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Common Fixed Point Theorems for T-Weak Contraction Mapping in a Cone Metric Space

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Abstract

In this paper, we prove some common fixed point theorems for T-Weak contraction mapping in complete cone metric spaces. Our results are generalizations of the results of J.R. Morales and E. Rojas[3].

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1. Introduction

Recently, Huang and Zhang [1] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. The results in [1] were generalized by Sh.Rezapour and R. Hamlbarani [2] omitting the assumption of normality on the cone. Subsequently, many authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cone, see for instance [14],[15] and [16].

It is well known that the classic Banach's contraction mapping theorem is a fundamental result in fixed point theory. Rhoades [17] made a comparison of various different types of contraction mappings. In 2009, A Beiranvand et al [5] introduced new classes of contractive functions T-contraction and T-contractive mappings and then they established and extended the Banach contraction principle. S Moradi [4] introduced the T-Kannan contractive mapping which extend the well known Kannan's fixed point theorem given in [12]. Morales and Rojas [3].[9],[10] have extended the concept of T-contraction mappings to cone metric space by proving fixed point theorems for T-Kannan, T-Chatterjea, T-Zamfirescu, T-Weakly contraction mappings.

In the present paper, we study the existence of a common fixed point for pair of T-Weak contraction mappings in the setting of complete cone metric spaces generalizing consequently the results given in [1] and [3] .

2. Preliminary Notes

First, we recall some standard notations and definitions in cone metric spaces with some of their properties (see [1]).

Definition 2.1 [1] Let E be a real Banach space and P a subset of $E.P$ is called a cone if and only if :

(i) P is closed , non-empty and $P\neq \{0\}$;

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leqslant$ on E with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in intP$, $int\,P$ denotes the interior of P .The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leqslant x \leqslant y$ implies $\|x\| \leqslant K\|y\|$

The least positive number K satisfying the above is called the normal constant of P.

Definition 2.2. [1] Let X be a non-empty set and $d : X \times X \to E$ a mapping such that

(i) $0 \leqslant d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space.

Example 2.3. [1] Let $E = R^2, P = \{(x, y) \in E : x, y \geqslant 0\}, X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (\mid x - y \mid, \alpha \mid x - y \mid)$, where $\alpha \geqslant 0$ is a constant. Then (X, d) is a cone metric space .

Definition 2.4. [1] Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$;

(i) (x_n) converges to x if for every $c \in E$ with $0 \ll c$ there is an n_o such that for all $n > n_o$, $d(x_n, x) \ll c$. We denote this by $lim_{n\to\infty} x_n = x$ or $x_n \to x, (n \to \infty)$.

(ii) If for any $c \in E$ with $0 \ll c$ there is an n_o such that for all $n, m \geqslant n_o$, $d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X. Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 2.5.[1] Let (X, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K. Let $(x_n), (y_n)$ be sequences in X and $x, y \in X$.

(i) (x_n) converges to x if and only if $lim_{n\to\infty} d(x_n, x) = 0$;

(ii) If (x_n) converges to x and (x_n) converges to y then $x = y$. That is the limit of (x_n) is unique;

(iii) If (x_n) converges to x , then (x_n) is Cauchy sequence ;

(iv) (x_n) is a Cauchy sequence if and only if $lim_{n,m\to\infty} d(x_n, x_m) = 0$;

(v) If $x_n \to x$ and $y_n \to y, (n \to \infty)$ then $d(x_n, y_n) \to d(x, y)$.

Definition 2.6(see[9]and[10]). Let (X, d) be a cone metric space, P a normal cone with normal constant K and $T : X \to X$. Then

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$, implies that $lim_{n\to\infty} T(x_n) = T(x)$, for all (x_n) in X ;

(ii) T is said to be subsequentially convergent, if we have, for every sequence (y_n) that $T(y_n)$ is convergent, implies (y_n) has a onvergent subsequence;

(iii) T is said to be sequentially convergent, if we have, for every sequence (y_n), if $T(y_n)$ is convergent, then (y_n) is also convergent.

3. Main Results

The results which we will give, are generalization of theorem 3.4 and 3.5 of [3].

In 2003, V. Berinde (see [6] and [7]) introduced a new class of contraction mappings on metric spaces, which are called Weak contractions. In sequel, J.R. Morales and E. Roas [3] extended these kind of mappings by introducing a new function T and are called T-Weak contraction.

Definition 3.1[3] Let (X, d) be a cone metric space and $T, S : X \to X$ two mappings. S is called a T-Weak contraction, if there exist a constant $\delta \in (0, 1)$ and some $L \geqslant 0$ such that $d(TSx, TSy) \leqslant \delta d(Tx, Ty) + Ld(Ty, TSx)$ for all $x, y \in X$.

If we take $T = I_d$ and and $E = \mathbb{R}_+$then we obtain the notion of Berinde [6].

Due to the symmetry of the metric, the T-Weak contractive condition implicitly include the following dual one:

$d(TSx, TSy) \leqslant \delta d(Tx, Ty) + Ld(Tx, TSy)$ for all $x, y \in X$.

Theorem 3.2. Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Moreover, let $T : X \to X$ be a continuous and one to one mapping and $R, S : X \to X$ be a pair of continuous T-Weak contraction mappings. Then

(i) For every $x_o \in X$,

$$lim_{n\to\infty} d(TR^{2n+1}x_o, TR^{2n+2}x_o) = 0$$

and $$lim_{n\to\infty} d(TS^{2n+2}x_o, TS^{2n+3}x_o) = 0$$

(ii) There is $\vartheta \in X$ such that

$$lim_{n\to\infty} TR^{2n+1}x_o = \vartheta = lim_{n\to\infty} TS^{2n+2}x_o.$$

(iii) If T is subsequentially convergent, then $(R^{2n+1}x_o)$ and $(S^{2n+2}x_o)$ have a convergent subsequences.

(iv) There is common fixed point $u \in X$ such that $Ru = u = Su$.

(v) If T is sequentially convergent, then for each $x_o \in X$ the iterate sequences $(R^{2n+1}x_o)$ and $(S^{2n+2}x_o)$ converge to u.

Proof: (i) Since R and S are pair of $T - Weak$ contraction mappings, then by Definition 3.1, there exist a constant $\delta \in (0, 1)$ and some $L \geqslant 0$ such that $d(TRx, TSy) \leqslant \delta d(Tx, Ty) + Ld(Ty.TRx)$ for all $x, y, \in X$.

Suppose $x_o \in X$ is an arbitrary point and the Picard iteration associated to R and $S.(x_{2n+1})$ and (x_{2n+2}) are defined by

$$x_{2n+2} = Rx_{2n+1} = R^{2n+1}x_o, n = 0, 1, 2, - - - - - - - -$$

and $$x_{2n+3} = Sx_{2n+2} = S^{2n+2}x_o, n = 0, 1, 2, - - - - - - - -$$

Thus,

$$d(TR^{2n+2}x_o, TR^{2n+1}x_o) \leqslant hd(TR^{2n+1}x_o, TR^{2n}x_o)$$

where $h = \frac{\delta}{1-2\delta} < 1$. Therefore, for all n we have

$$d(TR^{2n+2}x_o, TR^{2n+1}x_o) \leqslant h^{2n+1}d(TRx_o, Tx_o) \quad \textbf{(1)}$$

Similarly, we have

$$d(TS^{2n+3}x_o, TS^{2n+2}x_o) \leqslant k^{2n+2}(TSx_o, Tx_o) \quad \textbf{(2)}$$

From (1) and the fact the cone P is a normal cone we obtain that

$$\|d(TR^{2n+2}x_o, TR^{2n+1}x_o) \| \leqslant Kh^{2n+1} \| d(TRx_o, Tx_o) \|,$$

taking limit $n \to \infty$ in the above inequality, we can conclude that

$$\lim_{n\to\infty} d(TR^{2n+2}x_o, TR^{2n+1}x_o) = 0 \quad (3)$$

Similarly, from (2) we have,

$$\lim_{n\to\infty} d(TS^{2n+3}x_o, TS^{2n+2}x_o) = 0 \quad (4)$$

(ii) Now, for $m, n \in N$ with $m > n$, we get

$$d(TR^{2m+1}x_o, TR^{2n+1}x_o) \leqslant (h^{2n+1} + - - - - - - - - - - + h^{2m})d(TRx_o, Tx_o)$$

$$\leqslant \tfrac{h^{2n+1}}{1-h} d(TRx_o, Tx_o)$$

Again; as above, since P is a normal cone we obtain

$$\lim_{n,m\to\infty} d(TR^{2m+1}x_o, TR^{2n+1}x_o) = 0.$$

Hence, the fact that (X, d) is complete cone metric space, imply that $(TR^{2n+1}x_o)$ is a Cauchy sequence in X, therefore there is $\vartheta \in X$ such that

$$\lim_{n\to\infty} TR^{2n+1}x_o = \vartheta.$$

Similarly one can show easily that

$$\lim_{n\to\infty} TS^{2n+2}x_o = \vartheta.$$

(iii) if T is subsequentially convergent, $(R^{2n+1}x_o)$ has a convergent subsequence, so there is $u \in X$ and $\{(2n+1)_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} R^{(2n+1)_i}x_o = u.$$

(iv) Since T and R are continuous mappings, we obtain;

$$\lim_{i\to\infty} TR^{(2n+1)_i}x_o = Tu,$$

$$lim_{i\to\infty} TR^{(2n+1)_i+1}x_o = TRu$$

therefore, $Tu = \vartheta = TRu$, and since T is one-to-one, then $Ru = u$. So R has a fixed point.

Now, suppose that $Ru = u$ and $Ru_1 = u_1$.

$$d(TRu, TRu_1) \leqslant \delta d(Tu, Tu_1) + Ld(Tu_1, TRu)$$

$$d(Tu, Tu_1) \leqslant \delta d(Tu, Tu_1)$$

from the fact that $0 \leqslant \delta < 1$ and that T is one-to-one we obtain that $u = u_1$.

(v) It is clear that if T is sequentially convergent, then for each $x_o \in X$, the iterate sequence $(R^{2n+1}x_o)$converges to u.

Similarly, we can prove that $(S^{2n+2}x_o)$ converges to the fixed point of S i.e u.

or $lim_{i\to\infty} R^{(2n+1)_i}x_o = u = lim_{i\to\infty} S^{(2n+2)_i}x_o$.

This completes the proof of the theorem.

The next example shows that a $T-weak$ contraction may has infinitely fixed points.

Example 3.3 [8] Let $X = [0,1]$ be the unit interval with the usual metric and $T, S : X \to X$ the identity maps, that is, $Tx = Sx = x$ for all $x \in X$. Then, taking $0 \leqslant a < 1$ and $L \geqslant 1 - a$ we obtain

$$d(TSx, TSy) =\mid TSx - TSy \mid$$

$$\mid x - y \mid \leqslant a \mid x - y \mid + L \mid y - x \mid$$

which is valid for all $x, y \in [0,1]$. Thus the set of the fixed points F_s of the map S in the interval [0,1], i.e.

$$F_s = \{x \in [0,1]/Sx = x\} = [0,1].$$

In next result, we prove the existence of a unique common fixed point for pair of T-Weak contraction mappings in the setting of complete cone metric space with an additional contractive condition.

Theorem 3.4 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Moreover, let $T : X \to X$ be a continuous and one-to-one mapping and $R, S : X \to X$ be a pair of $T-Weak$ contraction for which there is $\sigma \in (0,1)$ and some $L_1 \geqslant 0$ such that

$$d(TRx, TSy) \leqslant \sigma d(Tx, Ty) + L_1 d(Tx, TRx)$$

for all $x, y, \in X$. Then;

(i) For every $x_o \in X$,

$$lim_{n\to\infty} d(TR^{2n+1}x_o, TR^{2n+2}x_o) = 0$$

and $$lim_{n\to\infty} d(TS^{2n+2}x_o, TS^{2n+3}x_o) = 0.$$

(ii) There is $\vartheta \in X$ such that

$$lim_{n\to\infty} TR^{2n+1}x_o = \vartheta = lim_{n\to\infty} TS^{2n+2}x_o.$$

(iii) If T is subsequentially convergent, then $(R^{2n+1}x_o)$ and $(S^{2n+2}x_o)$ have a convergent subsequences.

(iv) There is unique common fixed point $u \in X$ such that $Ru = u = Su$.

(v) If T is sequentially convergent, then for each $x_o \in X$ the iterate sequences $(R^{2n+1}x_o)$ and $(S^{2n+2}x_o)$ converge to u.

Proof: Assume R and S have two distinct fixed points $x^*, y^* \in X$. Then

$$d(Tx^*, Ty^*) = d(TRx^*, TSy^*)$$

$$\leqslant \sigma d(Tx^*, Ty^*) + L_1 d(Tx^*, TRx^*)$$

Thus, we get

$$d(Tx^*, Ty^*) \leqslant \sigma d(Tx^*, Ty^*) \Leftrightarrow (1-\sigma) d(Tx^*, Ty^*) \leqslant 0$$

Therefore, $d(Tx^*, Ty^*) = 0$. Since T is one-to-one then $x^* = y^*$.

The rest of the proof same as the proof of the Theorem 3.2.

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