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Fuzzy relation equations and Galois connections

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Abstract

In this paper, we study solutions of two types of fuzzy relation equations $A_i \to R = B_i$ and $R \to A_i = B_i$ in residuated lattices. We investigate the relations between Galois connections and solutions of fuzzy relation equations. Moreover, we give approximation solutions of two types of fuzzy relation equations.

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1 Introduction

Sanchez [10] introduced the theory of fuzzy relation equations with various types of composition: max-min, min-max, min-α. Fuzzy relation equations with new types of composition(pseudo t-norm [5], continuous t-norm [11], residuated lattice [3,6-9]) is developed [4]. On the other hand, concept lattices using Galois connections play an important role in information theory [1,3]. Diaz and Medina [3] introduced the relations between isotone Galois connection and solutions of fuzzy relation equations $A_i \odot R = B_i$.

In this paper, we study solutions of two types of fuzzy relation equations $A_i \to R = B_i$ and $R \to A_i = B_i$ in residuated lattices. We investigate the relations between Galois connections and solutions of fuzzy relation equations $A_i \to A_i = B_i$ and $R \to A_i = B_i$. Moreover, we give approximation solutions of two types of fuzzy relation equations.

2 Preliminaries

Definition 2.1 [12] A structure $(L, \vee, \wedge, \odot, \to, 0, 1)$ is called a *residuated lattice* if it satisfies the following conditions:

(R1) $(L, \vee, \wedge, 1, 0)$ is a bounded where 1 is the universal upper bound and 0 denotes the universal lower bound;
(R2) $(L, \odot, 1)$ is a commutative monoid;
(R3) it satisfies a residuation , i.e.

$$a \odot b \leq c \text{ iff } a \leq b \to c.$$

Remark 2.2 [12] A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \to b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice.

In this paper, we assume $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice.

Lemma 2.3 [12] For each $x, y, z, x_i, y_i \in L$, we have the following properties.
(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.
(2) $x \odot y \leq x \wedge y \leq x \vee y$.
(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.
(4) $x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$.
(5) $(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$.
(6) $(x \odot y) \to z = x \to (y \to z)$.
(7) $x \to (y \to z) = y \to (x \to z)$.
(8) $x \odot (x \to y) \leq y$.
(9) $(x \to y) \odot (y \to z) \leq x \to z$.
(10) $x \leq (x \to y) \to y$.
(11) $y \to z) \leq (x \odot y) \to (x \odot z)$ and $(y \to z) \leq (x \to y) \to (x \to z)$.
(12) $\bigwedge_{i\in\Gamma}(x_i \to y_i) \leq (\bigwedge_{i\in\Gamma} x_i) \to (\bigwedge_{i\in\Gamma} y_i)$.
(13) $\bigwedge_{i\in\Gamma}(x_i \to y_i) \leq (\bigvee_{i\in\Gamma} x_i) \to (\bigvee_{i\in\Gamma} y_i)$.
(14) $x \to y = 1$ iff $x \leq y$.

3 Fuzzy relation equations and Galois connections

Definition 3.1 (1) Let $A_i \in L^U$, $R \in L^{U\times V}$ and $B_i \in L^V$. We define fuzzy relation equations as follows

$$(A_i \odot R)(v) = \bigvee_{u\in U} (A_i(u) \odot R(u,v)) = B_i(v),\ i \in \{1, ..., n\} \qquad \mathbf{(1)}$$

$$(R \to A_i)(v) = \bigwedge_{u\in U} (R(u,v) \to A_i(u)) = B_i(v),\ i \in \{1, ..., n\}. \qquad \mathbf{(2)}$$

(2) Let $A^t_j \in L^V$, $R \in L^{U\times V}$ and $D_j \in L^U$ for $j \in \{1, ..., m\}$. We define fuzzy relation equations as follows

$$(A^t_j \to R)(u) = \bigwedge_{v\in V} (A^t_j(v) \to R(u,v)) = D_j(u),\ j \in \{1, ..., m\} \qquad \mathbf{(3)}$$

$$(R \to A_j^t)(u) = \bigwedge_{v \in V} (R(u,v) \to A_j^t(v)) = D_j(u),\ j \in \{1,...,m\}. \qquad \textbf{(4)}$$

Let $U = \{u_1, ..., u_m\}$ and $V = \{v_1, ..., v_n\}$ be two sets, $R \in L^{U \times V}$ an unknown fuzzy relation $A_1, ..., A_n \in L^U$ and $B_1, ..., B_n \in L^V$. If $v \in V$, $A_i(u_j) = a_{ij}$ for $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, $R(u_j, v) = x_j$, $B_j(v) = b_j$, then system **(1)** can be written by

$$\begin{array}{ll} a_{11} \odot x_1 \vee ... \vee a_{1m} \odot x_m & = b_1 \\ \vdots \quad \vdots & \vdots \\ a_{n1} \odot x_1 \vee ... \vee a_{nm} \odot x_m & = b_n \end{array} \qquad \textbf{(5)}$$

Put $h(x) = (h(x)_1, ..., h(x)_n)$ with $h(x)_i = \bigvee_k^m (a_{ik} \odot x_k)$ for $i \in \{1, ..., n\}$ and $x = (x_1, ..., x_m) \in L^m$.

The system **(2)** can be written by

$$\begin{array}{ll} x_1 \to a_{11} \wedge ... \wedge x_m \to a_{1m} & = b_1 \\ \vdots \quad \vdots & \vdots \\ x_1 \to a_{n1} \wedge ... \wedge x_m \to a_{nm} & = b_n \end{array} \qquad \textbf{(6)}$$

Put $f^{\to}(x) = (f^{\to}(x)_1, ..., f^{\to}(x)_n)$ with $f^{\to}(x)_i = \bigwedge_k^m (x_k \to a_{ik})$ for $i \in \{1, ..., n\}$ and $x = (x_1, ..., x_m) \in L^m$.

If $u \in U$, $A_j^t(v_i) = a_{ij}$ for $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, $R(u, v_i) = y_i$, $D_j(u) = d_j$,

The system **(3)** can be written by

$$\begin{array}{ll} a_{11} \to y_1 \wedge ... \wedge a_{n1} \to y_n & = d_1 \\ \vdots \quad \vdots & \vdots \\ a_{1m} \to y_1 \wedge ... \wedge a_{nm} \to y_n & = d_m \end{array} \qquad \textbf{(7)}$$

Put $g(y) = (g(y)_1, ..., g(y)_m)$ with $g(y)_j = \bigwedge_{l=1}^n (a_{lj} \to y_l)$ for $j \in \{1, ..., m\}$ and $y = (y_1, ..., y_n) \in L^n$.

The system **(4)** can be written by

$$\begin{array}{ll} y_1 \to a_{11} \wedge ... \wedge y_n \to a_{n1} & = d_1 \\ \vdots \quad \vdots & \vdots \\ y_1 \to a_{1m} \wedge ... \wedge y_n \to a_{nm} & = d_m \end{array} \qquad \textbf{(8)}$$

Put $g^{\to}(y) = (g^{\to}(y)_1, ..., g^{\to}(y)_m)$ with $g^{\to}(y)_j = \bigwedge_{p=1}^n (y_p \to a_{pj})$ for $j \in \{1, ..., m\}$ and $y = (y_1, ..., y_n) \in L^n$.

Definition 3.2 [3] Let $h : L^m \to L^n$ and $g : L^n \to L^m$ be an increasing function. The pair (h, g) is an isotone Galois connection if

$$h(a) \leq b \text{ iff } a \leq g(b),\ \forall a \in L^m, b \in L^n.$$

Theorem 3.3 [3] (1) *Let $h : L^m \to L^n$ and $g : L^n \to L^m$ be a function. Then the pair (h, g) is an isotone Galois connection iff $h(f(b)) \leq b, a \leq g(h(a)),\ \forall a \in L^m, b \in L^n$.*

(2) *(h, g) are an isotone Galois connection.*

(3) *(5) is solvable iff $h(g(b)) = b$ for $b = (b_1, ..., b_n)$. Moreover, if (5) is solvable with $h(x) = b$, then $g(b)$ is the greatest solution.*

(4) *(7) is solvable iff $g(h(d)) = d$ for $d = (d_1, ..., d_m)$. Moreover, if (7) is solvable with $g(y) = d$, then $h(d)$ is the least solution.*

Definition 3.4 Let $f : L^m \to L^n$ and $g : L^n \to L^m$ be a decreasing function. The pair (f, g) is an antitone Galois connection if

$$y \leq f(x) \text{ iff } x \leq g(y),\ \forall x \in L^m, y \in L^n.$$

Theorem 3.5 (1) *Let $f : L^m \to L^n$ and $g : L^n \to L^m$ be a function. Then the pair (f, g) is a an antitone Galois connection iff $y \leq f(g(y)), x \leq g(f(x)),\ \forall x \in L^m, y \in L^n$.*

(2) *$(f^{\to}, g^{\to})$ and $(g^{\to}, f^{\to})$ are an antitone Galois connections.*

(3) *(6) is solvable iff $f^{\to}(g^{\to}(b)) = b$ for $b = (b_1, ..., b_n)$. Moreover, if (6) is solvable with $f^{\to}(x) = b$, then $g^{\to}(b)$ is the greatest solution.*

(4) *(8) is solvable iff $g^{\to}(f^{\to}(d)) = d$ for $d = (d_1, ...d_m)$. Moreover, if (8) is solvable with $g^{\to}(y) = d$, then $f^{\to}(d)$ is the greatest solution.*

Proof (1) $(\Rightarrow)$. Since $g(y) \leq g(y)$, then $y \leq f(g(y))$. Since $f(x) \leq f(x)$, then $x \leq g(f(x))$.

$(\Rightarrow)$. If $x_1 \leq x_2$ and $x_2 \leq g(f(x_2))$, then $f(x_2) \leq f(x_1)$. If $y_1 \leq y_2$ and $y_2 \leq f(g(y_2))$, then $g(y_2) \leq g(y_1)$. Hence f and g are decreasing functions.

Let $y \leq f(x)$ be given. Then $g(y) \geq g(f(x)) \geq x$. Let $x \leq g(y)$ be given. Then $f(x) \geq f(g(y)) \geq y$. Hence the pair (f, g) is an antitone Galois connection.

(2) By Lemma 2.3(10), we have

$$\begin{array}{ll} f^{\to}(g^{\to}(b))_i & = \bigwedge_{j=1}^{m}(g^{\to}(b)_j \to a_{ij}) = \bigwedge_{j=1}^{m}(\bigwedge_{p=1}^{n}(b_p \to a_{pj}) \to a_{ij}) \\ & \geq \bigwedge_{j=1}^{m}((b_i \to a_{ij}) \to a_{ij}) \geq b_i. \\ g^{\to}(f^{\to}(d))_j & = \bigwedge_{k=1}^{n}(f^{\Rightarrow}(d)_k \to a_{kj}) = \bigwedge_{k=1}^{n}(\bigwedge_{p=1}^{n}(d_p \to a_{kp}) \to a_{kj}) \\ & \geq \bigwedge_{k=1}^{n}((d_j \to a_{kj}) \to a_{kj}) \geq d_j. \end{array}$$

By (1), $(f^{\to}, g^{\to})$ and $(g^{\to}, f^{\to})$ are antitone Galois connections.

(3) ($\Rightarrow$) Let $x = (x_1, ..., x_m)$ be a solution of (6). Since $\bigwedge_{k=1}^{m}(x_k \to a_{ik}) = b_i, i \in \{1, ..., n\}$, then

$$x_k \to a_{ik} \geq \bigwedge_{k=1}^{m}(x_k \to a_{ik}) = b_i, i \in \{1, ..., n\}.$$

Then $x_k \odot b_i \leq a_{ik}$. Thus $x_k \leq b_i \to a_{ik}$. Hence $x_k \leq \bigwedge_{p=1}^{n}(b_p \to a_{pk})$. So,

$$\begin{aligned} b_i &= \bigwedge_{k=1}^{m}(x_k \to a_{ik}) \geq \bigwedge_{k=1}^{m}(\bigwedge_{p=1}^{n}(b_p \to a_{pk}) \to a_{ik}) \\ &= f^{\to}(g^{\to}(b))_i \geq \bigwedge_{k=1}^{m}((b_i \to a_{ik}) \to a_{ik}) \geq b_i. \end{aligned}$$

Thus, $f^{\to}(g^{\to}(b))_i = b_i$. Hence $g^{\to}(b)$ is the greatest solution.

(4) ($\Rightarrow$) Let $y = (y_1, ..., y_n)$ be a solution of (8). Since $\bigwedge_{p=1}^{n}(y_p \to a_{pj}) = d_j, j \in \{1, ..., m\}$, then

$$y_p \to a_{pj} \geq \bigwedge_{p=1}^{n}(y_p \to a_{pj}) = d_j, j \in \{1, ..., m\}.$$

Then $d_j \odot y_p \leq a_{pj}$. Thus $y_p \leq d_j \to a_{pj}$. Hence $y_p \leq \bigwedge_{k=1}^{m}(d_k \to a_{pk})$. So,

$$\begin{aligned} d_j &= \bigwedge_{p=1}^{n}(y_p \to a_{pj}) \geq \bigwedge_{p=1}^{n}(\bigwedge_{k=1}^{m}(d_k \to a_{pk}) \to a_{pj}) \\ &= g^{\to}(f^{\to}(d))_j \geq \bigwedge_{p=1}^{n}((d_j \to a_{pj}) \to a_{pj}) \geq d_j. \end{aligned}$$

Thus, $g^{\to}(f^{\to}(d))_j = d_j$. Hence $f^{\to}(d)$ is the greatest solution.

Theorem 3.6 (1) *If* (5) *is solvable, then* $\bigwedge_{k=1}^{m}(a_{ik} \to a_{jk}) \leq b_i \to b_j$.
(2) *If* (7) *is solvable, then* $\bigwedge_{p=1}^{n}(a_{pj} \to a_{pi}) \leq d_i \to d_j$.
(3) *If* (6) *is solvable, then* $\bigwedge_{k=1}^{m}(a_{ik} \to a_{jk}) \leq b_i \to b_j$.
(4) *If* (8) *is solvable, then* $\bigwedge_{p=1}^{n}(a_{pi} \to a_{pj}) \leq d_i \to d_j$.

Proof (1) Since $\bigvee_{k=1}^{m}(a_{ik} \odot x_k) = b_i$, by Lemma 2.3 (11,13), we have

$$\begin{aligned} b_i \to b_j &= \bigvee_{k=1}^{m}(a_{ik} \odot x_k) \to \bigvee_{k=1}^{m}(a_{jk} \odot x_k) \\ &\geq \bigwedge_{k=1}^{m}((a_{ik} \odot x_k) \to (a_{jk} \odot x_k)) \\ &\geq \bigwedge_{k=1}^{m}(a_{ik} \to a_{jk}). \end{aligned}$$

(2) Since $g(y)_j = \bigwedge_{l=1}^{n}(a_{lj} \to y_l) = d_j$, by Lemma 2.3 (11,12), we have

$$\begin{aligned} d_i \to d_j &= \bigwedge_{l=1}^{n}(a_{li} \to y_l) \to \bigwedge_{l=1}^{n}(a_{lj} \to y_l) \\ &\geq \bigwedge_{l=1}^{n}((a_{li} \to y_l) \to (a_{lj} \to y_l)) \\ &\geq \bigwedge_{l=1}^{n}(a_{lj} \to a_{li}). \end{aligned}$$

(3) Since $\bigwedge_{k=1}^{m}(x_k \to a_{ik}) = b_i$, by Lemma 2.3 (11,12), we have

$$\begin{aligned} b_i \to b_j &= \bigwedge_{k=1}^{m}(x_k \to a_{ik}) \to \bigwedge_{k=1}^{m}(x_k \to a_{jk}) \\ &\geq \bigwedge_{k=1}^{m}((x_k \to a_{ik}) \to (x_k \to a_{jk})) \\ &\geq \bigwedge_{k=1}^{m}(a_{ik} \to a_{jk}). \end{aligned}$$

(4) Since $g^{\to}(y)_j = \bigwedge_{p=1}^{n}(y_p \to a_{pj}) = d_j$, by Lemma 2.3 (11,12), we have

$$\begin{array}{ll} d_i \to d_j & = \bigwedge_{p=1}^{n}(y_p \to a_{pi}) \to \bigwedge_{p=1}^{n}(y_p \to a_{pj}) \\ & \geq \bigwedge_{p=1}^{n}((y_p \to a_{pi}) \to (y_p \to a_{pj})) \\ & \geq \bigwedge_{p=1}^{n}(a_{pi} \to a_{pj}). \end{array}$$

Example 3.7 The structure $(L = [0,1], \odot, \to, 0, 1)$ is a residuated lattice defined binary operations $\odot$ (called Łukasiewicz conjection) and $\to$ on $L = [0,1]$ by

$$x \odot y = \max\{0, x + y - 1\}, \ x \to y = \min\{1 - x + y, 1\}.$$

(1)

$$\begin{array}{l} 0.4 \to y_1 \wedge 0.6 \to y_2 \wedge 0.8 \to y_3 = 0.1 \\ 0.2 \to y_1 \wedge 0.5 \to y_2 \wedge 0.4 \to y_3 = 0.7 \end{array} \qquad (\mathbf{9})$$

$$(a_{ij}) = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.5 \\ 0.8 & 0.4 \end{pmatrix}$$

Since $0.6 = \wedge_{p=1}^{3}(a_{p1} \to a_{p2}) \not\leq d_2 \to d_1 = 0.7$, by Theorem 3.2(2), $h(d) = (y_1, y_2, y_3) = (\vee_k^2(a_{1k} \odot d_k), \vee_k^2(a_{2k} \odot d_k), \vee_k^2(a_{3k} \odot d_k)) = (0, 0.2, 0.1)$ is not a solution of **(9)**.

(2)

$$\begin{array}{l} 0.4 \to y_1 \wedge 0.6 \to y_2 \wedge 0.8 \to y_3 = 0.6 \\ 0.2 \to y_1 \wedge 0.5 \to y_2 \wedge 0.4 \to y_3 = 0.7 \end{array} \qquad (\mathbf{10})$$

Then $h(d) = (y_1, y_2, y_3) = (\vee_k^2(a_{1k} \odot d_k), \vee_k^2(a_{2k} \odot d_k), \vee_k^2(a_{3k} \odot d_k)) = (0, 0.2, 0.4)$ is a solution of **(10)**

(3)

$$\begin{array}{l} y_1 \to 0.4 \wedge y_2 \to 0.6 \wedge y_3 \to 0.8 = 0.8 \\ y_1 \to 0.2 \wedge y_2 \to 0.5 \wedge y_3 \to 0.4 = 0.7 \end{array} \qquad (\mathbf{11})$$

Then $g^{\to}(d) = (y_1, y_2, y_3) = (\wedge_k^2(d_k \to a_{1k}), \wedge_k^2(d_k \to a_{2k}), \wedge_k^2(d_k \to a_{3k})) = (0.5, 0.8, 0.9)$ is a solution of **(11)**

Theorem 3.8 [3] *Let $h : L^m \to L^n$ and $g : L^n \to L^m$ be functions such that the pair (h, g) is an isotone Galois connection. Let $Cl(L^m) = \{x \in L^m \mid g(h(x)) = x\}$ and $Cl(L^n) = \{y \in L^n \mid h(g(y)) = y\}$ be given. Then $h : Cl(L^m) \to Cl(L^n)$ is a bijective function.*

Theorem 3.9 *Let $f^{\rightarrow}: L^m \to L^n$ and $g^{\rightarrow}: L^n \to L^m$ be functions such that the pair $(f^{\rightarrow}, g^{\rightarrow})$ is a antitone Galois connection. Let $cl(L^m) = \{x \in L^m \mid g^{\rightarrow}(f^{\rightarrow}(x)) = x\}$ and $cl(L^n) = \{y \in L^n \mid f^{\rightarrow}(g^{\rightarrow}(y)) = y\}$ be given. Then $f^{\rightarrow}: cl(L^m) \to cl(L^n)$ is a bijective function.*

Proof For $x \in cl(L^m)$;i.e. $g^{\rightarrow}(f^{\rightarrow}(x)) = x$, $f^{\rightarrow}(g^{\rightarrow}(f^{\rightarrow}(x))) \geq f^{\rightarrow}(x)$ and $g^{\rightarrow}(f^{\rightarrow}(x)) \geq x$ implies $f^{\rightarrow}(g^{\rightarrow}(f^{\rightarrow}(x))) \leq f^{\rightarrow}(x)$. Thus $f^{\rightarrow}(g^{\rightarrow}(f^{\rightarrow}(x))) = f^{\rightarrow}(x)$. So, $f^{\rightarrow}(x) \in cl(L^n)$. $f^{\rightarrow}$ is well defined.

If $f^{\rightarrow}(x) = f^{\rightarrow}(z)$ with $x = g^{\rightarrow}(f^{\rightarrow}(x))$ and $g^{\rightarrow}(f^{\rightarrow}(z)) = z$, then

$$\begin{aligned} z &= g^{\rightarrow}(f^{\rightarrow}(z)) = g^{\rightarrow}(f^{\rightarrow}(g^{\rightarrow}(f^{\rightarrow}(z)) \\ &= g^{\rightarrow}(f^{\rightarrow}(g^{\rightarrow}(f^{\rightarrow}(x)) = g^{\rightarrow}(f^{\rightarrow}(x)) = x. \end{aligned}$$

Hence $f^{\rightarrow}$ is injective.

For $y \in cl(L^n)$;i.e. $f^{\rightarrow}(g^{\rightarrow}(y)) = y$, there exists $g^{\rightarrow}(y) \in L^m$ such that $g^{\rightarrow}(y) = g^{\rightarrow}(f^{\rightarrow}(g^{\rightarrow}(y)))$. Hence $f^{\rightarrow}$ is surjective.

Definition 3.10 Let $A_i \in L^U$, $R \in L^{U\times V}$ and $B_i \in L^V$, $i \in \{1, ..., n\}$.

(1) The right upper approximation is defined as

$$\mathcal{R}_{ru} = \{R \in L^{U\times V} \mid A_i \to R \geq B_i, i \in \{1, ..., n\}\}.$$

(2) The left upper approximation is defined as

$$\mathcal{R}_{lu} = \{R \in L^{U\times V} \mid R \to A_i \geq B_i, i \in \{1, ..., n\}\}.$$

(3) The right quality $\delta_r(R)$ of approximation is defined as

$$\delta_r(R) = \bigwedge_{i=1}^{n} \bigwedge_{v\in V} ((A_i \to R)(v) \to B_i(v)).$$

(4) The left quality $\delta_l(R)$ of approximation is defined as

$$\delta_l(R) = \bigwedge_{i=1}^{n} \bigwedge_{v\in V} ((R \to A_i)(v) \to B_i(v)).$$

Definition 3.11 Let $A_i \in L^U$, $R \in L^{U\times V}$ and $B_i \in L^V$, $i \in \{1, ..., n\}$.

(1) A fuzzy relation R_b is a best approximation solution of $A_i \to R = B_i$ in the approximation space $L^{U\times V}$ (resp. $\mathcal{R}_{ru}$) with respect to $\delta_r(R)$ if

$$\delta_r(R_b) = \bigvee_{R\in L^{U\times V}} \delta_r(R), \;\; \delta_r(R_b) = \bigvee_{R\in \mathcal{R}_{ru}} \delta_r(R).$$

(2) A fuzzy relation R_b is a best approximation solution of $R \to A_i = B_i$ in the approximation space $L^{U\times V}$ (resp. $\mathcal{R}_{lu}$) with respect to $\delta_l(R)$ if

$$\delta_l(R_b) = \bigvee_{R\in L^{U\times V}} \delta_l(R), \;\; \delta_l(R_b) = \bigvee_{R\in \mathcal{R}_{lu}} \delta_l(R).$$

Theorem 3.12 *Define $\leq_{\delta_r}, \leq_{\delta_l}, \leq_{ru}, \leq_{lu}$ on the approximation space $L^{U\times V}$ (resp. $\mathcal{R}_{rl}$ or $\mathcal{R}_{ru}$) as follows:*

$$R_1 \leq_{\delta_r} R_2 \ \text{ iff } \ \delta_r(R_2) \leq \delta_r(R_1),$$

$$R_1 \leq_{\delta_l} R_2 \ \text{ iff } \ \delta_l(R_2) \leq \delta_l(R_1),$$

$$R_1 \leq_{ru} R_2 \ \text{ iff } \ A_i \to R_1 \leq A_i \to R_2, i \in \{1, ..., n\}, R_1, R_2 \in \mathcal{R}_{ru}$$

$$R_1 \leq_{lu} R_2 \ \text{ iff } \ R_1 \to A_i \leq R_2 \to A_i, i \in \{1, ..., n\}, R_1, R_2 \in \mathcal{R}_{lu}$$

Then $\leq_{\delta_r}, \leq_{\delta_l}, \leq_{ru}, \leq_{lu}$ are preorders. Moreover, $R_1 \leq_{lu} R_2$ implies $R_1 \leq_{\delta_l} R_2$ and $R_1 \leq_{ru} R_2$ implies $R_1 \leq_{\delta_r} R_2$.

Proof (1) Since $\delta_r(R) = \delta_r(R)$, $R \leq_{\delta_r} R$. Thus $\leq_{\delta_r}$ is reflexive.

If $R_1 \leq_{\delta_r} R_2$ and $R_2 \leq_{\delta_r} R_3$, then $\delta_r(R_2) \leq \delta_r(R_1)$ and $\delta_r(R_3) \leq \delta_r(R_2)$. Hence $\delta_r(R_3) \leq \delta_r(R_1)$;i.e. $R_1 \leq_{\delta_r} R_3$. Thus $\leq_{\delta_r}$ is transitive. So, $\leq_{\delta_r}$ is a preorder. Similarly, $\leq_{\delta_l}, \leq_{ru}, \leq_{lu}$ are preorders.

Since $R_1 \leq_{ru} R_2$ iff $B_i \leq A_i \to R_1 \leq A_i \to R_2, i \in \{1, ..., n\}, R_1, R_2 \in \mathcal{R}_{ru}$, then $\delta_r(R_2) = \bigwedge_{i=1}^n \bigwedge_{v\in V}((A_i \to R_2) \to B_i(v)) \leq \bigwedge_{i=1}^n \bigwedge_{v\in V}((A_i \to R_1)(v) \to B_i(v)) = \delta_r(R_1)$. So, $R_1 \leq_{\delta_r} R_2$. Similarly, $R_1 \leq_{lu} R_2$ implies $R_1 \leq_{\delta_l} R_2$.

Theorem 3.13 *Let $A_i \in L^U$, $R \in L^{U\times V}$ and $B_i \in L^V$, $i \in \{1, ..., n\}$.*

(1) $R^{\odot}(u, v) = \bigvee_{p=1}^n (A_p(u) \odot B_p(v))$ is the least element in $\mathcal{R}_{ru} = \{R \in L^{U\times V} \mid A_i \to R \geq B_i, i \in \{1, ..., n\}\}$ with respect to the ordinary order $\leq$.

(2) $R^{\to}(u, v) = \bigwedge_{p=1}^n (B_p(v) \to A_p(u))$ is the greatest element in $\mathcal{R}_{lu} = \{R \in L^{X\times Y} \mid R \to A_i \geq B_i, i \in \{1, ..., n\}\}$ with respect to the ordinary order $\leq$.

Proof (1) We have $R^{\odot} \in R_{ru}$ from:

$$\begin{array}{ll} (A_i \to R^{\odot})(v) & = \bigwedge_{u\in U}(A_i(u) \to R^{\odot}(u, v)) \\ & = \bigwedge_{u\in U}((A_i(u) \to \bigvee_{p=1}^n A_p(u) \odot B_p(v)) \\ & \geq \bigwedge_{u\in U}(A_i(u) \to A_i(u) \odot B_i(v)) \\ & \geq B_i(v). \end{array}$$

Let $R \in R_{ru}$ be given.

$$\begin{array}{l} (A_i \to R)(v) = \bigwedge_{u\in U}(A_i(u) \to R(u, v)) \geq B_i(v) \\ (\Rightarrow) R(u, v) \geq A_i(u) \odot B_i(v) \\ (\Rightarrow) R(u, v) \geq \bigvee_{i=1}^n (A_i(u) \odot B_i(v)). \end{array}$$

Thus $R \geq R^{\odot}$. So, $R^{\odot}$ is the least element in R_{ru}.

(2) We have $R^{\rightarrow} \in R_{lu}$ from:

$$\begin{aligned}(R^{\rightarrow} \to A_i)(v) &= \bigwedge_{x\in X}(R^{\rightarrow}(u,v) \to A_i(u)) \\ &= \bigwedge_{x\in X}(\bigwedge_{p=1}^{n}(B_p(v) \to A_p(u)) \to A_i(u)) \\ &\geq \bigwedge_{x\in X}((B_i(v) \to A_i(u)) \to A_i(u)) \\ &\geq B_i(v).\end{aligned}$$

Let $R \in R_{lu}$ be given.

$$\begin{aligned}&(R \to A_i)(v) = \bigwedge_{x\in X}(R(u,v) \to A_i(u)) \geq B_i(v) \\ &(\Rightarrow) R(u,v) \odot B_i(v) \leq A_i(u) \\ &(\Rightarrow) R(u,v) \leq B_i(v) \to A_i(u).\end{aligned}$$

Thus $R \leq R^{\rightarrow}$. So, $R^{\rightarrow}$ is the greatest element in R_{lu}.

Definition 3.14 Let $A_i \in L^U$, $R \in L^{U\times V}$ and $B_i \in L^V$, $i \in \{1, ..., n\}$.

(1) A fuzzy relation $R_b^{ru} \in \mathcal{R}_{ru}$ is a best approximation solution of $A_i \to R = B_i$ in the approximation space $\mathcal{R}_{ru}$ with respect to $\leq_{ru}$ if there is no fuzzy relation $R \in \mathcal{R}_{ru}$ such that $R \leq_{ru} R_b^{ru}$ and $A_i \to R \neq A_i \to R_b^{ru}$ for at least one $i \in \{1, 2, ..., n\}$.

(2) A fuzzy relation $R_b^{lu} \in \mathcal{R}_{lu}$ is a best approximation solution of $R \to A_i = B_i$ in the approximation space $\mathcal{R}_{lu}$ with respect to $\leq_{lu}$ if there is no fuzzy relation $R \in \mathcal{R}_{lu}$ such that $R \leq_{lu} R_b^{lu}$ and $R \to A_i \neq R_b^{lu} \to A_i$ for at least one $i \in \{1, 2, ..., n\}$.

Theorem 3.15 *Let $A_i \in L^U$, $R \in L^{U\times V}$ and $B_i \in L^V$, $i \in \{1, ..., n\}$.*

(1) *If the system of $A_i \to R = B_i$ is unsolvable with respect to an unknown $R \in L^{U\times V}$, then $R^{\odot}(u,v) = \bigvee_{p=1}^{n}(A_p(u) \odot B_p(v))$ is the best approximation solution in R_{ru} with respect to the preorder $\leq_{ru}$, that is, $R_b^{ru} = R^{\odot}$.*

(2) *If the system of $R \to A_i = B_i$ is unsolvable with respect to an unknown $R \in L^{U\times V}$, then $R^{\rightarrow}(u,v) = \bigwedge_{p=1}^{n}(B_p(v) \to A_p(u))$ is the best approximation solution in R_{lu} with respect to the ordinary order $\leq_{lu}$, that is, $R_b^{lu} = R^{\rightarrow}$.*

Proof (1) Suppose there exists a fuzzy relation $R \in \mathcal{R}_{ru}$ such that $R \leq_{ru} R^{\odot}$ and $A_i \to R \neq A_i \to R^{\odot}$ for at least one $i \in \{1, 2, ..., n\}$. Since $R \leq_{ru} R^{\odot}$,

$$R \leq_{ru} R^{\odot} \text{ iff } A_i \to R \leq A_i \to R^{\odot}, i \in \{1, ..., n\}, R, R^{\odot} \in \mathcal{R}_{ru}$$

Since $A_i \to R \neq A_i \to R^{\odot}$ for at least one $i \in \{1, 2, ..., n\}$, there exists $v \in V$ such that $(A_i \to R)(v) < (A_i \to R^{\odot})(v)$. By Theorem 3.13(1), since $R \geq R^{\odot}$, then $A_i \to R^{\odot} \leq A_i \to R$. It is a contradiction. Thus, $R_b^{ru} = R^{\odot}$.

(2) Suppose there exists a fuzzy relation $R \in \mathcal{R}_{lu}$ such that $R \leq_{lu} R^{\rightarrow}$ and $R \to A_i \neq R^{\rightarrow} \to A_i$ for at least one $i \in \{1, 2, ..., n\}$. Since $R \leq_{lu} R^{\rightarrow}$,

$$R \leq_{lu} R^{\rightarrow} \text{ iff } R \to A_i \leq R^{\rightarrow} \to A_i, i \in \{1, ..., n\}, R, R^{\rightarrow} \in \mathcal{R}_{lu}$$

Since $R \to A_i \neq R^{\rightarrow} \to A_i$ for at least one $i \in \{1, 2, ..., n\}$, there exists $v \in V$ such that $(R \to A_i)(v) < (R^{\rightarrow} \to A_i)(v)$. By Theorem 3.13(2), since $R \leq R^{\rightarrow}$, then $R^{\rightarrow} \to A_i \leq R \to A_i$. It is a contradiction. Thus, $R_b^{lu} = R^{\rightarrow}$.

Theorem 3.16 *Let $A_i \in L^U$, $R \in L^{U \times V}$ and $B_i \in L^V$, $i \in \{1, ..., n\}$.*

(1) *If the system of $A_i \to R = B_i$ is unsolvable with respect to an unknown $R \in L^{U \times V}$, then $R^{\odot}(u, v) = \bigvee_{p=1}^n (A_p(u) \odot B_p(v))$ is the best approximation solution in R_{ru} with respect to the approximation quality $\delta_r(R)$.*

(2) *If the system of $R \to A_i = B_i$ is unsolvable with respect to an unknown $R \in L^{U \times V}$, then $R^{\rightarrow}(u, v) = \bigwedge_{p=1}^n (B_p(v) \to A_p(u))$ is the best approximation solution in R_{lu} with respect to the approximation quality $\delta_l(R)$.*

Proof (1) Let $R \in \mathcal{R}_{ru}$ be a fuzzy relation. We will show that $\delta_r(R) \leq \delta_r(R^{\odot})$.

Put $A_i \to R = C_i$ for $i \in \{1, ..., n\}$. Since R is solvable, by Theorem 3.3 (5), $C_i \geq B_i$ and $A_i(u) \to \bigvee_{p=1}^n (A_p(u) \odot C_p(v)) = C_i(v)$ such that

$$R(u, v) \geq \bigvee_{p=1} (A_p(u) \odot C_p(v)) \geq \bigvee_{p=1} (A_p(u) \odot B_p(v)) = R^{\odot}(u, v).$$

Thus,

$$\begin{aligned} \delta_r(R) &= \textstyle\bigwedge_{i=1}^n \bigwedge_{v \in V} ((A_i \to R)(v) \to B_i(v)) \\ &\leq \textstyle\bigwedge_{i=1}^n \bigwedge_{v \in V} ((A_i \to R^{\odot})(v) \to B_i(v)) \\ &= \delta_r(R^{\odot}). \end{aligned}$$

(2) Let $R \in \mathcal{R}_{lu}$ be a fuzzy relation. We will show that $\delta_l(R) \leq \delta_l(R^{\rightarrow})$.

Put $R \to A_i = C_i$ for $i \in \{1, ..., n\}$. Since R is solvable, by Theorem 3.5(3), $C_i \geq B_i$ and $\bigwedge_{p=1} (C_p(v) \to A_p(u)) \to A_i(u) = C_i(v)$ such that

$$R(u, v) \leq \bigwedge_{p=1} (C_p(v) \to A_p(u)) \leq \bigwedge_{p=1} (B_p(v) \to A_p(u)) \leq R^{\rightarrow}(u, v).$$

Thus,

$$\begin{aligned} \delta_l(R) &= \textstyle\bigwedge_{i=1}^n \bigwedge_{v \in V} ((R \to A_i)(v) \to B_i(v)) \\ &\leq \textstyle\bigwedge_{i=1}^n \bigwedge_{v \in V} ((R^{\rightarrow} \to A_i)(v) \to B_i(v)) \\ &= \delta_l(R^{\rightarrow}). \end{aligned}$$

Example 3.17 Let the structure $(L = [0,1], \odot, \to, 0, 1)$ be as same in Example 3.7. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2\}$ be sets.

(1) Put

$$A_1 = (A_1(u_1), A_1(u_2), A_1(u_3)) = (0.5, 0.3, 0.9)$$
$$A_2 = (A_2(u_1), A_2(u_2), A_2(u_3)) = (0.7, 0.2, 0.4)$$
$$B_1 = (B_1(v_1), B_1(v_2)) = (0.3, 0.6)$$
$$B_2 = (B_2(v_1), B_2(v_2)) = (0.7, 0.5)$$

$$\begin{array}{l} 0.5 \to x_1 \wedge 0.3 \to x_2 \wedge 0.9 \to x_3 = 0.3 \\ 0.7 \to x_1 \wedge 0.2 \to x_2 \wedge 0.4 \to x_3 = 0.7 \end{array} \quad \mathbf{(12)}$$

Then $(x_1, x_2, x_3) = (R_1(u_1, v_1), R_1(u_2, v_1), R_1(u_3, v_1)) = (0.4, 0.2, 0.3)$ is a solution of **(12)**. Since $h(b)_i = \bigvee_{p=1}^{2}(b_p \odot a_{pi})$ for $b = (B_1(v_1), B_2(v_1)) = (0.3, 0.7)$, $h(b) = (R^{\odot}(u_1, v_1), R^{\odot}(u_2, v_1), R^{\odot}(u_3, v_1)) = (0.4, 0, 0.2)$ is the least solution of **(12)** ;i.e. $g(h(b)) = b$.

$$\begin{array}{l} 0.5 \to x_1 \wedge 0.3 \to x_2 \wedge 0.9 \to x_3 = 0.6 \\ 0.7 \to x_1 \wedge 0.2 \to x_2 \wedge 0.4 \to x_3 = 0.5 \end{array} \quad \mathbf{(13)}$$

Then $(x_1, x_2, x_3) = (R_1(u_1, v_2), R_1(u_2, v_2), R_1(u_3, v_2)) = (0.2, 0.1, 0.5)$ is a solution of **(10)**. Since $h(b)_i = \bigvee_{p=1}^{2}(b_p \odot a_{pi})$ for $b = (B_1(v_2), B_2(v_2)) = (0.6, 0.5)$, $h(b) = (R^{\odot}(u_1, v_2), R^{\odot}(u_2, v_2), R^{\odot}(u_3, v_2)) = (0.2, 0.1, 0.5)$ is the least solution of **(13)** ;i.e. $g(h(b)) = b$.

For $i \in \{1, 2\}$,

$$\begin{array}{l} 0.5 \to R(u_1, v_i) \wedge 0.3 \to R(u_2, v_i) \wedge 0.9 \to R(u_3, v_i) = B_1(v_i) \\ 0.7 \to R(u_1, v_i) \wedge 0.2 \to R(u_2, v_i) \wedge 0.4 \to R(u_3, v_i) = B_2(v_i) \end{array} \quad \mathbf{(14)}$$

we obtain:

$$R = R_1 = \begin{pmatrix} 0.4 & 0.2 \\ 0.2 & 0.1 \\ 0.3 & 0.5 \end{pmatrix} \quad R = R^{\odot} = \begin{pmatrix} 0.4 & 0.2 \\ 0 & 0.1 \\ 0.2 & 0.5 \end{pmatrix}$$

(2)

$$\begin{array}{l} x_1 \to 0.5 \wedge x_2 \to 0.3 \wedge x_3 \to 0.9 = 0.3 \\ x_1 \to 0.7 \wedge x_2 \to 0.2 \wedge x_3 \to 0.4 = 0.7 \end{array} \quad \mathbf{(15)}$$

Since $g^{\to}(b)_i = \bigwedge_{p=1}^{2}(b_p \to a_{pi})$ for $b = (B_1(v_1), B_2(v_1)) = (0.3, 0.7)$, $g^{\to}(b) = (R^{\to}(u_1, v_1), R^{\to}(u_2, v_1), R^{\to}(u_3, v_1)) = (1, 0.5, 0.7)$ is not a solution of **(15)** ;i.e. $f^{\to}(g^{\to}(b)) > b$.

$$\begin{array}{l} x_1 \to 0.5 \wedge x_2 \to 0.3 \wedge x_3 \to 0.9 = 0.6 \\ x_1 \to 0.7 \wedge x_2 \to 0.2 \wedge x_3 \to 0.4 = 0.5 \end{array} \quad \mathbf{(16)}$$

Then $(x_1, x_2, x_3) = (R_1(u_1, v_2), R_1(u_2, v_2), R_1(u_3, v_2)) = (0.9, 0.7, 0.8)$ is a solution of **(16)**. Since $g^{\rightarrow}(c)_i = \bigwedge_{p=1}^{2}(c_p \rightarrow a_{pi})$ for $c = (B_1(v_2), B_2(v_2)) = (0.6, 0.5)$, $g^{\rightarrow}(c) = (R^{\rightarrow}(u_1, v_2), R^{\rightarrow}(u_2, v_2), R^{\rightarrow}(u_3, v_2)) = (0.9, 0.7, 0.9)$ is the greatest solution of **(16)** ;i.e. $f^{\rightarrow}(g^{\rightarrow}(c)) = c$.

For $i \in \{1, 2\}$,

$$\begin{array}{l} R(u_1, v_i) \rightarrow 0.5 \wedge R(u_2, v_i) \rightarrow 0.3 \wedge R(u_3, v_i) \rightarrow 0.9 = B_1(v_i) \\ R(u_1, v_i) \rightarrow 0.7 \wedge R(u_2, v_i) \rightarrow 0.2 \wedge R(u_3, v_i) \rightarrow 0.4 = B_2(v_i) \end{array} \qquad (\mathbf{17})$$

Put

$$R_2 = \begin{pmatrix} 1 & 0.9 \\ 0.5 & 0.7 \\ 0.7 & 0.8 \end{pmatrix} \quad R^{\rightarrow} = \begin{pmatrix} 1 & 0.9 \\ 0.5 & 0.7 \\ 0.7 & 0.9 \end{pmatrix}$$

Then

$$\delta_l(R_2) = \delta_l(R^{\rightarrow}) = \bigwedge_{i=1}^{2} \bigwedge_{v \in V} ((R^{\rightarrow} \rightarrow A_i)(v) \rightarrow B_i(v)) = 0.8.$$

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