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On blow-up solution to a nonlinear dispersive wave equation

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Abstract

In this paper, comparing with the previous results, we present a new blow-up solution for strong solutions to the equation provided that the potential $(1-\partial_x^2)u_0$ changes sign on $\mathbf{R}$, which improves considerably the previous result.

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1 Introduction

In this article, we will consider the nonlinear dispersive wave equation

$$u_t - u_{txx} + (a+b)uu_x - au_xu_{xx} - buu_{xxx} + \lambda(u - u_{xx}) = 0, \tag{1}$$

where $a > 0, b > 0$ and $\lambda \geq 0$ are arbitrary constants, u is the fluid velocity in the x direction, $\lambda(u - u_{xx})$ represents the weakly dissipative term.

For $a = 2$, $b = 1$ and $\lambda = 0$ in Eq.(1), Eq.(1) becomes the famous Camassa-Holm equation [1, 2, 3]

$$u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \quad t > 0, \quad x \in R, \tag{2}$$

where $u(x,t)$ represents the free surface above a flat bottom. As a model to describe the shallow water motion, Eq.(2) has a bi-Hamiltonian structure and infinite conservation laws and is completely integrable.

For $a = 2$, $b = 1$ and $\lambda > 0$ in Eq.(1), Eq.(1) changes into weakly dissipative Camassa-Holm equation [4]

$$u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \lambda(u - u_{xx}) = 0, \quad t > 0, \quad x \in R. \tag{3}$$

For $a = 3$, $b = 1$ and $\lambda = 0$ in Eq.(1), Eq.(1) becomes the classical Degasperis-Procesi equation[5, 6]

$$u_t - u_{txx} + 4uu_x - 3u_xu_{xx} - uu_{xxx} = 0, \quad t > 0, \quad x \in R. \tag{4}$$

As a model to describe the shallow water wave, Degasperis-Procesi equation (4) also satisfies complete integrablity, bi-Hamiltonian structure, infinite conservation laws and peakon soliton solutions which are analogous to the Camassa-Holm. However, they are quite different model(see [5]).

For $a = 3$, $b = 1$ and $\lambda > 0$ in Eq.(1), Eq.(1) is read as weakly dissipative Degasperis-Procesi equation[7]

$$u_t - u_{txx} + 4uu_x - 3u_xu_{xx} - uu_{xxx} + \lambda(u - u_{xx}) = 0, t > 0, \quad x \in R. \tag{5}$$

In [8], Lai and Wu study global existence and blow-up to Eq.(1) with $\lambda = 0$ assumed that the potential $(1 - \partial_x^2)u_0$ does not change sign on $\mathbf{R}$. To our best knowledge, the blow-up solution of Eq.(1) under the condition $y_0 \leq 0$ for $x \leq x_0$ and $y_0 \geq 0$ for $x \geq x_0$ seems not have been investigated. Present paper is mainly concerned with blow-up solution to Eq.(1) provided that the potential $(1 - \partial_x^2)u_0$ changes sign on $\mathbf{R}$. Since Eq.(1) is a generalization of Camassa-Holm equation and Degasperis-Procesi equation, Eq.(1) loses some important conservation laws that they possess. In the paper, we mainly depend on some useful prior estimates from the equation and the good method presented in Liu and Yin [6] to obtain the blow-up solution for the equation.

2 Preliminary Notes

We denote by $*$ the convolution. Note that if $G(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbf{R}$, then $(1 - \partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbf{R})$ and $G * (u - u_{xx}) = u$. Using this identity, the Cauchy problem of Eq.(1) becomes

$$\begin{cases} u_t + buu_x + \partial_x G * [\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2] + \lambda u = 0, \quad t > 0, \quad x \in \mathbf{R}, \\ u(0, x) = u_0(x), \end{cases} \tag{6}$$

which is equivalent to

$$\begin{cases} y_t + buy_x + ayu_x + \lambda y = 0, \qquad t > 0, \qquad x \in \mathbf{R}, \\ y = u - u_{xx}, \\ u(0, x) = u_0(x). \end{cases} \tag{7}$$

3 Blow-up

We firstly recall the local well-posedness of solution and blow-up scenario for problem (6).

Lemma 2.1. [9] *Given* $u_0 \in H^s(s > \frac{3}{2})$*, there exist a maximal* $T = T(u_0)$ *and a unique solution* u *to problem (6), such that*

$$u = u(\cdot, u_0) \in C\Big([0,T); H^s(\mathbf{R})\Big) \bigcap C^1\Big([0,T); H^{s-1}(\mathbf{R})\Big).$$

Lemma 2.2. [9]*Let* $u_0 \in H^s, s \geq 2$*, and* u *be the corresponding solution to problem (6) with time* T*. Then* $T < \infty$ *if and only if*

$$\lim_{t\to T} \inf \Big\{ \inf_{x\in\mathbf{R}} \Big[u_x(t,x)\Big]\Big\} = -\infty.$$

Lemma 2.3. *Provided that* $2b \leq a \leq 3b$*,* $u_0 \in H^s(\mathbf{R})\bigcap L^1(\mathbf{R}), s \geq \frac{3}{2}$*, if* $y_0 = u_0 - u_{0xx}$ *satisfies* $y_0 \leq 0$ *for* $x \geq x_0$*,* $y_0 \geq 0$ *for* $x \leq x_0$ *and* $y(t,x) = u(t,x) - u_{xx}(t,x)$*, then, for* $t \in \mathbf{R}_+$*, it holds*

(i) $\| u \|_{L^1} \leq e^{-\lambda t} \| u_0 \|_{L^1}$ *on* R*,*

(ii) $\| u \|^2_{H^1} \leq \| u_0 \|^2_{H^1} \exp\Big(-2\lambda t + \frac{a-2b}{\lambda} \| u_0 \|_{L^1} (1 - e^{-\lambda t})\Big)$.

Proof. Since $q(t,x)$ is an increasing diffeomorphism of $\mathbf{R}$ with $q_x(t,x) > 0$ with respect to time t. We deduce from the assumption that for $t \in [0,T)$,

$$\begin{cases} y(t,x) \leq 0, & \text{if} \quad x \geq q(t,x_0), \\ y(t,x) \geq 0, & \text{if} \quad x \leq q(t,x_0), \end{cases} \tag{8}$$

and $y(t, q(t,x_0)) = 0$.

Integrating the first equation of problem (6) with respect to x in interval $(-\infty, q(t,x_0)]$ yields

$$\frac{d}{dt}\int_{-\infty}^{q(t,x_0)} u dx = -b\int_{-\infty}^{q(t,x_0)} uu_x dx - \int_{-\infty}^{q(t,x_0)} \partial_x(G * [\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2])dx$$
$$-\lambda \int_{-\infty}^{q(t,x_0)} udx. \tag{9}$$

On the other hand, integrating the first equation of problem (6) with respect to x in interval $[q(t,x_0), +\infty)$ leads to

$$\frac{d}{dt}\int_{q(t,x_0)}^{\infty} u dx = -b\int_{q(t,x_0)}^{\infty} uu_x dx - \int_{q(t,x_0)}^{\infty} \partial_x(G * [\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2])dx$$
$$-\lambda \int_{q(t,x_0)}^{\infty} udx. \tag{10}$$

Subtracting (10) from (9) yields

$$\frac{d}{dt}(\int_{-\infty}^{q(t,x_0)} udx - \int_{q(t,x_0)}^{\infty} udx) = -2G * [\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2]) \mid_{x=q(t,x_0)}$$
$$-bu^2 \mid_{x=q(t,x_0)} -\lambda(\int_{-\infty}^{q(t,x_0)} udx - \int_{q(t,x_0)}^{\infty} udx),$$

which results in

$$\frac{d}{dt}\int_{-\infty}^{\infty} | u | dx \leq -\lambda \int_{-\infty}^{\infty} | u | dx.$$

Hence, we get

$$\| u \|_{L^1} \leq e^{-\lambda t} \| u_0 \|_{L^1} .$$

This proves (i). Using $u = G * y$, one has

$$\begin{aligned} u(t,x) &= \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\xi}y(\xi)d\xi + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\xi}y(\xi)d\xi, \\ u_x(t,x) &= -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\xi}y(\xi)d\xi + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\xi}y(\xi)d\xi. \end{aligned} \tag{11}$$

Thus, we get

$$\begin{aligned} u(t,x) + u_x(t,x) &= e^{x}\int_{x}^{+\infty} e^{-\xi}y(\xi)d\xi \leq 0, \quad \text{if} \quad x \geq q(t,x_0), \\ u(t,x) - u_x(t,x) &= e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi)d\xi \geq 0, \quad \text{if} \quad x \leq q(t,x_0). \end{aligned} \tag{12}$$

From (12), we get $u_x \leq | u |$ on $\mathbf{R}$.

Multiplying Eq.(1) by u and integrating by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_R (u^2+u_x^2)dx &= -\lambda\int_R (u^2+u_x^2)dx - a\int_R uu_xu_{xx}dx - b\int_R u^2u_{xxx}dx \\ &= -\lambda\int_R (u^2+u_x^2)dx - \frac{2b-a}{2}\int_R u_x^3dx \\ &\leq -\lambda\int_R (u^2+u_x^2)dx + \frac{a-2b}{2}e^{-\lambda t} \| u_0 \|_{L^1} \| u \|_{H^1}^2 . \end{aligned}$$

Applying the Gronwall's inequality, one has

$$\| u \|_{H^1}^2 \leq \| u_0 \|_{H^1}^2 \exp\Big(-2\lambda t + \frac{a-2b}{\lambda} \| u_0 \|_{L^1} (1-e^{-\lambda t})\Big).$$

This completes the proof of Lemma 2.3.

Theorem 2.4. *Provided that* $2b \leq a \leq 3b$, $u_0 \in H^s(\mathbf{R})\bigcap L^1(\mathbf{R}), s > \frac{3}{2}$, *and* $y_0 = u_0 - u_{0xx}$ *satisfies* $y_0 \geq 0$ *for* $x \leq x_0$, $y_0 \leq 0$ *for* $x \geq x_0$ *and* $\lambda < -\frac{bc_0}{2(1+c_0)}u_0'(x_0)$, *where* $c_0 = \frac{u_0'^2-u_0^2}{\|u_0\|_{H^1}^2\exp[\frac{a-2b}{\lambda}(1-e^{-\lambda t})\|u_0\|_{L^1}]}$, *then the solution* $u(t,x)$ *to problem (6) blows up in finite time.*

Proof. Applying a simple density argument, we only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to problem (6) with the initial data $u_0 \in H^3(\mathbf{R})$. Differentiating the first equation of problem (6) with respect to x, we get

$$u_{tx} = \frac{b-a}{2}u_x^2 - buu_{xx} + \frac{a}{2}u^2 - G*[\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x^2)] - \lambda u_x. \tag{13}$$

Applying (6), we have

$$\begin{aligned}
\frac{d}{dt}u_x(t,q(t,x_0)) &= u_{tx}(t,q(t,x_0)) + u_{xx}(t,q(t,x_0))\frac{d}{dt}q(t,x_0) \\
&= u_{tx}(t,q(t,x_0)) + bu(t,q(t,x_0))u_{xx}(t,q(t,x_0)) \\
&= \frac{b-a}{2}u_x^2(t,q(t,x_0)) + \frac{a}{2}u^2(t,q(t,x_0)) \\
&\quad -G*[\frac{a}{2}u^2(t,q(t,x_0)) + \frac{3b-a}{2}u_x^2(t,q(t,x_0))] \\
&\quad -\lambda u_x(t,q(t,x_0)) \\
&= \frac{b-a}{2}u_x^2(t,q(t,x_0)) + \frac{a}{2}u^2(t,q(t,x_0)) \\
&\quad -G*[bu^2(t,q(t,x_0)) + \frac{b}{2}u_x^2(t,q(t,x_0))] \\
&\quad +\frac{a-2b}{2}G*(u_x^2(t,q(t,x_0)) - u^2(t,q(t,x_0))) \\
&\quad -\lambda u_x(t,q(t,x_0)).
\end{aligned} \tag{14}$$

Note that (see page 347 in [3])

$$\begin{aligned}
&e^{-x}\int_{-\infty}^{x} e^{\xi}[u_\xi^2(t,\xi) + 2u^2(t,\xi)]d\xi \geq u^2(t,x) \\
&e^{x}\int_{x}^{\infty} e^{-\xi}[u_\xi^2(t,\xi) + 2u^2(t,\xi)]d\xi \geq u^2(t,x).
\end{aligned} \tag{15}$$

It shows that $G*(\frac{1}{2}u_x^2+u^2)(t,x) \geq \frac{1}{2}u^2(t,x)$ for $\forall(t,x) \in [0,T)\times R$. Therefore, from (14), one has

$$\begin{aligned}
&\frac{d}{dt}u_x(t,q(t,x_0)) + \lambda u_x(t,q(t,x_0)) \\
&\leq \frac{b-a}{2}u_x^2(t,q(t,x_0)) + \frac{a-b}{2}u^2(t,q(t,x_0)) \\
&\qquad +\frac{a-2b}{2}G*(u_x^2(t,q(t,x_0)) - u^2(t,q(t,x_0))).
\end{aligned} \tag{16}$$

Due to $q(t,x)$ is an increasing diffeomorphism of R with $q_x(t,x) > 0$ with respect to t, for $t \in [0,T)$, we deduce from the assumption of Theorem 2.4 that

$$\begin{cases} y(t,x) \geq 0, & \text{if} \quad x \leq q(t,x_0), \\ y(t,x) \leq 0, & \text{if} \quad x \geq q(t,x_0) \end{cases} \tag{17}$$

and $y(t,q(t,x_0))=0$.

Set

$$M(t):=e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}y(t,\xi)d\xi,\quad t\in[0,T),$$
$$N(t):=e^{q(t,x_0)}\int_{q(t,x_0)}^{\infty}e^{-\xi}y(t,\xi)d\xi,\quad t\in[0,T).$$

From (17), for $t\in[0,T)$, we obtain

$$M(t,q(t,x_0))\geq 0\quad\text{and}\quad N(t,q(t,x_0))\leq 0. \tag{18}$$

On the other hand, taking into account equalities (11), for $t\in[0,T)$, we have

$$\begin{aligned}M(t,q(t,x_0))&=u(t,q(t,x_0))-u_x(t,q(t,x_0))\\ N(t,q(t,x_0))&=u(t,q(t,x_0))+u_x(t,q(t,x_0)).\end{aligned} \tag{19}$$

Therefore, from Eqs.(19), we get

$$M(t,q(t,x_0))N(t,q(t,x_0))=u^2(t,q(t,x_0))-u_x^2(t,q(t,x_0))\leq 0. \tag{20}$$

Note that $y(t,q(t,x_0))=0, t\in[0,T)$, it follows that

$$\begin{aligned}\frac{d}{dt}M(t,q(t,x_0))&=-q_t(t,x_0)M(t,q(t,x_0))\\ &\quad+e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}y_t(t,\xi)d\xi.\end{aligned} \tag{21}$$

Using (7) and integrating by parts, it holds

$$\begin{aligned}\int_{-\infty}^{q(t,x_0)}e^{\xi}y_t(t,\xi)d\xi&=-\lambda\int_{-\infty}^{q(t,x_0)}e^{\xi}y(t,\xi)d\xi-b\int_{-\infty}^{q(t,x_0)}e^{\xi}\Big(u(t,\xi)y(t,\xi)\Big)_{\xi}d\xi\\ &\quad-(a-b)\int_{-\infty}^{q(t,x_0)}e^{\xi}u_{\xi}(t,\xi)y(t,\xi)d\xi\\ &=-\lambda\int_{-\infty}^{q(t,x_0)}e^{\xi}y(t,\xi)d\xi+b\int_{-\infty}^{q(t,x_0)}e^{\xi}u^2(t,\xi)d\xi\\ &\quad-b\int_{-\infty}^{q(t,x_0)}e^{\xi}u(t,\xi)u_{\xi\xi}(t,\xi)d\xi-(a-b)\int_{-\infty}^{q(t,x_0)}e^{\xi}u_{\xi}(t,\xi)u(t,\xi)d\xi\\ &\quad+(a-b)\int_{-\infty}^{q(t,x_0)}e^{\xi}u_{\xi}(t,\xi)u_{\xi\xi}(t,\xi)d\xi\\ &=-\lambda\int_{-\infty}^{q(t,x_0)}e^{\xi}y(t,\xi)d\xi+\frac{a}{2}\int_{-\infty}^{q(t,x_0)}e^{\xi}u^2(t,\xi)d\xi\\ &\quad+\frac{3b-a}{2}\int_{-\infty}^{q(t,x_0)}e^{\xi}u_{\xi}^2(t,\xi)d\xi-be^{q(t,x_0)}u(t,q(t,x_0))u_x(t,q(t,x_0))\\ &\quad+\frac{a-b}{2}e^{q(t,x_0)}u_x^2(t,q(t,x_0))-\frac{a-2b}{2}e^{q(t,x_0)}u^2(t,q(t,x_0)).\end{aligned} \tag{22}$$

Substituting (22) into (21) leads to

$$\begin{aligned}
&\frac{d}{dt}M(t,q(t,x_0))+\lambda M(t,q(t,x_0))=-bu(t,x_0)M(t,q(t,x_0))\\
&+\frac{a}{2}e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}u^2(t,\xi)d\xi+\frac{3b-a}{2}e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}u_\xi^2(t,\xi)d\xi\\
&-bu(t,q(t,x_0))u_x(t,q(t,x_0))+\frac{a-b}{2}u_x^2(t,q(t,x_0))-\frac{a-2b}{2}u^2(t,q(t,x_0))\\
&=-bu^2(t,q(t,x_0))+be^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}(\frac{1}{2}u_\xi^2(t,\xi)+u^2(t,\xi))d\xi\\
&\quad-\frac{a-2b}{2}e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}(u_\xi^2(t,\xi)-u^2(t,\xi))d\xi\\
&\quad+\frac{a-b}{2}u_x^2(t,q(t,x_0))-\frac{a-2b}{2}u^2(t,q(t,x_0))\\
&\geq\frac{a-b}{2}(u_\xi^2(t,q(t,x_0))-u^2(t,q(t,x_0)))\\
&\quad-\frac{a-2b}{2}e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}(u_\xi^2(t,\xi)-u^2(t,\xi))d\xi\\
&=-\frac{a-b}{2}M(t,q(t,x_0))N(t,q(t,x_0))\\
&\quad+\frac{a-2b}{2}e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}(u^2(t,\xi)-u_\xi^2(t,\xi))d\xi, \qquad (23)
\end{aligned}$$

where we have used (15) and (20).

Since $e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)}e^{\xi}(u^2(t,\xi)-u_\xi^2(t,\xi))d\xi\geq M(t,q(t,x_0))N(t,q(t,x_0))$ (see page 815 in [6]). For $a-2b\geq 0$, we have

$$\frac{d}{dt}M(t,q(t,x_0))+\lambda M(t,q(t,x_0))\geq-\frac{b}{2}M(t,q(t,x_0))N(t,q(t,x_0))\geq 0. \quad (24)$$

In an analogous way, we get

$$\begin{aligned}
\frac{d}{dt}N(t,q(t,x_0))+\lambda N(t,q(t,x_0))\leq\frac{a-b}{2}(u^2(t,\xi)-u_x^2(t,\xi))\\
-\frac{a-2b}{2}e^{q(t,x_0)}\int_{q(t,x_0)}^{\infty}e^{-\xi}(u^2(t,\xi)-u_\xi^2(t,\xi))d\xi(25)
\end{aligned}$$

Due to $e^{q(t,x_0)}\int_{q(t,x_0)}^{\infty}e^{-\xi}(u^2(t,\xi)-u_\xi^2(t,\xi))d\xi\geq M(t,q(t,x_0))N(t,q(t,x_0))$ (see page 816 in [6]). For $a-2b\geq 0$, we obtain

$$\frac{d}{dt}N(t,q(t,x_0))+\lambda N(t,q(t,x_0))\leq\frac{b}{2}M(t,q(t,x_0))N(t,q(t,x_0))\leq 0. \quad (26)$$

For $t\in[0,T)$, we have

$$M(0)=e^{-x_0}\int_{-\infty}^{x_0}e^{\xi}y_0(\xi)d\xi>0$$

and

$$N(0) = e^{x_0} \int_{x_0}^{\infty} e^{-\xi} y_0(\xi) d\xi < 0.$$

Therefore, we derive from (24) and (26) that for $t \in [0, T)$

$$\begin{aligned} &M(t) \geq e^{-\lambda t} M(0) > 0, \\ &N(t) \leq e^{-\lambda t} N(0) < 0. \end{aligned} \tag{27}$$

Assume that solution $u(t)$ to problem (6) exists globally in time $t \in [0, \infty)$, i.e. $T = \infty$. Next, we show that it will lead to a contradiction.

From (24) and (27), we deduce

$$\frac{d}{dt} M(t) + \lambda M(t) \geq -\frac{b}{2} M(t) e^{-\lambda t} N(0). \tag{28}$$

In a similar way, it holds

$$\frac{d}{dt} N(t) + \lambda N(t) \leq \frac{b}{2} N(t) e^{-\lambda t} M(0). \tag{29}$$

Solving the inequalities (28) and (29), we obtain

$$\begin{aligned} M(t) &\geq M(0) \exp(-\lambda t + \frac{b}{2\lambda} N(0) e^{-\lambda t} - \frac{b}{2\lambda} N(0)), \\ N(t) &\leq N(0) \exp(-\lambda t - \frac{b}{2\lambda} M(0) e^{-\lambda t} + \frac{b}{2\lambda} M(0)). \end{aligned}$$

Noting that $M(0, x_0) > 0$ and $N(0, x_0) < 0$, and applying (20), one has

$$\begin{aligned} &u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0)) \\ &\quad \geq -\exp\Big(-2\lambda t + \frac{b}{2\lambda}(N(0) - M(0))(e^{-\lambda t} - 1)\Big) M(0) N(0). \end{aligned} \tag{30}$$

It follows from Lemma 2.3 that

$$\begin{aligned} &\frac{u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0))}{u^2(t, q(t, x_0))} \\ &\qquad \geq \frac{u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0))}{\| u(t, q(t, x_0)) \|_{H^1}^2} \\ &\qquad \geq -\frac{M(0) N(0)}{\| u_0 \|_{H^1}^2 \exp((1 - e^{-\lambda t}) \frac{a - 2b}{\lambda} \| u_0 \|_{L^1})}. \end{aligned}$$

The above inequality implies

$$u^2(t, q(t, x_0)) \leq \frac{1}{1 + c_0} u_x^2, \tag{31}$$

where

$$c_0 = \frac{u_0'^2(x_0) - u^2(x_0)}{\| u_0 \|_{H^1}^2 \exp((1-e^{-\lambda t})\frac{a-2b}{\lambda} \| u_0 \|_{L^1})} > 0.$$

Set $f(t) = u_x(t, q(t, x_0))$, from (16), we deduce

$$\frac{d}{dt} f(t) + \lambda f \leq -\frac{bc_0}{2(1+c_0)} f^2(t) < 0. \tag{32}$$

From (11), for $t \geq 0$, we have

$$\begin{aligned} f(t) &= u_x(t, q(t, x_0)) \\ &= -\frac{1}{2} e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\xi} y(\xi) d\xi + \frac{1}{2} e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} e^{\xi} y(\xi) d\xi < 0, \end{aligned}$$

where we have used (17). Therefore, From (32) we deduce

$$\frac{d}{dt}\Big(\frac{1}{f(t)}\Big) - \frac{\lambda}{f(t)} \geq \frac{bc_0}{2(1+c_0)}. \tag{33}$$

Solving (33) yields

$$\Big(\frac{1}{f(0)} + \frac{bc_0}{2\lambda(1+c_0)}\Big) e^{\lambda t} - \frac{bc_0}{2\lambda(1+c_0)} \leq \frac{1}{f(t)} < 0, \quad t \geq 0. \tag{34}$$

From the assumption of the Theorem, it holds

$$\frac{1}{f(0)} + \frac{bc_0}{2\lambda(1+c_0)} > 0.$$

It shows that $(\frac{1}{f(0)} + \frac{bc_0}{2\lambda(1+c_0)}\Big) e^{\lambda t} \to \infty$ as $t \to \infty$. Inequality (34) implies a contradiction. Therefore, we prove that $T < \infty$.

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