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On the Product of the Gamma Function and the Riemann Zeta Function

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Abstract

In 2011, W. T. Sulaiman gave inequalities involving the product of the gamma function and the Riemann zeta function. In this paper, we generalize the inequalities.

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1 Introduction

The gamma function Γ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

where $\mathrm{Re}(z) > 0$.

The Riemann zeta function ξ is defined by

$$\xi(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt,$$

where $s > 1$.

Now, we let h be the product of the gamma function and the Riemann zeta function, i.e., $h(x) = \Gamma(x)\xi(x)$ for all $x > 1$.

In 2011, Sulaiman [1] gave an inequality as follows.

$$h(1 + x + y) \leq h^{1/p}(1 + p(x + 1))h^{1/q}(1 + q(y - 1)) \qquad (1)$$

for all $x > -1, y > 1, p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

For any non-negative integer n, we denote h_n be the n-th derivative of h. In 2011, Sulaiman [1] gave two inequalities as follows.

$$h^2_{m+n}\left(\frac{x+y}{2}\right) \leq h_{2m}(x)h_{2n}(y) \tag{2}$$

and

$$h^3_{m+n+r}\left(\frac{x+y+z}{3}\right) \leq h_{3m}(x)h_{3n}(y)h_{3r}(z) \tag{3}$$

for all $x, y, z > 1$ and non-negative even integers n, m, r.

In this paper, we present the generalizations for inequalities (1), (2) and (3).

2 Results

Theorem 2.1. *Let* $x_1, x_2, ..., x_n > -1,\ y_1, y_2, ..., y_n > 1,\ p_1, p_2, ..., p_n > 1$ *and* $q_1, q_2, ..., q_n > 1$ *be such that* $\sum_{i=1}^{n}\left(\frac{1}{p_i} + \frac{1}{q_i}\right) = 1$. *Then*

$$h(1 + \sum_{i=1}^{n}(x_i + y_i)) \leq \prod_{i=1}^{n} h^{1/p_i}(1 + p_i(x_i + 1))h^{1/q_i}(1 + q_i(y_i - 1)). \tag{4}$$

Proof. By the assumption,

$$\begin{aligned}
h(1 + \sum_{i=1}^{n}(x_i + y_i)) &= \Gamma(1 + \sum_{i=1}^{n}(x_i + y_i))\xi(1 + \sum_{i=1}^{n}(x_i + y_i)) \\
&= \int_0^\infty \frac{t^{\sum_{i=1}^{n}(x_i+y_i)}}{e^t - 1}dt \\
&= \int_0^\infty \frac{t^{\sum_{i=1}^{n}(x_i+1)}t^{\sum_{i=1}^{n}(y_i-1)}}{e^t - 1}dt \\
&= \int_0^\infty \prod_{i=1}^{n}\left(\frac{t^{x_i+1}}{(e^t - 1)^{1/p_i}}\right)\left(\frac{t^{y_i-1}}{(e^t - 1)^{1/q_i}}\right)dt.
\end{aligned}$$

By the generalized Hölder inequality,

$$\begin{aligned}
h(1+\sum_{i=1}^{n}(x_i+y_i)) &\leq \prod_{i=1}^{n}\left(\int_0^\infty \frac{t^{p_i(x_i+1)}}{e^t-1}dt\right)^{1/p_i}\left(\int_0^\infty \frac{t^{q_i(y_i-1)}}{e^t-1}dt\right)^{1/q_i}\\
&= \prod_{i=1}^{n}\Gamma^{1/p_i}(1+p_i(x_i+1))\xi^{1/p_i}(1+p_i(x_i+1))\\
&\quad \times \Gamma^{1/q_i}(1+q_i(y_i-1))\xi^{1/q_i}(1+q_i(y_i-1))\\
&= \prod_{i=1}^{n}h^{1/p_i}(1+p_i(x_i+1))h^{1/q_i}(1+q_i(y_i-1)).
\end{aligned}$$

□

We note on Theorem 2.1 that if $n = 1$ then we obtain the inequality (1).

Theorem 2.2. *Let $x_1, x_2, ..., x_n > 1$ and let $k_1, k_2, ..., k_n$ be non-negative even integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$h_k^n\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) \leq \prod_{i=1}^{n} h_{nk_i}(x_i). \tag{5}$$

Proof. By the assumption,

$$\begin{aligned}
h_k\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) &= h^{(k)}\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)\\
&= \int_0^\infty \frac{(\log_e t)^k t^{\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)-1}}{e^t-1}dt\\
&= \int_0^\infty \frac{(\log_e t)^k t^{\sum_{i=1}^{n}\frac{x_i-1}{n}}}{e^t-1}dt\\
&= \int_0^\infty \prod_{i=1}^{n}\frac{(\log_e t)^{k_i} t^{\frac{x_i-1}{n}}}{(e^t-1)^{1/n}}dt\\
&= \int_0^\infty \prod_{i=1}^{n}\left(\frac{(\log_e t)^{nk_i} t^{x_i-1}}{e^t-1}\right)^{1/n} dt.
\end{aligned}$$

By the generalized Hölder inequality,

$$\begin{aligned} h_k\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) &\leq \prod_{i=1}^{n}\left(\int_0^\infty \frac{(\log_e t)^{nk_i}t^{x_i-1}}{e^t-1}dt\right)^{1/n} \\ &= \prod_{i=1}^{n}\left(h^{(nk_i)}(x_i)\right)^{1/n} \\ &= \prod_{i=1}^{n}\left(h_{nk_i}(x_i)\right)^{1/n} \\ &= \left(\prod_{i=1}^{n} h_{nk_i}(x_i)\right)^{1/n}. \end{aligned}$$

This implies the inequality (5). □

We note on Theorem 2.2 that (i) if $n = 2$ then we obtain the inequality (2), and (ii) if $n = 3$ then we obtain the inequality (3).

Corollary 2.3. *Let $x > 1$ and let $k_1, k_2, ..., k_n$ be non-negative even integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$h_k^n(x) \leq \prod_{i=1}^{n} h_{nk_i}(x).$$

Proof. This follows from Theorem 2.2 in case $x_1 = x_2 = ... = x_n$ □

References

[1] W. T. Sulaiman, Turan inequalites for the Riemann zeta functions, AIP Conf. Proc., **1389** (2011), 1793–1797.

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