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Two Inequalities for the Riemann Zeta Functions

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Abstract

In this paper, we present two inequalities involving the Riemann zeta function.

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1 Introduction

For any $s > 1$, we denote

$$\xi(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt,$$

where Γ is the gamma function. The function ξ is called the Riemann zeta function. In 2006, Laforgia and Natalini [1] gave an inequality that

$$(s+1)\frac{\xi(s)}{\xi(s+1)} \geq s\frac{\xi(s+1)}{\xi(s+2)},$$

for all $s > 1$. In 2011, Sulaiman [2] gave two inequalities as follows.

$$\xi(x^\alpha y)\Gamma(x^\alpha y) - \xi(xy)\Gamma(xy) \geq y\left(\xi(x^\alpha)\Gamma(x^\alpha) - \xi(x)\Gamma(x)\right) \tag{1}$$

for all $x, y, \alpha > 1$.

$$\xi\left(\frac{x}{p} + \frac{y}{q}\right) \leq \frac{\Gamma^{1/p}(x)\,\Gamma^{1/q}(y)}{\Gamma\left(\frac{x}{p} + \frac{y}{q}\right)} \xi^{1/p}(x)\xi^{1/q}(y) \tag{2}$$

for all $x, y, p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In this paper, we present the generalizations for inequalities (1) and (2) .

2 Results

Theorem 2.1. *Let $\alpha \geq 1$ and let f, g be functions such that $f, g > 1$ and $f' \geq 0$. Then, for any $x > 1$ and for any y,*

$$\begin{aligned}\xi(f(x^\alpha)g(y))\Gamma(f(x^\alpha)g(y)) - \xi(f(x)g(y))\Gamma(f(x)g(y))& \\ \geq g(y)\left(\xi(f(x^\alpha))\Gamma(f(x^\alpha)) - \xi(f(x))\Gamma(f(x))\right).& \qquad (3)\end{aligned}$$

Proof. For any x, y, we denote

$$h_y(x) = \xi(f(x)g(y))\Gamma(f(x)g(y)) - g(y)\left(\xi(f(x))\Gamma(f(x))\right),$$

then

$$\begin{aligned} h_y(x) &= \int_0^\infty \frac{t^{f(x)g(y)-1}}{e^t - 1} dt - g(y) \int_0^\infty \frac{t^{f(x)-1}}{e^t - 1} dt \\ &= \int_0^\infty \frac{t^{f(x)g(y)-1} - g(y)t^{f(x)-1}}{e^t - 1} dt. \end{aligned}$$

For fixed y,

$$\begin{aligned} h_y'(x) &= \int_0^\infty \frac{g(y)f'(x)\left(\log_e t\right) t^{f(x)g(y)-1} - g(y)f'(x)\left(\log_e t\right) t^{f(x)-1}}{e^t - 1} dt \\ &= g(y) \int_0^\infty \frac{f'(x)}{e^t - 1} \left(\log_e t\right) \left(t^{f(x)g(y)-1} - t^{f(x)-1}\right) dt \\ &\geq 0 \end{aligned}$$

for all x.

Hence, for any y, we obtain that h_y is non-decreasing, and then

$$h_y(x^\alpha) \geq h_y(x)$$

for all $x > 1$.

Then

$$\begin{aligned}\xi(f(x^\alpha)g(y))\Gamma(f(x^\alpha)g(y)) - g(y)\left(\xi(f(x^\alpha))\Gamma(f(x^\alpha))\right)& \\ \geq \xi(f(x)g(y))\Gamma(f(x)g(y)) - g(y)\left(\xi(f(x))\Gamma(f(x))\right)&\end{aligned}$$

for all $x > 1$ and for all y.

This implies the inequality (3). $\square$

We note on Theorem 2.1 that if both f and g are the identity function then we obtain the inequality (1).

Theorem 2.2. *Let* $x_1, x_2, ..., x_n > 1$ *and* $p_1, p_2, ..., p_n > 1$ *be such that* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. *Then*

$$\xi\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right) \leq \frac{\prod_{i=1}^{n} \Gamma^{1/p_i}(x_i)\, \xi^{1/p_i}(x_i)}{\Gamma\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)}. \tag{4}$$

Proof. By the definition of ξ and the assumption,

$$\begin{aligned}
\xi\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right) &= \frac{1}{\Gamma\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)} \int_0^\infty \frac{t^{\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)-1}}{e^t - 1} dt \\
&= \frac{1}{\Gamma\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)} \int_0^\infty \frac{t^{\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)-\left(\sum_{i=1}^{n} \frac{1}{p_i}\right)}}{e^t - 1} dt \\
&= \frac{1}{\Gamma\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)} \int_0^\infty \frac{t^{\left(\sum_{i=1}^{n} \frac{x_i-1}{p_i}\right)}}{e^t - 1} dt \\
&= \frac{1}{\Gamma\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)} \int_0^\infty \prod_{i=1}^{n} \frac{t^{\frac{x_i-1}{p_i}}}{(e^t - 1)^{1/p_i}} dt \\
&= \frac{1}{\Gamma\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right)} \int_0^\infty \prod_{i=1}^{n} \left(\frac{t^{x_i-1}}{e^t - 1}\right)^{1/p_i} dt.
\end{aligned}$$

By the generalized Hölder inequality,

$$\xi\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right) \le \frac{1}{\Gamma\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\prod_{i=1}^{n}\left(\int_0^\infty \frac{t^{x_i-1}}{e^t-1}dt\right)^{1/p_i}$$
$$= \frac{1}{\Gamma\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right)}\prod_{i=1}^{n}\left(\Gamma\left(x_i\right)\xi(x_i)\right)^{1/p_i}.$$

This implies the inequality (4). □

We note on Theorem 2.2 that if $n = 2$ then we obtain the inequality (2).

3 Open Problem

In fact, we have the generalized Hölder inequality for $\sum_{i=1}^{n}\frac{1}{p_i} = \frac{1}{r}$; $r \ge 1$. In the proof of Theorem 2.2, we use the generalized Hölder inequality in case $r = 1$. Now, we pose a question that how to generalize inequality (4) if we use the generalized Hölder inequality in case $r > 1$.

References

[1] A. Laforgia and P. Natalini, Turan-type inequalities for some special functions, J. Ineq. Pure Appl. Math., **7** (2006), Art. 32.

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