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Three Inequalities for the Incomplete Zeta Functions

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Abstract

In this paper, we present three inequalities involving the incomplete zeta function.

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1 Introduction

The Riemann zeta function ξ is defined by

$$\xi(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt,$$

where Γ is the gamma function and $s > 1$.

The incomplete zeta function $\xi_{a,b}$ (see [1]) is defined, for $0 \leq a < b$, by

$$\xi_{a,b}(s) = \frac{1}{\Gamma(s)} \int_a^b \frac{t^{s-1}}{e^t - 1} dt,$$

where Γ is the gamma function and $s > 1$.

In 2011, Sulaiman [1] presented two inequalities involving the incomplete zeta function.

Now, we denote $h_{a,b}(x) = \Gamma(x)\xi_{a,b}(x)$ for all $x > 1$. Then

$$h_{a,b}(x) = \int_a^b \frac{t^{x-1}}{e^t - 1} dt$$

for all $x > 1$.

In this paper, we will present three inequalities involving the incomplete zeta function.

2 Results

Theorem 2.1. *Let f, g be functions such that $f \geq g > 1$ and $f' = g' \geq 0$. Define $H_{a,b}(x) = h_{a,b}(f(x)) - h_{a,b}(g(x))$. Then $H_{a,b}$ is non-decreasing.*

Proof. For any x,

$$H_{a,b}(x) = \int_a^b \frac{t^{f(x)-1} - t^{g(x)-1}}{e^t - 1} dt,$$

and then

$$\begin{aligned} H'_{a,b}(x) &= \int_a^b \frac{(\log_e t) f'(x) t^{f(x)-1} - (\log_e t) g'(x) t^{g(x)-1}}{e^t - 1} dt \\ &= \int_a^b \frac{f'(x)}{e^t - 1} (\log_e t)(t^{f(x)-1} - t^{g(x)-1}) dt \\ &\geq 0. \end{aligned}$$

Hence, $H_{a,b}$ is non-decreasing. □

Theorem 2.2. *Let $\alpha \geq 1$. Then, for any $x, y > 1$,*

$$h_{a,b}(x^\alpha y) - h_{a,b}(xy) \geq y \left(h_{a,b}(x^\alpha) - h_{a,b}(x) \right). \tag{1}$$

Proof. For any $x, y > 1$, we define

$$f_y(x) = h_{a,b}(xy) - y h_{a,b}(x),$$

then

$$\begin{aligned} f_y(x) &= \int_a^b \frac{t^{xy-1}}{e^t - 1} dt - y \int_a^b \frac{t^{x-1}}{e^t - 1} dt \\ &= \int_a^b \frac{t^{xy-1} - y t^{x-1}}{e^t - 1} dt. \end{aligned}$$

For fixed $y > 1$,

$$\begin{aligned} f'_y(x) &= \int_a^b \frac{(\log_e t) y t^{xy-1} - (\log_e t) y t^{x-1}}{e^t - 1} dt \\ &= \int_a^b \frac{y}{e^t - 1} (\log_e t) \left(t^{xy-1} - t^{x-1} \right) dt \\ &\geq 0 \end{aligned}$$

for all $x > 1$.

Hence, for any $y > 1$, we obtain that f_y is non-decreasing, and then

$$f_y(x^\alpha) \geq f_y(x)$$

for all $x > 1$.

Then

$$h_{a,b}(x^\alpha y) - yh_{a,b}(x^\alpha) \geq h_{a,b}(xy) - yh_{a,b}(x)$$

for all $x, y > 1$.

This implies the inequality (1). $\square$

Theorem 2.3. *Let $\alpha \geq 1$ and $0 < y < 1$. Then, for any $x > 1$,*

$$h_{a,b}(x+y) \geq h_{a,b}(x) - h_{a,b}(1-y). \tag{2}$$

Proof. For any $x > 1$, we define

$$g(x) = h_{a,b}(x+y) - h_{a,b}(x) + h_{a,b}(1-y).$$

Then

$$\begin{aligned} g'(x) &= h'_{a,b}(x+y) - h'_{a,b}(x) \\ &= \int_a^b \frac{(\log_e t)t^{x+y-1}}{e^t - 1} dt - \int_a^b \frac{(\log_e t)t^{x-1}}{e^t - 1} dt \\ &= \int_a^b \frac{1}{e^t - 1} (\log_e t) \left(t^{x+y-1} - t^{x-1}\right) dt \\ &\geq 0 \end{aligned}$$

for all $x > 1$.

Hence, g is non-decreasing.

For all $x > 1$,

$$\begin{aligned} g(x) &= \int_a^b \frac{t^{x+y-1}}{e^t - 1} dt - \int_a^b \frac{t^{x-1}}{e^t - 1} dt + \int_a^b \frac{t^{-y}}{e^t - 1} dt \\ &= \int_a^b \frac{1}{e^t - 1} \left(t^{x+y-1} - t^{x-1} + t^{-y}\right) dt \\ &\geq \int_a^b \frac{1}{e^t - 1} \left(t^{1+y-1} - t^{1-1} + t^{-y}\right) dt \\ &= \int_a^b \frac{1}{e^t - 1} \left(t^y - 1 + t^{-y}\right) dt \\ &= \int_a^b \frac{t^{-y}}{e^t - 1} \left((t^y)^2 - t^y + 1\right) dt \\ &\geq 0. \end{aligned}$$

Hence, $h_{a,b}(x+y) - h_{a,b}(x) + h_{a,b}(1-y) \geq 0$ for all $x > 1$.
This implies the inequality (2). □

Theorem 2.4. *Let $x, y, p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$h_{a,b}\left(\frac{x}{p} + \frac{y}{q}\right) \leq h_{a,b}^{1/p}(x) h_{a,b}^{1/q}(y). \tag{3}$$

Proof. By the definition of $h_{a,b}$ and the assumption,

$$\begin{aligned} h_{a,b}\left(\frac{x}{p} + \frac{y}{q}\right) &= \int_a^b \frac{t^{\left(\frac{x}{p}+\frac{y}{q}\right)-1}}{e^t - 1} dt \\ &= \int_a^b \left(\frac{t^{\frac{x-1}{p}}}{(e^t-1)^{1/p}}\right)\left(\frac{t^{\frac{y-1}{q}}}{(e^t-1)^{1/q}}\right) dt. \end{aligned}$$

By the Hölder inequality,

$$h_{a,b}\left(\frac{x}{p} + \frac{y}{q}\right) \leq \left(\int_a^b \frac{t^{x-1}}{e^t-1} dt\right)^{1/p} \left(\int_a^b \frac{t^{y-1}}{e^t-1} dt\right)^{1/q}.$$

This implies the inequality (3). □

References

[1] W. T. Sulaiman, Turan inequalites for the Riemann zeta functions, AIP Conf. Proc., **1389** (2011), 1793–1797.

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