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On the n-th Derivative of the Incomplete Zeta Functions

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Abstract

New inequalities involving the n-th derivative of the incomplete zeta function are presented.

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1 Introduction

For any $0 \leq a < b$ and $s > 1$, the incomplete zeta function $\xi_{a,b}$ (see [1]) is defined by

$$\xi_{a,b}(s) = \frac{1}{\Gamma(s)} \int_a^b \frac{t^{s-1}}{e^t - 1} dt,$$

where Γ is the gamma function.

Now, for any $0 \leq a < b$, we define $h_{a,b}(x) = \Gamma(x)\xi_{a,b}(x)$ for all $x > 1$. Then, for any $0 \leq a < b$,

$$h_{a,b}(x) = \int_a^b \frac{t^{x-1}}{e^t - 1} dt$$

for all $x > 1$.

We note on the n-th derivative of $h_{a,b}$ that, for any $0 \leq a < b$,

$$h_{a,b}^{(n)}(x) = \int_a^b \frac{(\log_e t)^n t^{x-1}}{e^t - 1} dt$$

for all $x > 1$.

In this paper, new inequalities involving the n-th derivative of the incomplete zeta function are presented.

2 Results

Theorem 2.1. *Let $0 \leq a < b$ and $y > 0$ and let n be a positive odd integer. Then*

$$h_{a,b}^{(n)}(x+y) \geq h_{a,b}^{(n)}(x) \tag{1}$$

for all $x > 1$.

Proof. For any $x > 1$,

$$\begin{aligned} h_{a,b}^{(n)}(x+y) - h_{a,b}^{(n)}(x) &= \int_a^b \frac{(\log_e t)^n t^{x+y-1}}{e^t - 1} dt - \int_a^b \frac{(\log_e t)^n t^{x-1}}{e^t - 1} dt \\ &= \int_a^b \frac{(\log_e t)^n}{e^t - 1} \left(t^{x+y-1} - t^{y-1}\right) dt \\ &= \int_a^b \frac{(\log_e t)^{n-1}}{e^t - 1} (\log_e t) \left(t^{x+y-1} - t^{y-1}\right) dt \\ &\geq 0. \end{aligned}$$

This implies the inequality (1). □

Corollary 2.2. *Let n be a positive odd integer and $0 \leq a < b$. Then $h_{a,b}^{(n)}$ is non-decreasing.*

Proof. Let $z > x > 1$. Then $z = x + y$ for some $y > 0$.

By Theorem 2.1, we have

$$h_{a,b}^{(n)}(x) \leq h_{a,b}^{(n)}(x+y) = h_{a,b}^{(n)}(z).$$

Hence, $h_{a,b}^{(n)}$ is non-decreasing. □

Corollary 2.3. *Let $0 \leq a < b$. Assume that $1 < x \leq z$. Then*

$$\Gamma(z)\xi'_{a,b}(z) - \xi_{a,b}(x)\Gamma'(x) \geq \Gamma(x)\xi'_{a,b}(x) - \xi_{a,b}(z)\Gamma'(z).$$

Proof. By Corollary 2.2, we have $h'_{a,b}(x) \leq h'_{a,b}(z)$. Then

$$\Gamma(x)\xi'_{a,b}(x) + \xi_{a,b}(x)\Gamma'(x) \leq \Gamma(z)\xi'_{a,b}(z) + \xi_{a,b}(z)\Gamma'(z).$$

Then

$$\Gamma(x)\xi'_{a,b}(x) - \xi_{a,b}(z)\Gamma'(z) \leq \Gamma(z)\xi'_{a,b}(z) - \xi_{a,b}(x)\Gamma'(x).$$

□

Theorem 2.4. *Let $0 \leq a < b$ and $x_1, x_2, ..., x_n > 1$ and let $k_1, k_2, ..., k_n$ be non-negative even integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$\left(h_{a,b}^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(x_i). \tag{2}$$

Proof. By the assumption,

$$\begin{aligned}
h_{a,b}^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) &= \int_a^b \frac{(\log_e t)^k t^{\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)-1}}{e^t - 1} dt \\
&= \int_a^b \frac{(\log_e t)^k t^{\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)-\left(\sum_{i=1}^{n} \frac{1}{n}\right)}}{e^t - 1} dt \\
&= \int_a^b \frac{(\log_e t)^k t^{\sum_{i=1}^{n} \frac{x_i - 1}{n}}}{e^t - 1} dt \\
&= \int_a^b \frac{(\log_e t)^{\sum_{i=1}^{n} k_i} t^{\sum_{i=1}^{n} \frac{x_i - 1}{n}}}{\left((e^t - 1)^{1/n}\right)^n} dt \\
&= \int_a^b \frac{\prod_{i=1}^{n} (\log_e t)^{k_i} \prod_{i=1}^{n} t^{\frac{x_i - 1}{n}}}{\left((e^t - 1)^{1/n}\right)^n} dt \\
&= \int_a^b \prod_{i=1}^{n} \frac{(\log_e t)^{k_i} t^{\frac{x_i - 1}{n}}}{(e^t - 1)^{1/n}} dt \\
&= \int_a^b \prod_{i=1}^{n} \left(\frac{(\log_e t)^{nk_i} t^{x_i - 1}}{e^t - 1}\right)^{1/n} dt.
\end{aligned}$$

By the generalized Hölder inequality,

$$\begin{aligned}
h_{a,b}^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) &\leq \prod_{i=1}^{n} \left(\int_a^b \frac{(\log_e t)^{nk_i} t^{x_i - 1}}{e^t - 1} dt\right)^{1/n} \\
&= \prod_{i=1}^{n} \left(h^{(nk_i)}(x_i)\right)^{1/n} \\
&= \left(\prod_{i=1}^{n} h^{(nk_i)}(x_i)\right)^{1/n}.
\end{aligned}$$

This implies the inequality (2). ∎

Corollary 2.5. *Let $0 \leq a < b$ and $x > 1$ and let $k_1, k_2, ..., k_n$ be non-negative even integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$\left(h_{a,b}^{(k)}(x)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(x).$$

Proof. This follows from Theorem 2.4 in case $x_1 = x_2 = ... = x_n$ □

Theorem 2.6. *Let $1 < a < b$ and $x_1, x_2, ..., x_n > 1$ and let $k_1, k_2, ..., k_n$ be non-negative integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$\left(h_{a,b}^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(x_i). \quad (3)$$

Proof. This proof is similar to the proof of Theorem 2.4. □

Corollary 2.7. *Let $1 < a < b$ and $x > 1$ and let $k_1, k_2, ..., k_n$ be non-negative integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$\left(h_{a,b}^{(k)}(x)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(x).$$

Proof. This follows from Theorem 2.6 in case $x_1 = x_2 = ... = x_n$ □

References

[1] W. T. Sulaiman, Turan inequalites for the Riemann zeta functions, AIP Conf. Proc., **1389** (2011), 1793–1797.

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