clearing denominators gives the Lorentzian Pyhtagorean triple

$$(a, b, c) = \left(2uv, u^2+v^2, u^2-v^2\right). \tag{37}$$

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