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Theory of algebraic functions on the Riemann Sphere

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Abstract

The Riemann sphere (S) is defined as the complex plane together with the point at infinity. Algebraic functions are defined as subsets of $S \times S$ such that a bivariate polynomial on S is zero. It is shown that the set of algebraic functions is closed under addition, multiplication, composition, inversion, union, and differentiation. Singular points are defined as points where the function is not locally 1 to 1. A general method is given for calculating the singular point parameters i.e. a topological winding number ratio, a strength coefficient, and location in $S \times S$, and it is argued that the topology of an algebraic function depends only on the winding number ratios of all its singular points. After showing how most of these singular point parameters can be calculated under the closure operations and that a function without singular points is linear, it follows that the set of all quadruples of singular point parameters uniquely determine an algebraic function.

Mathematics Subject Classification: 30-xx

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1 Introduction

Algebraic functions can be loosely described as functions that can be expressed in terms of polynomials and inverses of polynomial functions. There are difficulties with manipulating them for example there is more than one way to represent them resulting from the use of algebraic identities, and because solutions of polynomial equations of degree 5 and more cannot in general be represented in explicit form there is in general no explicit formula for them. Just as analysis in the complex number plane (the algebraic completion of the real number line) provides many methods and results not available in real analysis, for example the use of contour integration in evaluation of integrals and solutions of some differential equations, so also complex analysis can be applied to the characterisation of algebraic functions, which to my knowledge has not been done systematically.

An important aspect of complex analysis is the addition of the point at infinity to the complex plane giving the Riemann sphere. Riemann pointed out that this gives a structure which is topologically equivalent to the surface of a sphere known as the Riemann Sphere or the extended complex plane. The relevant 1 to 1 correspondence is for example the projection from the sphere to the plane passing through its "equator" defined by a straight line passing through the "north pole" of the sphere and the pair of associated points, one on the sphere and the other on the plane. The "north pole" is itself mapped to the point ∞ and the "south pole" is mapped to 0. This is also motivated by the fact that the function $z \to 1/z$ is now a 1 to 1 mapping with $0 \to \infty$ and $\infty \to 0$.

The behaviour of ∞ in algebra is defined by $\infty + x = \infty$ where $x \neq \infty$, $\infty.x = \infty$ provided $x \neq 0$, $1/\infty = 0$, and if x is a real number then $\infty \uparrow x = 0$ if $x < 0$, and is ∞ if $x > 0$, and is 1 if $x = 0$, where $\uparrow$ denotes the exponentiation operator. In particular note that there is no distinction between $+\infty$ and $-\infty$ therefore it is not correct that $\infty + \infty = \infty$. These are different directions of approach to the same point ∞. The point ∞ is exceptional and it will always be treated separately. The values of all variables will be assumed to be not ∞ unless explicitly stated otherwise.

In this paper algebraic functions will be considered as mappings from the Riemann sphere to itself, P_i will always denote polynomials, and z, w and t will denote variables in the Riemann sphere. Algebraic functions will usually be multivalued so should perhaps be referred to as relations, but I will use "function" because of the familiarity of this term in applications. Because these functions are multi-valued, a singular point ideally should be specified by the pair (z, w), because one z value could correspond to several points (z, w) having different properties. The simpler notation z will be used if the pair (z, w) to which z refers is clear from the context.

It it well known that rational functions can be characterised by their behaviour at poles and their behaviour at large values of the independent variable (e.g. from using polynomial division and the partial fraction expansion). In this paper I extend this idea and introduce singular points, special cases of which are branch points and poles, and provide a method to find and characterise them and provide some convincing arguments to show that the topological structure of the "graph" of an algebraic function is completely characterised by the set of 'winding number ratios' (denoted by r), one at each of the singular points. To these parameters can be added corresponding 'strength coefficients' (denoted by A) giving the magnitude of each singularity. After defining the set of algebraic functions and establishing some of its closure properties, a general method for calculating A and r at any singular point of an algebraic function is described that depends on knowing the leading multiple derivatives of the function implicitly determining the dominant behaviour of the algebraic function at this point. I also give rules for calculating a subset of these parameters from the corresponding parameters involved in the closure relations, that in most cases is the whole set. I then showed that an algebraic function without singular points has to be a linear function. This is where the Riemann sphere construction is important because otherwise for example all polynomials would be included. This result combined with the rules for calculating singular points finally shows the uniqueness of an algebraic function with a given set of singular point parameters.

2 Some simple examples

It is well known that rational functions $R(z) = P_1(z)/P_2(z)$ are characterised by their behaviour at singular points and their behaviour at infinity (which is determined by the additive polynomial term). This result follows from their partial fraction representation. I consider here the case of no repeated roots in P_2. If degree$(P_1) \geq$ degree(P_2), use polynomial long division to find polynomials P_3 and P_4 such that $P_1/P_2 = P_3 + P_4/P_2$ where degree$(P_4) <$ degree(P_2). Now consider the fractional part $w = P_4(z)/P_2(z)$, where $P_2(z) = C\prod_{i=1}^{k}(z - z_i)$ and C is a non-zero constant and the factors of P_2 each have multiplicity 1, then it follows that

$$\frac{P_4(z)}{P_2(z)} = \frac{P_4(z)}{C\prod_{i=1}^{k}(z - z_i)} = \frac{1}{C}\sum_{j=1}^{k}\frac{A_j}{(z - z_j)}. \tag{1}$$

To establish this, clear the fractions to get $P_4(z) = \sum_{j=1}^{k} A_j \prod_{i=1,i\neq j}^{k}(z - z_i)$. Put $z = z_l$ where z_l is one of the z_i for $1 \leq i \leq k$ then $P_4(z_l) = \sum_{j=1}^{k} A_j \prod_{i=1,i\neq j}^{k}(z_l - z_i)$ for some coefficients A_j. The product is zero if $l = i$ for some i such that $i \neq j$ and $1 \leq i \leq k$, so only the $j = l$ term is non-zero therefore

$$A_l = \frac{P_4(z_l)}{\prod_{i=1,i\neq l}^{k}(z_l - z_i)}. \tag{2}$$

Substituting back gives

$$P_4(z) = \sum_{j=1}^{k}\left[\frac{P_4(z_j)\prod_{i=1,i\neq j}^{k}(z - z_i)}{\prod_{i=1,i\neq j}^{k}(z_j - z_i)}\right]. \tag{3}$$

This holds for $z = z_l$ for any l such that $1 \leq l \leq k$ because if $l \neq j$, the product in the numerator is zero, leaving just 1 term, an identity. So (3) holds at $k = \text{degree}(P_2) \geq \text{degree}(P_4) + 1$ distinct points everywhere because both sides of (3) are polynomials in z with degrees $\leq k - 1$. This establishes the right hand equality of (1) and the formula (2) for the coefficients. The original rational function considered, $R(z)$, has singular points (poles) at $z = z_j$ for $1 \leq j \leq k$ which are the zeros of the denominator, and the polynomial term is defined by the behaviour of $R(z)$ at infinity i.e. $P_3(z)$.

In an attempt to extend these ideas to algebraic functions, note that algebraic functions, regarded as functions mapping the extended complex plane to itself, in general have topological properties because they can have branch points as well as poles. Branch points are points about which algebraic functions become multivalued. Branch points and poles are examples of singular points. The topology of an algebraic function i.e. the topology of its graph G (the set of pairs of points (z, w) such that the algebraic function relates z to w and is a multi-sheeted 2-dimensional surface in 4 dimensions, 2 for each of z and w), is related to the behaviours of G in the neighbourhoods of its singular points. It is useful to be able to codify such behaviours in terms of a standard set of parameters, which is one objective of this paper. The following example may make this clear.

Consider the function

$$w = \sqrt{(z^2 - 1)} \tag{4}$$

of the complex variable z which can also be written as $\sqrt{(z+1)}.\sqrt{(z-1)}$. Put $z = 1 + \epsilon.e^{i\theta}$ where $0 \leq \theta \leq 2\pi$, then $\sqrt{(z-1)} = \pm\epsilon^{1/2}.e^{i\theta/2}$ where ϵ is an arbitrary small positive real number. Therefore one of the factors of w changes sign when z goes round a small circle about 1 once, and hence w itself has the same behaviour because the other factor returns to its original value (and is almost constant throughout). Thus it takes two such circuits about 1 in the z plane to correspond to one complete circuit in the w plane (in the same direction). A similar argument applies at $z = -1$ for the first factor of w . This situation is described by saying that $w(z)$ has branch points at $z = 1$ and $z = -1$ each with a turns ratio of 1/2 i.e. 2:1. For this example, it is clear that -1 and 1 are the only finite singular points in the z plane because each of the 2 factors has just 1 finite singular point. Both these points have $w = 0$. In general, a singular point (z, w) is a point in G that is not a regular point in G, and a regular point in G is a point in a neighbourhood of which a small circle in z corresponds to a small circle in w i.e. the turns ratio is 1:1. In other words it could be said that in a neighbourhood of a regular point, the function is locally one-to-one. Equation (4) is equivalent to the inverse equation $z = \sqrt{(w^2 + 1)}$. The similarity of this with equation (4) suggests that this gives a very similar analysis, which indeed shows that $w = i$ and $w = -i$ are the only finite singular points in the w plane and each have

$z = 0$. Hence a table can constructed giving the finite singular points and turns ratios of this function as follows. The variable A will be explained later. This

A	z(turns in z)	w(turns in w)
$1/2$	$1(2)$	$0(1)$
$-1/2$	$-1(2)$	$0(1)$
$-i/2$	$0(1)$	$i(2)$
$-i/2$	$0(1)$	$-i(2)$

Table 1: Singular points and turns ratios of $w = \sqrt{(z^2 - 1)}$

analysis omits the behaviour at (∞, ∞), which is characterised by $w \approx \pm z$. In this example, note that the point $z = 0$ must be included as a singular point, although with regard to the independent variable z, this point is an optimum point and it was first found to be a singular point only after considering the inverse relation with w as the independent variable. Also one value of z always corresponds to 2 values of w and one value of w always corresponds to 2 values of z except when the argument of the squareroot is zero. These points coincide with the singular points.

3 Attempts to characterise the topology of algebraic function mappings

In this section I give some topological arguments showing that the topological properties of the multisheeted surface representing an algebraic function is completely characterised by the turns ratios at each singular point. I also extend the concept of a turns ratio to arbitrary closed paths. This description may be hard to follow if the reader is not familiar with topological concepts.

There is a concept of equivalence of paths in the z plane for an algebraic function: two paths from z_1 to z_2 (P_1 and P_2) are equivalent if and only if one can be continuously deformed to the other keeping the endpoints fixed, without the path ever crossing a singular point. The motivation for this is that if a singular point is crossed to get from P_1 to P_2, then by tracing P_1 followed by P_2 in reverse, it is possible that the arrival at z_1 will be on a different branch of the function from the starting point so it is reasonable to say that P_1 and P_2 are not equivalent. Avoiding crossing a singular point guarantees that this will not happen.

Any circuit (i.e. a closed path) that passes through an arbitrary fixed point z_0 is obviously equivalent to a concatenated sequence of circuits that are shrunk to hairpin loops that only surround one singular point each, and each circuit doubles back to the starting point z_0. Therefore to specify the equivalence class of (i.e. equivalence class containing) such a circuit needs only the sequence of singular points each with an exponent of 1 or -1 (corresponding to anticlockwise or clockwise traversal round the singular point respectively). Note that a circuit is allowed to cross itself at 1 or more points. The sequence is often called a *word* in the algebra (in this case generated by the basic circuits passing through z_0 going anticlockwise once round each singular point in the z plane). The effect of following such a closed circuit in the z plane is to possibly move to another sheet in the multi-sheeted surface G, $(w_1(z), \dots w_i(z))$, of the algebraic function i.e. there is a permutation on the sheets associated with every closed circuit in the z plane starting and ending at z_0. Therefore it is reasonable to suppose that the topology of G is characterised by the relation between the equivalence class E of a circuit (i.e. the *word*) and the mapping M associated with it amongst the different values of w corresponding to z_0. M is defined in the obvious way i.e. $M(w_i)$ is w_j obtained by starting at (z_0, w_i) and passing round the projection to G of a representative path in E with w varying continuously with z throughout, and finishing at (z_0, w_j). If there are n values of w, $\{w_1, w_2 \dots w_n\}$ associated with z_0, the equation $w_{P(i)} = M(w_i)$ defines permutation P on the index set $\{1, 2, \dots n\}$, so once a labelling of the different values of w corresponding to z_0 is fixed, the topology defines the mapping $E \to P$.

It is obvious that this result is independent of the fixed arbitrary point z_0 after defining a 1 to 1 correspondence between the values of w at any two non-singular points z_{01} and z_{02} used as the starting and ending points of circuits in the z plane. This can be done with an open path connecting them not passing through any singular points. Therefore the topology is uniquely determined by the permutations on the function values resulting from single loop paths round the singular points which are clearly determined by the winding number ratios i.e. the topological behaviour of G in the neighbourhoods of singular points.

I return again to the example given by equation (4). All the circuits in this example are described in the anticlockwise direction. The straight line segment from 0 to 1 is mapped to the two straight line segments from $\pm i$ to 0, so a circuit that passes once round the singular points of z i.e. 0 and 1 only, is equivalent to a pair of closely spaced straight parallel lines from 0 to 1 terminated by almost complete small circles around 0 and 1. If this circuit is started from 0 just prior to the small circle around 0, this is mapped to a pair of open paths in w. One starts at i, and passes round i twice and then straight to near 0, round 0 half a circle going to the opposite side of it, then straight to $-i$. The other path is its inversion through 0, and completes a closed circuit in the w plane. This shows that if the circuit that passes round 0 and 1 is described twice in the z plane, this will correspond to a circuit in the w plane that passes round i twice, round 0 once, and round $-i$ twice. This kind of argument can be clearly extended to other cases with different turns ratios at the singular points.

By a similar argument, combining equivalence classes of circuits together, it is obvious that the turns ratio associated with a large circle in the z plane (small circle in a neighbourhood of ∞ in the Riemann sphere) i.e. the singular behaviour at $z = \infty$ is derivable from the behaviour of the function at all finite points of the z plane.

Singular points arise where the number of values of the function changes, and only at such points. This is obvious and follows from the definition below in Section 5.1, but consider the following argument. At a typical (regular) point z_1, the function maps a small circle round z_1 to a set of small circles (each described once) centred on a set of points $w_1 \ldots w_n$ say (turns ratio = 1), where $w_1 \ldots w_n$ are the images of z_1. As z_1 changes, $w_1 \ldots w_n$ change, but the topology of the situation cannot change (because it is a discrete change) unless two of more of the $w_1 \ldots w_n$ coincide. When this happens it is possible to predict the turns ratios that will occur. For example two circles with $r = 1$ coming together will lead to a single circuit (a circle when it is large enough compared with the distance between the w_i) with $r = 1/2$ because the circuits will merge initially to a dumbbell shape with a narrow neck having both images of a point on the small circle in the z plane. This change is associated with the appearance of a singular point because now $r \neq 1$.

4 Definition and closure properties of the set of algebraic functions

I define algebraic functions here in a way that is very natural as is shown by the closure properties in the next section, and includes all the examples found in elementary algebra. An algebraic function relating the complex variables z and w is defined by the relation $P(z, w) = 0$ where P is a polynomial in both its arguments i.e.

$$\sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij} z^i . w^j = 0 \tag{5}$$

where n and m are non-negative integers. The surface G is the set of all such points (z, w) satisfying (5). Examples include single variable polynomials $w = P(z) \equiv \sum_{i=0}^{k} a_i z^i$, their inverses $z = P(w)$, and rational functions discussed above i.e. $w = P_1(z)/P_2(z)$ where P_1 and P_2 are single variable polynomials.

In order to establish closure properties of the set of algebraic functions, I will use an elimination theorem which can be roughly stated as follows:
From a set of polynomial equations (A) $P_i(z_1, z_2, \ldots z_n) = 0$, it is possible, after specifying an appropriate ordering for the variables and their power products,

to obtain another equivalent set of polynomial equations (B) involving the same variables such that all polynomial equations derivable from the set (B) (i.e. linear combinations of the set (B) with coefficients that are also polynomials in all the variables) that only involve the first i variables, are themselves derivable from the subset of (B) that involves only the first i variables, for all values i from 1 to the size of the system (B). For this purpose (B) can be chosen as the reduced Gröbner basis of (A). The theory of Gröbner bases (for an introduction see for example [1], but [2] devotes a whole chapter to it) provides the general theory and procedure for doing this i.e. there is an algorithm (e.g. Buchberger's algorithm) to reduce any such set of equations to a Gröbner basis. This can be followed by further reduction of each polynomial with respect to the remaining ones to get the reduced Gröbner basis that is unique for a given order of the power products. If the order of the variables is $z_1, z_2, \ldots z_n$ lowest first and the power products are ordered lexicographically, in the reduced Gröbner basis z and w must be chosen as the first two variables respectively so that in the reduced Gröbner basis, the two-variable polynomial only involves z and w.

Using this result, any algebraic expression involving the two variables z and w can be expressed implicitly as $P(z, w) = 0$ where P is a polynomial in the two variables. The procedure is to introduce as many new variables as necessary to reduce the given expression to a multivariate polynomial. This together with the defining equations for the new variables constitute a set of polynomial equations which can be reduced as above to the unique reduced Gröbner basis from which the polynomial involving z and w only is the implicit form of the algebraic relationship between z and w required. For example $(1-wz)^5+(w^2+z^{1/2}(1+w))^{1/2} = 0$ can be expressed in the form $P(z, w) = 0$. It is only necessary to introduce the variables $s = 1 - wz, t = z^{1/2}, u = (w^2 + z^{1/2}(1 + w))^{1/2}$ then P is obtained by eliminating s, t, and u from the system $s + wz - 1 = 0, t^2 - z = 0, s^5 + u = 0, u^2 - w^2 - t(1 + w) = 0$.

Theorem: If f_1 and f_2 are algebraic functions, then (a) $f_1 + f_2$, (b) $f_1 \times f_2$, (c) $f_1 \circ f_2$ the composition of f_1 and f_2, (d) f_1^{-1} the inverse of f_1, (e) $f_1 \cup f_2$ i.e. the set theoretic union of f_1 and f_2, and (f) f_1' the derivative of f_1 are also algebraic functions.

Note that I define the inverse of f by $(z, w) \in G_f \Leftrightarrow (w, z) \in G_{f^{-1}}$ where G_f is the graph of the algebraic function f , and the composition $f \circ g$ by $(z, w) \in G_{fog} \Leftrightarrow \exists s \in$ complex plane $\cup\infty$ such that $(z, s) \in G_f$ and $(s, w) \in G_g$.

Proof. There are 2-variable polynomials P_1 and P_2 such that $w = f_1(z) \Leftrightarrow P_1(z, w) = 0$ and $w = f_2(z) \Leftrightarrow P_2(z, w) = 0$. If $w = f_1(z) + f_2(z)$ then the relation between z and w can be expressed by following set of equations from which u and v must be eliminated:

$$\begin{aligned} w - u - v &= 0 \\ P_1(z, u) &= 0 \\ P_2(z, v) &= 0 \end{aligned}$$

The elimination theorem guarantees that there is another polynomial P_3 such that $P_3(z, w) = 0$ results from the stated elimination, thus $f_1 + f_2$ is an algebraic function. Likewise $f_1 \times f_2$ is also shown to be an algebraic function. If $w = f_1(f_2(z))$, this relation is expressed by the set of equations $w = f_1(u)$ and $u = f_2(z)$, alternatively as $P_1(u, w) = 0$ and $P_2(z, u) = 0$ from which again the elimination theorem shows that u can be eliminated giving $P_4(z, w) = 0$ where P_4 is another polynomial in 2 variables, showing that the relation between w and z is algebraic. It is now trivial to show that f_1^{-1} is also an algebraic function. By $f_1 \cup f_2$ I denote the union of the relations defined by f_1 and f_2 . Let G_1 be the set of pairs (z, w) such that $P_1(z, w) = 0$ and likewise let G_2 be the set of pairs (z, w) such that $P_2(z, w) = 0$, then $G_1 \cup G_2$ is the set of pairs (z, w) such that $P_1(z, w) = 0$ or $P_2(z, w) = 0$ which is the same as $P_1(z, w) \times P_2(z, w) = 0$, (because the polynomials are finite whenever z and w are finite), which is of the same form with a polynomial $P_5 = P_1 \times P_2$ showing that this relation between z and w is also algebraic. By differentiation, $\partial P_1/\partial z + \partial P_1/\partial w.dw/dz = 0$. Regarding dw/dz as a new variable s, and eliminating w between this equation and $P_1(z, w) = 0$ gives the equation $P_6(z, s) = 0$ where P_6 is another bivariate polynomial. This equation shows that dw/dz it is an algebraic function. □

These results show that the definition of an algebraic function used in this paper has all the properties that one would expect and justifies the term "algebraic".

5 Characterising singular points

5.1 The case when z_0 and w_0 are both finite

The point (z_0, w_0) is a regular point of an algebraic function if the function is locally one-to-one there i.e. if there are two sufficiently small neighbourhoods c_1 and c_2 containing z_0 and w_0 respectively such that the function restricted to $z \in c_1$ and $w \in c_2$ is a one-to-one correspondence. At a regular point $dw/dz \neq 0$ and $\neq \infty$ because then $w - w_0 \approx dw/dz \times (z - z_0)$ for w close to w_0 and z close to z_0, so the turns ratio is 1:1 and dw/dz is unique.

Any point (z_0, w_0) without this property is a singular point. For any such point dw/dz is 0 or ∞ or is not unique, and/or the turns ratio is not 1:1. Notice that this definition implies that points where two or more branches of the multivalued function coincide are classified as singular points. This could be the result of a single formula for example the n branches of $w = z^{1/n}$ coincide at $(0, 0)$, or as a result of several formulae for example $w = 2z-1$ and $w = z^2$ given by $P(z, w) \equiv (w - z^2).(w - 2z + 1) = 0$ has a singular point at $(1, 1)$. The general case of a singular point at $z = z_0$, $w = w_0$ in the surface G is defined by a relation of the form $(z - z_0)^p \propto (w - w_0)^q$ for the same z_0 and w_0 where p and q are coprime positive integers. It follows that $1/p$ turns in the z plane about z_0 correspond to 1 turn in $(z - z_0)^p$ about 0 or 1 turn in $(w - w_0)^q$ and to $1/q$ turns in w about w_0 i.e. q turns in z corresponds to p turns in w so the turns ratio is $q : p$ i.e. p/q. The exponent $r = p/q$ in the equivalent relation

$$(w - w_0) = A(z - z_0)^r \tag{6}$$

for w close to w_0 and z close to z_0 where $A \neq 0$ is a scale factor. The dominating behaviour given by (6) for z near z_0 and w near w_0 defines the turns ratio r and strength coefficient A.

From equation (5), it follows that $\frac{\partial P}{\partial z} + \frac{\partial P}{\partial w}.\frac{dw}{dz} = 0$. Therefore, because polynomials are finite for finite arguments, for all points (z_0, w_0) satisfying (5) the following implications hold:
$dw/dz = 0 \implies \partial P/\partial z = 0$ and $dw/dz = \infty \implies \partial P/\partial w = 0$
but the converses are not necessarily true. Also dw/dz is finite and non-zero if and only if $\partial P/\partial z$ and $\partial P/\partial w$ are non-zero (and they must be finite). Therefore all finite singular points will be found among the solutions of either of the two following pairs of polynomial equations

$$P(z, w) = 0, \frac{\partial P}{\partial z}(z, w) = 0. \tag{7}$$

$$P(z, w) = 0, \frac{\partial P}{\partial w}(z, w) = 0. \tag{8}$$

Therefore it is useful to consider (7) and (8) separately and their intersection.

5.2 The case when z_0 or w_0 or both are infinity

If the turns ratio r is negative, then a small circle in z corresponds to a large circle in w which may be thought of as small circle in the extended complex plane (Riemann sphere) about the point ∞. This suggests that a similar kind of classification of singular points should be applicable when z_0 and/or w_0 is infinity.

Because the mapping z to $1/z$ is one to one in the extended complex plane, any behaviour of a function near infinity can be mapped to equivalent behaviour of a function near zero, and so characterised as above for finite points. It may be possible to define singular points to include or to exclude the case $r = -1$. I will choose to include it i.e. small circles in the definition of a regular point will not include a "circle" around ∞ unless both "circles" are of this type. Therefore the point (∞, ∞) will not be considered as a singular point if $r = 1$, unless it is also an intersection point, but for other values of r it is a singular point.

In order to characterise the behaviour of the relationship between z and $w(z)$ defined implicitly by $P(z, w) = 0$ in the neighbourhood of (∞, w_0), introduce Q defined by $Q(z, w) \equiv P(1/z, w)$ and examine the function defined implicitly by $Q(z, w) = 0$ in the neighbourhood of $(0, w_0)$. If the pair (A, r) characterises this relationship asymptotically, then $w - w_0 = Az^r$, so in terms of the original variables, i.e. replacing z by $1/z$ gives $w - w_0 = Az^{-r}$ near (∞, w_0) i.e. the pair $(A, -r)$ characterises this singular point. Likewise if the behaviour near (z_0, ∞) is required, introduce the function defined implicitly by $R(z, w) \equiv P(z, 1/w) = 0$. Suppose this yields the behaviour given by (A, r) at $(z_0, 0)$, then its behaviour is dominated by $w = A(z - z_0)^r$, so in terms of the original variables $w = A^{-1}(z - z_0)^{-r}$ so it is characterised by $(A^{-1}, -r)$. Both transformations can be applied if the behaviour near (∞, ∞) is required. Then in terms of the transformed variables, the singular point is at $(0, 0)$, so $w = Az^r$, and in terms of the original variables, $w^{-1} = Az^{-r}$ i.e. $w = A^{-1}z^r$. To make these results easy to remember, note that in each of these cases the functional form to be fitted that characterises the behaviour in the neighbourhood of the singular point is obtained from the formula for the general finite case (6) by substituting 0 for z_0 and w_0 when they are infinite (and A and r are transformed).

It is clear that r is positive when z_0 and w_0 are finite because a small circle is mapped to a small circle (described r times), and negative when one of z_0 and w_0 is infinite (small circle is mapped to a large circle or vice versa), and positive when both are infinite. For any singular point, the derivative $dw/dz = Ar(z - z_0)^{r-1}$ approaches ∞ if $r < 1$ and approaches 0 if $r > 1$ as $z \to z_0$ if z_0 is finite, but if $z_0 = \infty$ then we have $dw/dz = Arz^{r-1}$ which approaches 0 if $r < 1$ and ∞ if $r > 1$ as $z \to \infty$. Therefore if $dw/dz = 0$ at the singular point, either z_0 is finite and $r > 1$ or $z_0 = \infty$ and $r < 1$. Similarly if $dw/dz = \infty$ at the singular point, either z_0 is finite and $r < 1$ or $z_0 = \infty$ and $r > 1$. When $r = 1$ the point is a regular point unless there is an intersection with another branch of the function there. The variable r can never be zero because this gives a constant term which would be included in w_0. The transformations described in this section allow the method of the following section to be extended to the case where z_0 or w_0 is ∞.

6 General method for calculating A and r for a finite singular point

I now describe a procedure to find the scale factor A and turns ratio r for the singular point (z_0, w_0) where the relation between w and z is defined by $P(z, w) = 0$, and $w_0 \neq \infty$ and $z_0 \neq \infty$. Examples are given in this section showing that the result depends on the pattern of leading non-zero multiple derivatives of P with respect to z and w at (z_0, w_0). In general, the relationship between Δz and Δw at the singular point (z_0, w_0) is determined by equating the power series

$$P(z_0 + \Delta z, w_0 + \Delta w) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Delta z^i \Delta w^j}{i!j!} \left(\frac{\partial}{\partial z}\right)^i \left(\frac{\partial}{\partial w}\right)^j P \Bigg|_{z=z_0, w=w_0} \tag{9}$$

to zero where the $i = j = 0$ term is $P(z_0, w_0) = 0$, and from the preceding section at least one of the two first derivative terms with $i = 0, j = 1$ and with $i = 1, j = 0$ is also zero. Of the remaining terms, a non-zero (i, j) term dominates a (k, l) term as $\Delta z \to 0$ and $\Delta w \to 0$ regardless of the relationship between Δz and Δw if and only if $k \geq i$ and $l \geq j$ with at least one of these being a strict inequality. The dominated terms may be removed from the equation and the relationship between Δz and Δw of the form

$$\Delta w = A \Delta z^r \tag{10}$$

is assumed, where Δw and $\Delta z \to 0$. A and r are chosen so that the leading terms in equation (9) cancel. If $r = p/q$ in lowest terms where p and q are integers, then as Δz passes once round zero, Δw changes by the factor $e^{2\pi i p/q}$, so the same formula (10) holds with A multiplied by $e^{2\pi i p/q}$. Therefore whenever (10) holds for one value of A it actually holds for the q distinct q-th roots of A^q one corresponding to each branch of the function in the neighbourhood of (z_0, w_0).

The simplest case is when $P = \frac{\partial P}{\partial z} = 0$ and $\frac{\partial^2 P}{\partial z^2} \neq 0$. In this case the leading terms of (9) are

$$0 = P(z_0 + \Delta z, w_0 + \Delta w) = \frac{\Delta z^2}{2!} \frac{\partial^2 P}{\partial z^2} + \Delta w \frac{\partial P}{\partial w}$$

and these terms dominate all other terms, therefore substituting in equation (10) gives $\frac{\Delta z^2}{2} \frac{\partial^2 P}{\partial z^2} + A \Delta z^r \frac{\partial P}{\partial w} = 0$. For this to be valid for small Δz requires that $r = 2$ therefore $A = -\frac{1}{2} \frac{\partial^2 P}{\partial z^2} / \frac{\partial P}{\partial w}$. Now suppose that both the first derivative terms are zero at the singular point and the 3 second derivative terms are non-zero there. Then the leading terms of (9) are $\frac{\Delta z^2}{2} \frac{\partial^2 P}{\partial z^2} + \Delta z \Delta w \frac{\partial^2 P}{\partial z \partial w} + \frac{\Delta w^2}{2} \frac{\partial^2 P}{\partial w^2} = 0$. Substituting in equation (10) gives $\frac{\Delta z^2}{2} \frac{\partial^2 P}{\partial z^2} + A \Delta z^{r+1} \frac{\partial^2 P}{\partial z \partial w} + \frac{1}{2} A^2 \Delta z^{2r} \frac{\partial^2 P}{\partial w^2} =$

0. Only when $r = 1$ do any of these exponents of Δz agree, and then they all agree. This is a quadratic equation for A with solution(s) $A = \left\{-\frac{\partial^2 P}{\partial w \partial z} \pm \sqrt{\left(\frac{\partial P}{\partial w \partial z}\right)^2 - \frac{\partial^2 P}{\partial z^2}\frac{\partial^2 P}{\partial w^2}}\right\} / \frac{\partial^2 P}{\partial w^2}$. In the unequal root case, two values of Δz are each associated with a value of Δw and vice versa. Because $r = 1$, (z_0, w_0) behaves like a regular point except that there can be 2 distinct values for the derivative $A = dw/dz$. This is therefore a case where 2 different branches of the function cross. These examples suggest that the same kind of analysis can be applied whenever any set of leading terms in (9) expresses the relationship between Δz and Δw and the result is asymptotically a set of relations of the form (10) with different pairs of values (A, r) at different singular points (z_0, w_0). Further, note that it is possible that different values of r can occur at the same singular point for example $P = (z - w^2)(2z - w) = 0$ gives $(A, r) = (2, 1), (1, 0.5), \text{and} (-1, 0.5)$ at $(0, 0)$.

I give one more complex example to illustrate the generality of the technique. Suppose that for the following set of pairs (i, j) the derivative $\partial^{i+j} P / \partial z^i \partial w^j = 0$ at the singular point: for $j = 0$ with $1 \le i \le 7$ and $j = 1$ with $0 \le i \le 4$ and $j = 2$ and $j = 3$ each with $i = 0$ and 1. Then by plotting the points (i, j) it is obvious that the following set of non-zero (i, j) terms dominate all others in equation (9): $(8, 0), (5, 1), (2, 2), (0, 4)$ and the equation reduces to

$$\frac{\Delta w^4}{4!}\frac{\partial^4 P}{\partial w^4} + \frac{\Delta z^2}{2!}\frac{\Delta w^2}{2!}\frac{\partial^4 P}{\partial z^2 \partial w^2} + \frac{\Delta z^5 \Delta w}{5!}\frac{\partial^6 P}{\partial z^5 \partial w} + \frac{\Delta z^8}{8!}\frac{\partial^8 P}{\partial z^8} = 0$$

with other terms that are dominated being omitted. Again looking for solutions of the form (10) gives an equation after elimination of Δw with the following exponents of Δz: $4r$, $2r + 2$, $r + 5$, 8. Equating these in pairs gives 4 distinct values of r, which are:

1	giving exponents 4, 4, 6, 8
5/3	giving exponents 20/3, 16/3, 20/3, 8
2	giving exponents 8, 6, 7, 8
3	giving exponents 12, 8, 8, 8

Only in the first and last of these is there more than one term with the same leading (smallest) power of Δz repeated, so only in these cases is the set of dominant terms cancelling. (When the the domimant terms cancel, another term in the expression for Δw could be sought that cancels some of the next leading terms in the equation, but this will not be necessary in this paper.) For $r = 1$ the equation of the leading terms is $\frac{A^4}{4!}\frac{\partial^4 P}{\partial w^4} + \frac{A^2}{4}\frac{\partial^4 P}{\partial z^2 \partial w^2} = 0$, from which $A = \pm i\sqrt{3! \frac{\partial^4 P}{\partial z^2 \partial w^2} / \frac{\partial^4 P}{\partial w^4}}$. Because $r = 1$ this is an example of a singular point arising from an intersection of two branches of the function that each separately would not have a singular point here. The double root $A = 0$ is no use in this analysis but it seems to indicate that there are 2 other solutions that have different values of r. The other case is $r = 3$ which gives the following equation for the leading terms:

$$\frac{A^2}{4}\frac{\partial^4 P}{\partial z^2 \partial w^2} + \frac{A}{5!}\frac{\partial^6 P}{\partial z^5 \partial w} + \frac{1}{8!}\frac{\partial^8 P}{\partial z^8} = 0$$

which has either 1 or 2 distinct solutions for A. If in a situation like this, A and r are known but P is unknown, the equation showing the cancellation of the leading terms will be clearly linear and homogeneous (i.e. without a constant term) with respect to the coefficients of P, therefore if sufficient such relations are known, P could be determined up to a constant factor, which is sufficient to determine the associated algebraic function uniquely. It is natural to conjecture that this happens if all the singular points classified by A and r and their locations (z, w) are known.

7 Rules for deriving the parameters for singular points for functions

In this section rules will be derived for calculating the singular point parameters (z, w, A, r) of algebraic functions resulting from the operations $+$, $\times$, composition, inversion, and differentiation applied to other algebraic functions with known singular point parameters. The aim of this was to make it easy to calculate the parameters of any singular point of any algebraic function. The A_i and r_i for the assumptions and results of these theorems are not 0 and not ∞, and $r \neq 1$ because $r = 1$ makes it a regular point. Unfortunately as will be seen, singular points can arise where neither of the operands had a singular point so it is not easy to see how these calculations could be done only with the knowledge of the singular point parameters of the function. The cases where the singular points do not coincide are basically trivial because one of the functions is essentially locally a constant.

For each function with a singular point, it is necessary to consider the four combinations of finite and infinite values of z_0 and w_0. These determine the corresponding conditions on r as well as the formula for w as shown in Table 2.

z_0	w_0	r	w
$z_0 \neq \infty$	$w_0 \neq \infty$	$r > 0$	$w = w_0 + A(z - z_0)^r$
$z_0 \neq \infty$	$w_0 = \infty$	$r < 0$	$w = A(z - z_0)^r$
$z_0 = \infty$	$w_0 \neq \infty$	$r < 0$	$w = w_0 + Az^r$
$z_0 = \infty$	$w_0 = \infty$	$r > 0$	$w = Az^r$

Table 2: Asymptotic behaviour at a single singular point

7.1 The Addition Rule

Suppose that algebraic functions $f_1(z)$ and $f_2(z)$ have singular points at the same point z_0 given by (z_0, w_1, A_1, r_1) and (z_0, w_2, A_2, r_2) respectively then the singular point of the total defined by (z_0, w_0, A_0, r_0) is determined by adding the expressions for w for each of the eight combination of z_0, w_1, and w_2 being finite and infinite, and picking out the parameters in the leading term in the sum. The result is $w_0 = w_1 + w_2$ when w_1 and w_2 are both finite, and 0 otherwise, and as in Table 2, $r_1 > 0$ if z_1 and w_1 are both finite or both infinite, and $r_1 < 0$ otherwise. Likewise $r_2 > 0$ if z_1 and w_2 are both finite or both infinite, and $r_2 < 0$ otherwise. The parameter r_0 is the leading exponent which is given by $r_0 = \begin{cases} \min(r_1, r_2) \text{ if } z_1 \neq \infty \\ \max(r_1, r_2) \text{ if } z_1 = \infty \end{cases}$. The parameter A_0 is the coefficient of the leading term so if $r_1 \neq r_2$ then

$$A_0 = \left\{ \begin{matrix} A_1 \text{ if } r_0 = r_1 \\ A_2 \text{ if } r_0 = r_2 \end{matrix} \right\}.$$

otherwise $A_0 = A_1 + A_2$, except that if this gives $A_0 = 0$ then A_0 must be recalculated and there is no formula for it here because it depends on other terms not mentioned. The algebra includes the case $r_1 = r_2 = 1$ which means f_1 and f_2 have regular points at $z = z_0$, and shows that in this case $r_0 = 1$ i.e. $f_1 + f_2$ also has a regular point at z_0 with the sole exception of the case when $A_0 = 0$ which might give a singular point.

If the the point (z_0, w_2) is not a singular point, then $w = w_1 + w_2 + A_1(z - z_0)^{r_1}$ for z near z_0 so the singular point of the total is given by $(z_0, w_1 + w_2, A_1, r_1)$.

7.2 The Multiplication Rule

With the same notation for the singular point parameters as in the addition rule, again in all eight cases, the leading term of the product $f_1(z)f_2(z)$ must be expressed in the form of Table 2 and the parameters picked out. The result is $w_0 = w_1 w_2$ if w_1 and w_2 are both finite, and 0 otherwise, and $r_0 =$

$$\left\{ \begin{matrix} \min(r_1, r_2) \text{ if} & z_0 \neq \infty, & w_1 \neq \infty \text{ and} & w_2 \neq \infty \\ \max(r_1, r_2) \text{ if} & z_0 = \infty, & w_1 \neq \infty \text{ and} & w_2 \neq \infty \\ r_1, \text{ if} & & w_1 = \infty \text{ and} & w_2 \neq \infty \\ r_2, \text{ if} & & w_1 \neq \infty \text{ and} & w_2 = \infty \\ r_1 + r_2, \text{ if} & & w_1 = \infty \text{ and} & w_2 = \infty \end{matrix} \right\} \text{ and}$$

$$A_0 = \left\{ \begin{array}{ll} \left\{ \begin{array}{c} w_1 A_2 \text{ if } r_2 < r_1 \\ w_2 A_1 \text{ if } r_2 > r_1 \\ w_1 A_2 + w_2 A_1 \text{ if } r_1 = r_2 \end{array} \right\} & \text{if } w_1 \neq \infty \text{ and } w_2 \neq \infty \\ w_2 A_1 & \text{if } w_1 = \infty \text{ and } w_2 \neq \infty \\ w_1 A_2 & \text{if } w_1 \neq \infty \text{ and } w_2 = \infty \\ A_1 A_2 & \text{if } w_1 = \infty \text{ and } w_2 = \infty \end{array} \right\}.$$

If (z_0, w_2) is not a singular point, $w = w_1 w_2 + A_1 w_2 (z - z_0)^{r_1}$ for z near z_0 so the singular point of the product is given by $(z_0, w_1 w_2, A_1 w_2, r_1)$.

Again putting $r_1 = r_2 = 1$ gives a pair of regular points. In these cases, $A_1 = \frac{dw_1}{dz}|_{z_0}$ and $A_2 = \frac{dw_2}{dz}|_{z_0}$. The product can be singular only if the product has $r_0 \neq 1$ or $A_0 = 0$. This can happen only if $w_1 = w_2 = \infty$ (giving $r_0 = 2$) or ($w_1 = 0$ and $w_2 = \infty$) or ($w_2 = 0$ and $w_1 = \infty$) or ($w_1 \frac{dw_2}{dz}|_{z_0} + w_2 \frac{dw_1}{dz}|_{z_0} \equiv \frac{d}{dz}(w_1 w_2) = 0$ and w_1 and w_2 are finite).

7.3 The Composition Rule

Let $f_1 : z \to w$ and $f_2 : w \to t$ have singular points given by (z_0, w_0, A_1, r_1) and (w_0, t_0, A_2, r_2) respectively, and if z_0, w_0 and t_0 are finite, then asymptotically

$$\begin{array}{c} w = w_0 + A_1 (z - z_0)^{r_1} \\ t = t_0 + A_2 (w - w_0)^{r_2} \end{array} \tag{11}$$

from which $t = t_0 + A_2 {A_1}^{r_2} (z - z_0)^{r_1 r_2}$. This form must be modified by replacing z_0 and t_0 by 0 if they are ∞ according to Table 2. The result is in each case $f_2(f_1(z)) = w_0 + A_0 (z - z_0)^{r_0}$ for z close to z_0 where w_0=t_0, $A_0 = A_2 {A_1}^{r_2}$, and $r_0 = r_1 r_2$, i.e. the composition of f_1 followed by f_2 has a singular point given by $(z_0, t_0, A_2 {A_1}^{r_2}, r_1 r_2)$.

If (z_0, w_0) is a regular point, it is necessary to consider the first order taylor expansion of f at z_0 which is given by $w = w_0 + \frac{dw}{dz}|_{z_0}(z - z_0)$, where $\frac{dw}{dz}|_{z_0} \neq 0$ by assumption. Now combining this with the second of equations 11 gives $t = t_0 + A_2 \left(\frac{dw}{dz}|_{z_0}(z - z_0) \right)^{r_2}$ so the resulting singular point is $(z_0, t_0, A_2 \left(\frac{dw}{dz}|_{z_0} \right)^{r_2}, r_2)$. Likewise if the point (w_0, t_0) is a regular point, then $t - t_0 = \frac{dt}{dw}|_{w_0}(w - w_0)$ so $t - t_0 = \frac{dt}{dw}|_{w_0} A_1 (z - z_0)^{r_1}$, so the singular point is $(z_0, t_0, \frac{dt}{dw}|_{w_0} A_1, r_1)$.

For a pair of regular points, A_1 and A_2 are $\neq 0$ and $\neq \infty$ and $r_1 = r_2 = 1$, therefore the same is true for $A_2 A_1^{r_2}$ and $r_1 r_2$ showing that the composition of a pair of regular points is also regular.

This rule is much simpler than the first two and unlike them it is complete in sense that there are no cases where further calculations are needed using more information about f_1 and f_2 to get the result. Another simple rule that is easy to derive is the

7.4 The Inversion Rule

Suppose $f : z \to w$ has a singular point given by (z_0, w_0, A, r) then asymptotically $w = w_0 + A(z - z_0)^r$ for w close to w_0 and z close to z_0. This can be rewritten as $z - z_0 = ((w - w_0)/A)^{1/r}$ i.e. the inverse function $f^{-1} : w \to z$ has the singular point given by $(w_0, z_0, A^{-1/r}, 1/r)$.

For a regular point, A is finite and non-zero and $r = 1$ so $1/r = 1$ and $A^{1/r} = A^{-1}$ is finite and non-zero, so the inverse of a regular point is regular.

7.5 The Derivative Rule

If for z close to z_0 we have $w = w_0 + A_0(z - z_0)^{r_0}$ then asymptotically $\frac{dw}{dz} = A_0 r_0 (z - z_0)^{r_0 - 1}$ so the result is again of the same form with w_0 replaced by 0, A_0 replaced by $A_0 r_0$, and r_0 replaced by $r_0 - 1$ i.e. the singular point corresponding to the derivative of the singular point (z_0, w_0, A_0, r_0) is given by $(z_0, 0, A_0 r_0, r_0 - 1)$, where again z_0 must be replaced by 0 if it is ∞.

The derivative of a regular point can be singular because then $r_0 = 1$ so the singular behaviour of the point is determined by $(z_0, 0, A_0, 0)$ showing that its singular or non-singular nature cannot be determined without further calculation.

8 Characterising algebraic functions by their singular points

Theorem 1: If an algebraic function has no zero (including at ∞) then it is the union of a set of non-zero constant functions.

Note: the converse is obvious so these two conditions are equivalent.

Proof. Suppose an algebraic function has no zero. Let the function be determined by w as a function of z such that $P(z, w) = 0$ then the condition is that $P(z, 0) = 0$ has no solution for z. Expanding the polynomial P as powers of w i.e. $P(z, w) = \sum_{i=0}^{s} w^i R_i(z)$ where R_i are single variable polynomials (with finite coefficients) gives the condition (with z finite) $R_0(z) = 0$ has no solution for z, so $R_0(z)$ must be a non-zero constant. Now we must add the condition that $z = \infty$ is not a solution. But this cannot be avoided because $w = 0$ and $R_i(\infty) = \infty$ and zero times ∞ can be any value, unless the R_i are all constants. Then P is independent of z and the algebraic function reduces to a set of constants i.e. $P(z, w) = 0$ if and only if $w = w_1$ or $w = w_2$ or $\ldots w = w_s$ where each of these constants is not zero. □

Theorem 2: If an algebraic function has no singular points, then it is a linear function i.e. a first degree polynomial.

I will consider the consequences of (7) and (8) each having no solution points (z, w) where z and w are finite, together with the condition that there are no singular points when z or w is ∞. These latter cases occur whenever either z is finite and w is infinite or vice versa, leading to a negative turns ratio r. (When the proof is complete it will be clear that these conditions imply that there is no singular point at (∞, ∞).)

Proof. In general one can write P in terms of its zeros as a polynomial in w i.e.

$$P(z,w) = f_0(z)\prod_{i=1}^{k}(w - f_i(z)) = 0 \tag{12}$$

where $f_0(z)$ is the coefficient of w^k in P and is therefore a polynomial in z. The single valued functions $f_i(z)$ for $1 \le i \le k$ are the branches of the algebraic function defined by equation (5) and are by assumption finite (because otherwise there would be a singular point with $dw/dz = \infty$ there) together with their first derivatives for all finite z. If $f_0(z_1) = 0$, with $z_1 \neq \infty$, by differentiating (12) we have

$$\left.\frac{\partial P}{\partial w}\right|_{z=z_1} = f_0(z_1)\sum_{l=1}^{k}\prod_{i=1, i\neq l}^{k}(w - f_i(z_1)) = 0 \tag{13}$$

contradicting the assumption of no finite singular points, therefore $f_0()$ is nowhere zero for finite arguments and must be a non-zero constant function. Then we can divide by it and differentiate with respect to z and w to obtain

$$\sum_{l=1}^{k}\left\{-f_l'(z)\prod_{i=1, i\neq l}^{k}(w - f_i(z))\right\} \neq 0 \tag{14}$$

and

$$\sum_{l=1}^{k}\prod_{i=1, i\neq l}^{k}(w - f_i(z)) \neq 0 \tag{15}$$

respectively because (z, w) is not a singular point. It follows from (12) that w must be equal to one of the $f_i(z)$ say $w = f_j(z)$. Because all the $f_i(z)$ are finite for finite z (and so w is finite), at most one non-zero term remains in the sums in (14) and (15), namely the $l = j$ terms, so

$$f_j'(z)\prod_{i=1, i\neq j}^{k}(w - f_i(z)) \neq 0 \tag{16}$$

and

$$\prod_{i=1, i\neq j}^{k}(w - f_i(z)) \neq 0. \tag{17}$$

Then equation (17) implies $w \neq f_i(z)$ for all $i \neq j$. Because j is arbitrary, $f_1(z), \dots f_k(z)$ are all different. $\{f_j(z)\}$ is a multivalued algebraic function and its derivative is also an algebraic function. From (16) and (17) this derivative is never zero if z is finite. This derivative is also not zero when z is infinite because otherwise there would be a singular point there. It follows from Theorem 1 that $f'_j(z)$ is a set of constants B_j so $f_j(z) = B_j z + C_j$ for $1 \leq j \leq k$ and $B_j \neq 0$ for all j. But because the $f_j(z)$ are always all different for finite z, the B_j are all equal (say $= B$), so $f_j(z) = Bz + C_j$ where $B \neq 0$. Finally, to avoid a singular point at (∞, ∞) resulting from the intersection of these functions, it follows that $k = 1$ and $f(z) = Bz + C$. □

Theorem 3: An algebraic function is uniquely determined by its singular points.

Proof. Suppose that two algebraic functions $w = f_1(z)$ and $w = f_2(z)$ have the same set of singular points $\{z_i, w_i, A_i, r_i\}$. Then $f_2^{-1}(.)$ has singular points $\{w_i, z_i, A_i^{-1/r_i}, 1/r_i\}$ and $f_1(f_2^{-1}(.))$ has singular points

$$\{z_i, z_i, A_i\left(A_i^{-1/r_i}\right)^{r_i}, 1\} = \{z_i, z_i, 1, 1\}$$

by the inversion and composition rules, which means no singular points because $r = 1$ in each quadruple. Therefore by Theorem 2 above, $f_1(f_2^{-1}(z)) = \alpha z + \beta$ for all z, for some constants α and β. Therefore $f_2^{-1}(z) = s$ which is equivalent to $f_2(s) = z$ implies $f_1(s) = \alpha z + \beta$. But $f_1(.)$ and $f_2(.)$ have the same set of singular points, so by the addition and multiplication rules, $f_1(.)$ has the singular points $\{z_i, \alpha w_i + \beta, \alpha A_i, r_i\}$ which must be same set as above, so $\alpha = 1$ and $\beta = 0$ and so $f_2(s) = z$ implies $f_1(s) = z$. Likewise the converse can be proved, hence finally $f_1(z) = f_2(z)$ for all z. □

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