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STRONG INSERTION OF A CONTINUOUS FUNCTION BETWEEN TWO COMPARABLE b-CONTINUOUS FUNCTIONS

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Abstract

A sufficient condition in terms of lower cut sets are given for the strong insertion of a continuous function between two comparable $b-$continuous real-valued functions.

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1 Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [3]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [11], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael [3].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [10]. A subset A of a topological space (X, τ) is called *semi-open* [10] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

D. Andrijevic introduced a new class of generalized open sets in a topological space, so called $b-$open sets [1]. This type of sets discussed by A. A. El-Atik under the name of $\gamma-$open sets [5]. This class is closed under arbitrary union. The class of $b-$open sets contains all semi-open sets and preopen sets. The class of $b-$open sets generates the same topology as the class of preopen sets. D. Andrijevic studied several fundamental and interesting properties of $b-$open sets. A subset A of a topological space (X, τ) is called $b-$*open* if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ [1]. A set A is called $b-$*closed* if its complement is $b-$open or equivalently if $Cl(Int(A)) \cap Int(Cl(A)) \subseteq A$.

Recall that a real-valued function f defined on a topological space X is called $A-$*continuous* [12] if the preimage of every open subset of R belongs to A, where A is a collection of subset of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [4, 6].

Hence, a real-valued function f defined on a topological space X is called $b-$*continuous* if the preimage of every open subset of R is a $b-$open subset of X.

Results of Katětov [7, 8] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a sufficient condition for the strong insertion of a continuous function between two comparable $b-$continuous real-valued functions.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [9].

A property P defined relative to a real-valued function on a topological space is a $c-$*property* provided that any constant function has property P and provided that the sum of a function with property P and any continuous function also has property P. If P_1 and P_2 are $c-$property, the following terminology is used:(i) A space X has the *weak insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a continuous function h such that $g \leq h \leq f$.(ii) A space X has the *strong insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, is given a sufficient condition for the weak insertion property. Also for a space with the weak insertion property, we give a sufficient condition for the space to have the strong insertion property.

2 Main Results

Before giving a sufficient condition for insertability of a continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space, the family of all $b-$open and $b-$closed will be denoted by $bO(X, \tau)$ and $bC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). Respectively, we define the *b-closure* and *b-interior* of a set A, denoted by $bCl(A)$ and $bInt(A)$ as follows:
$bCl(A) = \cap\{F : F \supseteq A, F \in bC(X, \tau)\}$ **and**
$bInt(A) = \cup\{O : O \subseteq A, O \in bO(X, \tau)\}$.

Proposition 2.1. (D. Andrijevic [1]) (i) The union of any family of $b-$open sets is a $b-$open set.
(ii) The intersection of an open and a $b-$open is a $b-$open set.
Hence, by Proposition 2.1, we have $bCl(A)$ is $b-$closed and $bInt(A)$ is $b-$open.
The following first two definitions are modifications of conditions considered in [7, 8].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:
1) If $A_i\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.
2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.
3) If $A\ \rho\ B$, then $Cl(A) \subseteq B$ and $A \subseteq Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.
We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on a topological space X with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$

and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a continuous function h defined on X such that $g \leq h \leq f$.

Proof. Theorem 1 of [**7**].

3 Applications

The abbreviation bc is used for $b-$continuous.

Corollary 3.1. **If for each pair of disjoint $b-$closed sets F_1, F_2 of X , there exist open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 =$, then X has the weak insertion property for (bc, bc).**

Proof. Let g and f be real-valued functions defined on the X, such that f and g are bc, and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $bCl(A) \subseteq bInt(B)$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $b-$closed set and since $\{x \in X : g(x) < t_2\}$ is a $b-$open set, it follows that $bCl(A(f,t_1)) \subseteq bInt(A(g,t_2))$. Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1.

Corollary 3.2. **If for each pair of disjoint $b-$closed sets F_1, F_2, there exist open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 =$ then every $b-$continuous function is continuous.**

Proof. Let f be a real-valued $b-$continuous function defined on the X. By setting $g = f$, then by Corollary 3.1, there exists a continuous function h such that $g = h = f$.

Corollary 3.3. **If for each pair of disjoint $b-$closed sets F_1, F_2 of X , there exist open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 =$ then X has the strong insertion property for (bc, bc).**

Proof. Let g and f be real-valued functions defined on the X, such that f and g are bc, and $g \leq f$. By setting $h = (f+g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by

Corollary 3.2, since g and f are continuous functions hence h is a continuous function.

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