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Exp-Function Method for the Some Nonlinear Partial Differential Equations

Yavuz UGURLU
Firat University, Department of Mathematics, 23119 ELAZIG, TURKEY.
matematikci_23@yahoo.com.tr
Tel: 00.90. 424. 237 00 00/3517, Fax: 00.90. 424. 233 00 62.

Abstract

In this study, we implement the exp-function method for the analytic solutions of the Cahn Allen, the clannish random walker's parabolic and the Fitzhugh–Nagumo equations.

Key Words: Cahn Allen equation; clannish random walker's parabolic equation; Fitzhugh–Nagumo equation; Exp-function method.
AMS classification: 02.30.Jr; 02.60.Cb; 02.70.Wz.

1. Introduction

The mathematical modeling of events in nature can be explained by differential equations.These equations are mathematical models of complex physical occurrences that arise in engineering, chemistry, biology, mechanics and physics. So, nonlinear phenomena play a crucial role in applied mathematics and physics. Calculating exact and numerical solutions of nonlinear equations in mathematical physics play an important role in soliton theory [1, 2]. Recently, it has become more interesting that obtaining exact solutions of nonlinear partial differential equations through using symbolical computer programs such as Maple, Matlab, Mathematica that facilitate complex and tedious algebrical computations. It is too important to find exact solutions of nonlinear partial differential equations. Therefore, various effective methods have been developed to understand the mechanisms of these physical models, to help physicists and engineers and to ensure knowledge for physical problems and its applications. Most of these methods are based on finding balance term with balancing of the highest order linear and nonlinear term. So, these methods can be only applied to nonlinear partial differential equation. Some of these methods are: Tanh function method by Malfliet in 1992 [3], automatic Tanh function method by Parkers and Duffy in 1996 [4], extended Tanh function method by Fan in 2000 [5], jacobi elliptic function method by Fu, Liu and Zhao in 2001 [6], modified extended Tanh function method by Elwakil in 2002 [7], generalized extended Tanh function method by Zheng in 2003 [8], modified jacobi elliptic function method by Shen and Pan in 2003 [9], improved Tanh function method by Chen and Zhang in 2004 [10], generalized jacobi elliptic function method by Chen and Hong-Qing in 2004 [11], jacobi elliptic function rational expansion method by Chen, Wang and Li in 2004 [12], the weierstrass elliptic function expansion method by Chen and Yan in 2006 [13], the exp-method by He in 2006 [14], G'/G expansion method by Wang, Li and Zhang in 2008 [15], extended G'/G expansion method by Guo and Zhou in 2010 [16], generalized G'/G expansion method by Lü in 2010 [17].

In this study, we implement the exp-method for the Cahn Allen equation [18], the clannish random walker's parabolic equation [19] and the Fitzhugh–Nagumo equation [20]. The exp-method was firstly presented by He [14] and was implemented by He and Wu in 2006. The method is generally used for nonlinear partial differential equations, but it is Zhou [21] who first applied the method to the differential-difference equation with great success. Following

Zhu, Dai obtained some excellent results for the discrete nonlinear Schrödinger equation and the hybrid lattice equation [22]. Xu and Zhang [23, 24] contributed much to the development of the method. Separately, Xu obtained periodic solutions [25] by using the exp function method. Oziş and Köroğlu [26] studied the exp function method for traveling wave solution. Wu and He obtained solitary solutions, periodic solutions and compacton-like solutions [27] by using the exp function method. Kaya and Inan found some solutions for the various nonlinear evolution equations [28] with the exp function method. We aim to find some traveling wave solutions of Cahn Allen equation, clannish random walker's parabolic equation and the Fitzhugh–Nagumo equation by using the exp-function method. The method was further developed some other scientists [29-34].

2. An Analysis of the Method and applications

Before starting to give the exp-function method, we give a simple description of the exp-function method. For doing this, it is considered in a two-variable general form of nonlinear PDE

$$Q(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{1}$$

with $u\,(x,t) = u\,(\xi)$, $\xi = kx + wt$, we get a nonlinear ODE for $u\,(\xi)$

$$Q'(u, u', u'', u''', \ldots) = 0, \tag{2}$$

where k and w are constants. We assume the solution of the Eq. (2) as following

$$u\,(\xi) = \frac{\sum\limits_{n=-d}^{c} a_n \exp\,(n\xi)}{\sum\limits_{m=-q}^{p} b_m \exp\,(m\xi)}, \tag{3}$$

where c, d, p and q are positive integer, which are unknown and to be further determined, a_n and b_m are unknown constants. We suppose that the solution of Eq. (2) can be expressed as

$$u\,(\xi) = \frac{a_c \exp\,(c\xi) + ... + a_{-d} \exp\,(-d\xi)}{b_p \exp\,(p\xi) + ... + b_{-q} \exp\,(-q\xi)}, \tag{4}$$

where c, d, p and q are positive integer that can be determined by balancing the highest order derivative and with the highest nonlinear terms into Eq. (2). Substituting solution (4) into Eq. (2) yields a set of algebraic equations for $\exp\,(\xi)$; then all coefficients of $\exp\,(\xi)$ have to vanish. After this separated algebraic equation, we can find a_n and b_m constants.

Example 1. Let's consider nonlinear parabolic partial differential equation given by

$$u_t = u_{xx} - u^n + u, \tag{5}$$

for $n = 3$, Eq.(5) becomes Cahn Allen equation [18]. This equation arises in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. To solve this example, we can use transformation $\xi = kx + wt$ (where, k and w are the wave number and the wave speed, respectively) then Eq. (5) becomes to an ordinary differential equation

$$wu' - k^2 u'' + u^3 - u = 0, \tag{6}$$

when balancing u^3 with u''

$$\frac{c_1 \exp\left[(3p + c)\,\xi\right] + ...}{c_2 \exp\left[4p\xi\right] + ...} = \frac{c_3 \exp\left[c\xi\right] + ...}{c_4 \exp\left[p\xi\right] + ...},$$

then gives $p = c$. Similarly, to determine values of d and q when balancing u^3 with u''

$$\frac{...d_1 \exp\left[-\left(3q + d\right)\xi\right]}{...d_2 \exp\left[-4q\xi\right]} = \frac{...d_3 \exp\left[-d\xi\right]}{...d_4 \exp\left[-q\xi\right]},$$

then gives $q = d$.

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (4) reduces to

$$u\left(\xi\right) = \frac{a_1 \exp\left(\xi\right) + a_0 + a_{-1} \exp\left(-\xi\right)}{\exp\left(\xi\right) + b_0 + b_{-1} \exp\left(-\xi\right)}, \tag{7}$$

substituting Eq. (7) into Eq. (6) yields a set of algebraic equations for $a_0, a_1, a_{-1}, b_0, b_{-1}, w, k$. Algebraic equations system can be written as following

$$\begin{gathered}
\frac{1}{A}\left(-a_1 + a_1^3 = 0\right), \\
\frac{1}{A}\left(a_{-1}^3 - a_{-1} b_{-1}^2 = 0\right), \\
\frac{1}{A}\left(-a_0 - k^2 a_0 - w a_0 + 3 a_0 a_1^2 - 2 a_1 b_0 + k^2 a_1 b_0 + w a_1 b_0\right) = 0, \\
\frac{1}{A}\left(3 a_{-1}^2 a_0 - a_0 b_{-1}^2 - k^2 a_0 b_{-1}^2 + w a_0 b_{-1}^2 - 2 a_{-1} b_{-1} b_0 + k^2 a_{-1} b_{-1} b_0 - w a_{-1} b_{-1} b_0\right) = 0, \\
\frac{1}{A}\left(\begin{array}{c} 3 a_{-1} a_0^2 + 3 a_{-1}^2 a_1 - 2 a_{-1} b_{-1} + 4 k^2 a_{-1} b_{-1} - 2 w a_{-1} b_{-1} - a_1 b_{-1}^2 - 4 k^2 a_1 b_{-1}^2 + \\ +2 w a_1 b_{-1}^2 - 2 a_0 b_{-1} b_0 + k^2 a_0 b_{-1} b_0 + w a_0 b_{-1} b_0 - a_{-1} b_0^2 - k^2 a_{-1} b_0^2 - w a_{-1} b_0^2 = 0 \end{array}\right), \\
\frac{1}{A}\left(\begin{array}{c} -a_{-1} - 4 k^2 a_{-1} - 2 w a_{-1} + 3 a_0^2 a_1 + 3 a_{-1} a_1^2 - 2 a_1 b_{-1} + 4 k^2 a_1 b_{-1} + 2 w a_1 b_{-1} - \\ -2 a_0 b_0 + k^2 a_0 b_0 - w a_0 b_0 - a_1 b_0^2 - k^2 a_1 b_0^2 + w a_1 b_0^2 = 0 \end{array}\right), \\
\frac{1}{A}\left(\begin{array}{c} a_0^3 + 6 a_{-1} a_0 a_1 - 2 a_0 b_{-1} + 6 k^2 a_0 b_{-1} - 2 a_{-1} b_0 - 3 k^2 a_{-1} b_0 - 3 w a_{-1} b_0 - \\ -2 a_1 b_{-1} b_0 - 3 k^2 a_1 b_{-1} b_0 + 3 w a_1 b_{-1} b_0 - a_0 b_0^2 = 0 \end{array}\right),
\end{gathered} \tag{8}$$

where $A = \left(e^{\xi} + b_0 + b_{-1} e^{-\xi}\right)^3$. It is solved algebraic equations system with the aid of Mathematica and it is obtained values $a_0, a_1, a_{-1}, b_0, b_{-1}, w, k$. If these values substitute into Eq. (7), we obtain traveling wave solutions of Eq.(5) as following

Family 1

$$k = -\frac{1}{\sqrt{2}},\ w = -\frac{3}{2},\ a_{-1} = -b_{-1},\ a_0 = \frac{1}{2}\left(-b_0 \pm \sqrt{-4 b_{-1} + b_0^2}\right),\ a_1 = 0,\ b_{-1} \neq 0,$$

$$u_1(x,t) = \frac{\frac{1}{2}\left(-b_0 \pm \sqrt{-4b_{-1}+b_0^2}\right) - b_{-1}\exp\left(\frac{1}{\sqrt{2}}x+\frac{3}{2}t\right)}{\exp\left(-\frac{1}{\sqrt{2}}x-\frac{3}{2}t\right)+b_0+b_{-1}\exp\left(\frac{1}{\sqrt{2}}x+\frac{3}{2}t\right)} \tag{9}$$

$$= \frac{-2\left(\cosh\left(\frac{x}{\sqrt{2}}+\frac{3t}{2}\right)+\sinh\left(\frac{x}{\sqrt{2}}+\frac{3t}{2}\right)\right)b_{-1}-b_0\pm\sqrt{-4b_{-1}+b_0^2}}{2\left(\cosh\left(\frac{x}{\sqrt{2}}+\frac{3t}{2}\right)-\sinh\left(\frac{x}{\sqrt{2}}+\frac{3t}{2}\right)+\left(\cosh\left(\frac{x}{\sqrt{2}}+\frac{3t}{2}\right)+\sinh\left(\frac{x}{\sqrt{2}}+\frac{3t}{2}\right)\right)b_{-1}+b_0\right)}$$

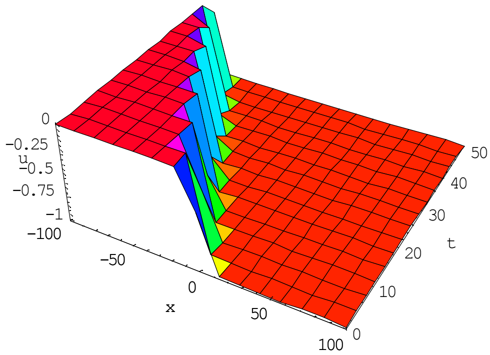

Figure 1. *Traveling wave solution of equation* (5) *for solution* (9), $b_0 = -1, b_{-1} = -1$

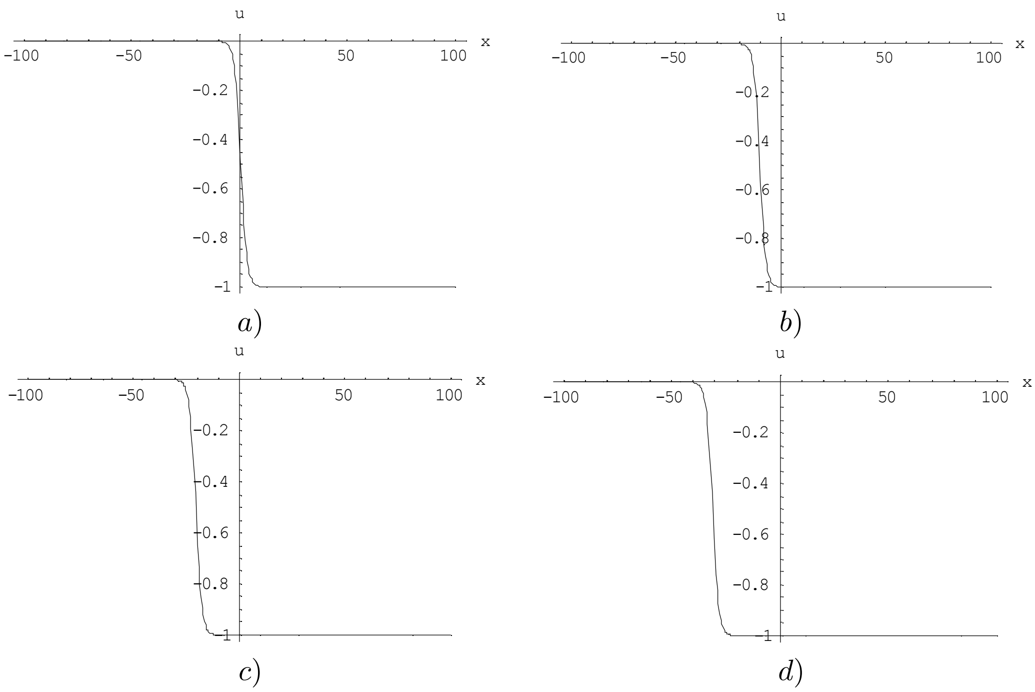

Figure 2. *Traveling wave solution of equation* (5) *for solution* (9), *a*) $t = 0$, *b*) $t = 5$, *c*) $t = 10$, *d*) $t = 15$ $(b_0 = -1, b_{-1} = -1)$

Family 2

$$k = -\frac{1}{\sqrt{2}},\ w = -\frac{3}{2},\ a_{-1} = b_{-1},\ a_0 = \frac{1}{2}\left(b_0 \pm \sqrt{-4b_{-1} + b_0^2}\right),\ a_1 = 0,\ b_{-1} \neq 0,$$

$$u_2(x,t) = \frac{\frac{1}{2}\left(b_0 \pm \sqrt{-4b_{-1} + b_0^2}\right) + b_{-1}\exp\left(\frac{1}{\sqrt{2}}x + \frac{3}{2}t\right)}{\exp\left(-\frac{1}{\sqrt{2}}x - \frac{3}{2}t\right) + b_0 + b_{-1}\exp\left(\frac{1}{\sqrt{2}}x + \frac{3}{2}t\right)} \tag{10}$$

Family 3

$$k = -\frac{1}{\sqrt{2}},\ w = \frac{3}{2},\ a_{-1} = 0,\ a_0 = \frac{1}{2}\left(\pm b_0 - \sqrt{-4b_{-1} + b_0^2}\right),\ a_1 = \pm 1,\ b_{-1} \neq 0,$$

$$u_3(x,t) = \frac{\pm\exp\left(-\frac{1}{\sqrt{2}}x + \frac{3}{2}t\right) + \frac{1}{2}\left(\pm b_0 - \sqrt{-4b_{-1} + b_0^2}\right)}{\exp\left(-\frac{1}{\sqrt{2}}x + \frac{3}{2}t\right) + b_0 + b_{-1}\exp\left(\frac{1}{\sqrt{2}}x - \frac{3}{2}t\right)}. \tag{11}$$

Family 4

$$k = -\frac{1}{\sqrt{2}},\ w = \frac{3}{2},\ a_{-1} = 0,\ a_0 = \frac{1}{2}\left(\pm b_0 + \sqrt{-4b_{-1} + b_0^2}\right),\ a_1 = \pm 1,\ b_{-1} \neq 0,$$

$$u_4(x,t) = \frac{\pm\exp\left(-\frac{1}{\sqrt{2}}x + \frac{3}{2}t\right) + \frac{1}{2}\left(\pm b_0 + \sqrt{-4b_{-1} + b_0^2}\right)}{\exp\left(-\frac{1}{\sqrt{2}}x + \frac{3}{2}t\right) + b_0 + b_{-1}\exp\left(\frac{1}{\sqrt{2}}x - \frac{3}{2}t\right)}. \tag{12}$$

Family 5

$$k = \frac{1}{2\sqrt{2}},\ w = \frac{3}{4},\ a_{-1} = 0,\ a_0 = 0,\ a_1 = \pm 1,\ b_0 = 0,\ b_{-1} \neq 0,$$

$$u_5(x,t) = \frac{\pm\exp\left(\frac{1}{2\sqrt{2}}x + \frac{3}{4}t\right)}{\exp\left(\frac{1}{2\sqrt{2}}x + \frac{3}{4}t\right) + b_{-1}\exp\left(-\frac{1}{2\sqrt{2}}x - \frac{3}{4}t\right)}. \tag{13}$$

Family 6

$$k = \frac{1}{\sqrt{2}},\ w = -\frac{3}{2},\ a_{-1} = -b_{-1},\ a_0 = \frac{1}{2}\left(-b_0 \pm \sqrt{-4b_{-1} + b_0^2}\right),\ a_1 = 0,\ b_{-1} \neq 0,$$

$$u_6(x,t) = \frac{\frac{1}{2}\left(-b_0 \pm \sqrt{-4b_{-1} + b_0^2}\right) - b_{-1}\exp\left(-\frac{1}{\sqrt{2}}x + \frac{3}{2}t\right)}{\exp\left(\frac{1}{\sqrt{2}}x - \frac{3}{2}t\right) + b_0 + b_{-1}\exp\left(-\frac{1}{\sqrt{2}}x + \frac{3}{2}t\right)}. \tag{14}$$

Family 7

$$k=\frac{1}{\sqrt{2}},\ w=-\frac{3}{2},\ a_{-1}=b_{-1},\ a_0=\frac{1}{2}\left(b_0\pm\sqrt{-4b_{-1}+b_0^2}\right),\ a_1=0,\ b_{-1}\neq 0,$$

$$u_7\left(x,t\right)=\frac{\frac{1}{2}\left(b_0\pm\sqrt{-4b_{-1}+b_0^2}\right)+b_{-1}\exp\left(-\frac{1}{\sqrt{2}}x+\frac{3}{2}t\right)}{\exp\left(\frac{1}{\sqrt{2}}x-\frac{3}{2}t\right)+b_0+b_{-1}\exp\left(-\frac{1}{\sqrt{2}}x+\frac{3}{2}t\right)}. \tag{15}$$

Family 8

$$k=\pm\frac{1}{\sqrt{2}},\ w=\frac{3}{2},\ a_{-1}=0,\ a_0=0,\ a_1=\pm 1,\ b_{-1}=0,\ b_0\neq 0,$$

$$u_8\left(x,t\right)=\frac{\pm\exp\left(\pm\frac{1}{\sqrt{2}}x+\frac{3}{2}t\right)}{\exp\left(\pm\frac{1}{\sqrt{2}}x+\frac{3}{2}t\right)+b_0}. \tag{16}$$

Family 9

$$k=\pm\frac{1}{\sqrt{2}},\ w=-\frac{3}{2},\ a_{-1}=0,\ a_0=\pm b_0,\ a_1=0,\ b_{-1}=0,\ b_0\neq 0,$$

$$u_9\left(x,t\right)=\frac{\pm b_0}{\exp\left(\pm\frac{1}{\sqrt{2}}x-\frac{3}{2}t\right)+b_0}. \tag{17}$$

Family 10

$$k=\pm\frac{1}{2\sqrt{2}},\ w=-\frac{3}{4},\ a_{-1}=\pm b_{-1},\ a_0=0,\ a_1=0,\ b_0=0,\ b_{-1}\neq 0,$$

$$u_{10}\left(x,t\right)=\frac{\pm b_{-1}\exp\left(\mp\frac{1}{2\sqrt{2}}x+\frac{3}{4}t\right)}{\exp\left(\pm\frac{1}{2\sqrt{2}}x-\frac{3}{4}t\right)+b_{-1}\exp\left(\mp\frac{1}{2\sqrt{2}}x+\frac{3}{4}t\right)}. \tag{18}$$

Remark 1. Taşcan and Bekir obtained some solutions for Cahn Allen equation by using the first integral method [18]. When our results compare to their results, it is seem that our solution (17) is same with their solutions u_1 and u_3 in (3.20) . (in our study, when $b_0 = 1$ and in their study, when $c_0 = 0$). Moreover, We have different solutions from their results by using Exp-function method.

Example 2. The clannish random walker's parabolic equation is derived for the motion of the two interacting populations which tend to be clannish, which they wish to live near those of their own kind. The equation is written as following

$$u_t-u_{xx}+\alpha\left(u^2\right)_x-\alpha u_x=0, \tag{19}$$

where α is aconstant. To investigate the traveling wave solution of the Eq. (19), we use the transformation $u\left(x,t\right)=u\left(\xi\right),\ \xi=kx+wt$. Then Eq. (19) becomes

$$wu^{'}-k^2u^{''}+2\alpha kuu^{'}-\alpha ku^{'}=0, \tag{20}$$

and integrating (20) yields, we yield following equation

$$wu - k^2 u' + \alpha k u^2 - \alpha k u = 0, \tag{21}$$

where integration constant is taken as zero. When balancing u^2 with u'

$$\frac{c_1 \exp\left[(p+c)\xi\right] + ...}{c_2 \exp\left[2p\xi\right] + ...} = \frac{c_3 \exp\left[2c\xi\right] + ...}{c_4 \exp\left[2p\xi\right] + ...}$$

then gives $p = c$. Similarly, to determine values of d and q when balancing u^2 with u'

$$\frac{...d_1 \exp\left[-(q+d)\xi\right]}{...d_2 \exp\left[-2q\xi\right]} = \frac{...d_3 \exp\left[-2d\xi\right]}{...d_4 \exp\left[-2q\xi\right]},$$

then gives $q = d$.

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (4) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(\xi) + b_0 + b_{-1} \exp(-\xi)}, \tag{22}$$

substituting Eq. (22) into Eq. (21) yields a set of algebraic equations for $a_0, a_1, a_{-1}, b_0, b_{-1}, \alpha, w$, k. Algebraic equations system can be expressed as following

$$\begin{gathered}
\frac{1}{B}\left(wa_1 - k\alpha a_1 + k\alpha a_1^2 = 0\right), \\
\frac{1}{B}\left(k\alpha a_{-1}^2 + wa_{-1}b_{-1} - k\alpha a_{-1}b_{-1} = 0\right), \\
\frac{1}{B}\left(k^2 a_0 + wa_0 - k\alpha a_0 + 2k\alpha a_0 a_1 - k^2 a_1 b_0 + wa_1 b_0 - k\alpha a_1 b_0 = 0\right), \\
\frac{1}{B}\left(2k\alpha a_{-1}a_0 - k^2 a_0 b_{-1} + wa_0 b_{-1} - k\alpha a_0 b_{-1} + k^2 a_{-1} b_0 + wa_{-1} b_0 - k\alpha a_{-1} b_0 = 0\right), \\
\frac{1}{B}\left(\begin{array}{c} 2k^2 a_{-1} + wa_{-1} - k\alpha a_{-1} + k\alpha a_0^2 + 2k\alpha a_{-1} a_1 - 2k^2 a_1 b_{-1} + wa_1 b_{-1} - k\alpha a_1 b_{-1} + \\ + wa_0 b_0 - k\alpha a_0 b_0 = 0 \end{array}\right),
\end{gathered} \tag{23}$$

where $B = \left(e^{\xi} + b_0 + b_{-1}e^{-\xi}\right)^2$. It is solved algebraic equations system with the aid of Mathematica and it is obtained values $a_0, a_1, a_{-1},\ b_0, b_{-1}, \alpha, w, k$. If these values substitute into Eq. (22), we write traveling wave solutions of Eq. (19) as following

Solution 1

$$w = -k(k-\alpha),\ a_{-1} = 0,\ a_1 = 0,\ b_{-1} = 0,\ b_0 = \frac{\alpha a_0}{k},\ k \neq 0,$$

$$\begin{aligned}
u_1(x,t) &= \frac{a_0}{\exp(kx - k(k-\alpha)t) + \frac{\alpha a_0}{k}} \\
&= \frac{k a_0}{k\left(\cosh\left(k(x+\alpha t - kt)\right) + \sinh\left(k(x+\alpha t - kt)\right)\right) + \alpha a_0}
\end{aligned} \tag{24}$$

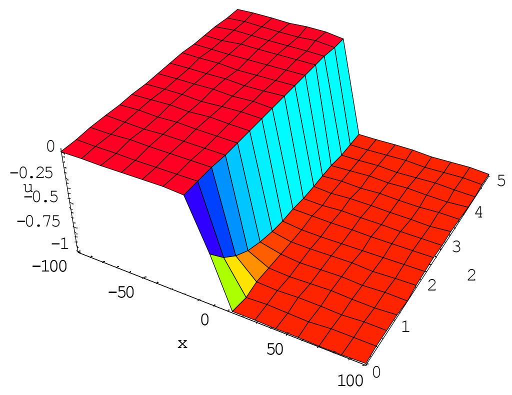

Figure 3. Traveling wave solution of equation (19) *for solution* (24), $a_0 = -1, \alpha = 1, k = -1$

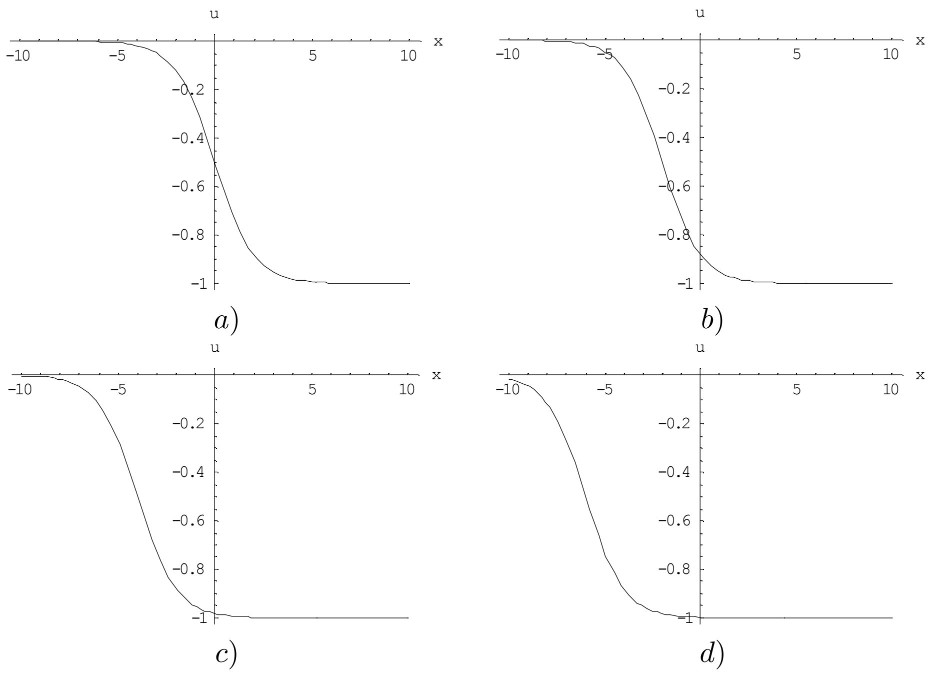

Figure 4. Traveling wave solution of equation (19) *for solution* (24), *a*) $t = 0$, *b*) $t = 1$, *c*) $t = 2$, *d*) $t = 3$, $(a_0 = -1, \alpha = 1, k = -1)$

Solution 2

$$w = -k(k-\alpha), \; a_{-1} = a_0\left(-\frac{\alpha a_0}{k} + b_0\right), \; a_1 = 0, \; b_{-1} = \frac{\alpha a_0(-\alpha a_0 + k b_0)}{k^2}, \; a_0 \neq 0, \; k \neq 0,$$

$$u_2(x,t) = \frac{a_0 + a_0\left(-\frac{\alpha a_0}{k} + b_0\right)\exp(-kx + k(k-\alpha)t)}{\exp(kx - k(k-\alpha)t) + b_0 + \frac{\alpha a_0(-\alpha a_0 + k b_0)}{k^2}\exp(-kx + k(k-\alpha)t)}. \tag{25}$$

Solution 3

$$w = k\left(-2k+\alpha\right),\ a_0 = 0,\ a_1 = 0,\ b_{-1} = \frac{\alpha a_{-1}}{2k},\ b_0 = 0,\ k \neq 0,$$

$$u_3\left(x,t\right) = \frac{a_{-1}\exp\left(-kx-k\left(-2k+\alpha\right)t\right)}{\exp\left(kx+k\left(-2k+\alpha\right)t\right)+\frac{\alpha a_{-1}}{2k}\exp\left(-kx-k\left(-2k+\alpha\right)t\right)}. \tag{26}$$

Solution 4

$$w = k\left(k+\alpha\right),\ a_{-1} = 0,\ a_0 = 0,\ a_1 = -\frac{k}{\alpha},\ b_{-1} = 0,\ k \neq 0,\ \alpha \neq 0,$$

$$u_4\left(x,t\right) = \frac{-\frac{k}{\alpha}\exp\left(kx+k\left(k+\alpha\right)t\right)}{\exp\left(kx+k\left(k+\alpha\right)t\right)+b_0}. \tag{27}$$

Solution 5

$$w = k\left(k+\alpha\right),\ a_{-1} = 0,\ a_1 = -\frac{k}{\alpha},\ b_{-1} = -\frac{\alpha a_0\left(\alpha a_0+kb_0\right)}{k^2},\ k \neq 0,\ \alpha \neq 0,\ a_0 \neq 0,$$

$$u_5\left(x,t\right) = \frac{-\frac{k}{\alpha}\exp\left(kx+k\left(k+\alpha\right)t\right)+a_0}{\exp\left(kx+k\left(k+\alpha\right)t\right)+b_0-\frac{\alpha a_0\left(\alpha a_0+kb_0\right)}{k^2}\exp\left(-kx-k\left(k+\alpha\right)t\right)}. \tag{28}$$

Solution 6

$$w = k\left(2k+\alpha\right),\ a_{-1} = 0,\ a_0 = 0,\ a_1 = -\frac{2k}{\alpha},\ b_0 = 0,\ k \neq 0,\ \alpha \neq 0,$$

$$u_6\left(x,t\right) = \frac{-\frac{2k}{\alpha}\exp\left(kx+k\left(2k+\alpha\right)t\right)}{\exp\left(kx+k\left(2k+\alpha\right)t\right)+b_{-1}\exp\left(-kx-k\left(2k+\alpha\right)t\right)}. \tag{29}$$

Remark 2. Uğurlu and Kaya obtained periodic solutions and soliton solutions for the clannish random walker's parabolic equation by using improved tanh function method [19]. The solutions of the clannish random walker's parabolic equation obtained in this study are different from their solutions.

Example 3. Let's consider Fitzhugh–Nagumo equation

$$u_t - u_{xx} - u\left(u-\alpha\right)\left(1-u\right) = 0, \tag{30}$$

where α is an arbitrary constant. Eq. (30) is an important nonlinear reaction–diffusion equation and applied to model the transmission of nerve impulses, also used in biology and the area of population genetics, in circuit theory. To investigate the traveling wave solution of the equation (30), we use the transformation $\xi = kx + wt$. Then Eq. (30) becomes

$$wu^{'} - k^2u^{''} - \left(1+\alpha\right)u^2 + u^3 + \alpha u = 0, \tag{31}$$

and when balancing u^3 with $u^{''}$

$$\frac{c_1\exp\left[\left(3p+c\right)\xi\right]+...}{c_2\exp\left[4p\xi\right]+...} = \frac{c_3\exp\left[c\xi\right]+...}{c_4\exp\left[p\xi\right]+...},$$

then gives $p = c$. Similarly, to determine values of d and q when balancing u^3 with u''

$$\frac{...d_1 \exp\left[-\left(3q+d\right)\xi\right]}{...d_2 \exp\left[-4q\xi\right]} = \frac{...d_3 \exp\left[-d\xi\right]}{...d_4 \exp\left[-q\xi\right]},$$

then gives $q = d$.

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (4) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(\xi) + b_0 + b_{-1} \exp(-\xi)}, \tag{32}$$

substituting Eq. (32) into Eq. (31) yields a set of algebraic equations for $a_0, a_1, a_{-1}, b_0, b_{-1}, w, k$. Algebraic equations system can be written as following

$$\frac{1}{C}\left(\alpha a_1 - a_1^2 - \alpha a_1^2 + a_1^3 = 0\right),$$

$$\frac{1}{C}\left(a_{-1}^3 - a_{-1}^2 b_{-1} - \alpha a_{-1}^2 b_{-1} + \alpha a_{-1} b_{-1}^2 = 0\right),$$

$$\frac{1}{C}\left(\begin{array}{c} -k^2 a_0 - w a_0 + \alpha a_0 - 2a_0 a_1 - 2\alpha a_0 a_1 + 3a_0 a_1^2 + k^2 a_1 b_0 + w a_1 b_0 + 2\alpha a_1 b_0 - a_1^2 b_0 - \\ -\alpha a_1^2 b_0 = 0 \end{array}\right),$$

$$\frac{1}{C}\left(\begin{array}{c} 3a_{-1}^2 a_0 - 2a_{-1} a_0 b_{-1} - 2\alpha a_{-1} a_0 b_{-1} - k^2 a_0 b_{-1}^2 + w a_0 b_{-1}^2 + \alpha a_0 b_{-1}^2 - a_{-1}^2 b_0 - \alpha a_{-1}^2 b_0 + \\ + k^2 a_{-1} b_{-1} b_0 - w a_{-1} b_{-1} b_0 + 2\alpha a_{-1} b_{-1} b_0 = 0 \end{array}\right),$$

$$\frac{1}{C}\left(\begin{array}{c} -a_{-1}^2 - \alpha a_{-1}^2 + 3a_{-1} a_0^2 + 3a_{-1}^2 a_1 + 4k^2 a_{-1} b_{-1} - 2w a_{-1} b_{-1} + 2\alpha a_{-1} b_{-1} - a_0^2 b_{-1} - \\ -\alpha a_0^2 b_{-1} - 2a_{-1} a_1 b_{-1} - 2\alpha a_{-1} a_1 b_{-1} + 4k^2 a_1 b_{-1}^2 + 2w a_1 b_{-1}^2 + \alpha a_1 b_{-1}^2 - 2a_{-1} a_0 b_0 - \\ -2\alpha a_{-1} a_0 b_0 + k^2 a_0 b_{-1} b_0 + w a_0 b_{-1} b_0 + 2\alpha a_0 b_{-1} b_0 - k^2 a_{-1} b_0^2 - w a_{-1} b_0^2 + \alpha a_{-1} b_0^2 = 0 \end{array}\right),$$

$$\frac{1}{C}\left(\begin{array}{c} -4k^2 a_{-1} - 2w a_{-1} + \alpha a_{-1} - a_0^2 - \alpha a_0^2 - 2a_{-1} a_1 - 2\alpha a_{-1} a_1 + 3a_0^2 a_1 + 3a_{-1} a_1^2 + \\ +4k^2 a_1 b_{-1} + 2w a_1 b_{-1} + 2\alpha a_1 b_{-1} - a_1^2 b_{-1} - \alpha a_1^2 b_{-1} + k^2 a_0 b_0 - w a_0 b_0 + 2\alpha a_0 b_0 - \\ -2a_0 a_1 b_0 - 2\alpha a_0 a_1 b_0 - k^2 a_1 b_0^2 + w a_1 b_0^2 + \alpha a_1 b_0^2 = 0 \end{array}\right), \tag{33}$$

$$\frac{1}{C}\left(\begin{array}{c} -2a_{-1} a_0 - 2\alpha a_{-1} a_0 + a_0^3 + 6a_{-1} a_0 a_1 + 6k^2 a_0 b_{-1} + 2\alpha a_0 b_{-1} - 2a_0 a_1 b_{-1} - 2\alpha a_0 a_1 b_{-1} - \\ -3k^2 a_{-1} b_0 - 3w a_{-1} b_0 + 2\alpha a_{-1} b_0 - a_0^2 b_0 - \alpha a_0^2 b_0 - 2a_{-1} a_1 b_0 - 2\alpha a_{-1} a_1 b_0 - \\ -3k^2 a_1 b_{-1} b_0 + 3w a_1 b_{-1} b_0 + 2\alpha a_1 b_{-1} b_0 + \alpha a_0 b_0^2 = 0 \end{array}\right)$$

where $C = \left(e^{\xi} + b_0 + b_{-1} e^{-\xi}\right)^3$. It is solved algebraic equations system with the aid of Mathematica and it is obtained values $a_0, a_1, a_{-1}, b_0, b_{-1}, \alpha, w, k$. If these values substitute into Eq. (32), we write traveling wave solutions of Eq. (30) as following

Solution 1

$$\begin{aligned} k &= -\frac{1}{\sqrt{2}},\ w = \frac{1}{2}\left(-1+2\alpha\right),\ a_{-1} = b_{-1},\ a_0 = \frac{1}{2}\left(b_0 \pm \sqrt{-4b_{-1} + b_0^2}\right),\ a_1 = 0, \\ \alpha &\neq 0,\ b_{-1} \neq 0, \end{aligned}$$

$$u_1(x,t) = \frac{\frac{1}{2}\left(b_0 \pm \sqrt{-4b_{-1} + b_0^2}\right) + b_{-1} \exp\left(\frac{1}{\sqrt{2}}x - \frac{1}{2}\left(-1+2\alpha\right)t\right)}{\exp\left(-\frac{1}{\sqrt{2}}x + \frac{1}{2}\left(-1+2\alpha\right)t\right) + b_0 + b_{-1} \exp\left(\frac{1}{\sqrt{2}}x - \frac{1}{2}\left(-1+2\alpha\right)t\right)}. \tag{34}$$

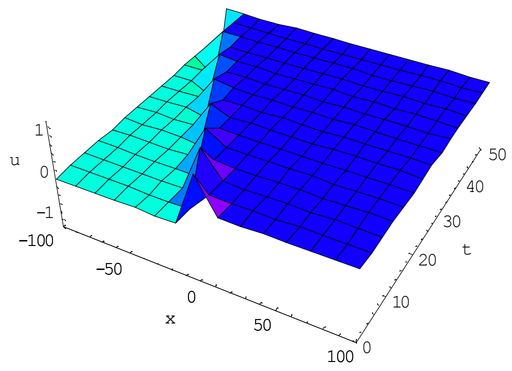

Figure 5. *Traveling wave solution of equation* (30) *for solution* (34), $\alpha = -1, b_0 = -1, b_{-1} = -1$

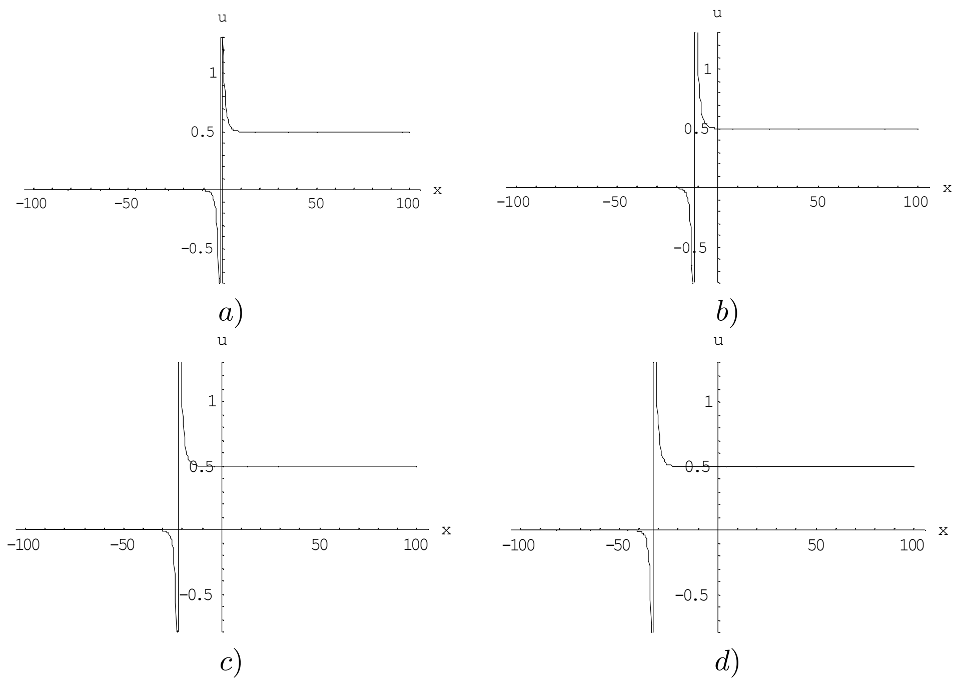

Figure 6. *Traveling wave solution of equation* (30) *for solution* (34), *a*) $t = 0$, *b*) $t = 5$, *c*) $t = 10$, *d*) $t = 15$, $(\alpha = -1, b_0 = -1, b_{-1} = -1)$

Solution 2

$k = -\frac{\alpha}{\sqrt{2}}$, $w = \frac{1}{2}\left(2a - \alpha^2\right)$, $a_{-1} = \alpha b_{-1}$, $a_0 = \frac{1}{2}\left(\alpha b_0 \pm \sqrt{-4\alpha^2 b_{-1} + \alpha^2 b_0^2}\right)$, $a_1 = 0$, $\alpha \neq 0$, $b_{-1} \neq 0$,

$$u_2(x,t) = \frac{\frac{1}{2}\left(\alpha b_0 \pm \sqrt{-4\alpha^2 b_{-1} + \alpha^2 b_0^2}\right) + \alpha b_{-1} \exp\left(\frac{\alpha}{\sqrt{2}}x - \frac{1}{2}\left(2a - \alpha^2\right)t\right)}{\exp\left(-\frac{\alpha}{\sqrt{2}}x + \frac{1}{2}\left(2a - \alpha^2\right)t\right) + b_0 + b_{-1}\exp\left(\frac{\alpha}{\sqrt{2}}x - \frac{1}{2}\left(2a - \alpha^2\right)t\right)}. \tag{35}$$

Solution 3

$$k=-\frac{1}{\sqrt{2}},\ w=\frac{1}{2}\left(1-2a\right),\ a_{-1}=0,\ a_0=0,\ a_1=1,\ b_{-1}=0,\ \ -2+\alpha\neq 0,\ b_0\neq 0,$$

$$u_3\left(x,t\right)=\frac{\exp\left(-\frac{1}{\sqrt{2}}x+\frac{1}{2}\left(1-2a\right)t\right)}{\exp\left(-\frac{1}{\sqrt{2}}x+\frac{1}{2}\left(1-2a\right)t\right)+b_0}. \tag{36}$$

Remark 3. Li and Guo obtained some exact solutions for the Fitzhugh–Nagumo equation by using the first integral method [35]. Our solutions are different solutions from Li and Guo's solutions.

3. Conclusions

In this paper, we apply the exp-function method with aid of Mathematica. We obtain some solutions of Cahn Allen equation, clannish random walker's parabolic equation and the Fitzhugh–Nagumo equation by using the exp-function method. The method can be used to many other nonlinear equations or coupled ones. In addition, this method is also computerizable which allows us to perform complicated and tedious algebraic calculation on a computer.

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